CS 188: Artificial Intelligence

Constraint Satisfaction Problems II

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Today

- Efficient Solution of CSPs
- Local Search
Constraint Satisfaction Problems

$N$ variables

$\text{domain } D$

constraints
Standard Search Formulation

- Standard search formulation of CSPs
  - States defined by the values assigned so far (partial assignments)
    - Initial state: the empty assignment, {}
    - Successor function: assign a value to an unassigned variable
    - Goal test: the current assignment is complete and satisfies all constraints
  - We started with the straightforward, naïve approach, then improved it
Backtracking Search
Backtracking Search

1. fix ordering
2. check constraints as you go
Explain it to your rubber duck!

Why is it ok to fix the ordering of variables?

Why is it good to fix the ordering of variables?
Filtering

Keep track of domains for unassigned variables and cross off bad options
Filtering: Forward Checking

- Filtering: Keep track of domains for unassigned variables and cross off bad options
- Forward checking: Cross off values that violate a constraint when added to the existing assignment
Filtering: Constraint Propagation

- Forward checking propagates information from assigned to unassigned variables, but doesn't provide early detection for all failures:
  - NT and SA cannot both be blue!
  - Why didn’t we detect this yet?
  - *Constraint propagation:* reason from constraint to constraint
Consistency of A Single Arc

- An arc $X \rightarrow Y$ is consistent iff for every $x$ in the tail there is some $y$ in the head which could be assigned without violating a constraint.

Delete from the tail!
Arc Consistency of an Entire CSP

- A simple form of propagation makes sure all arcs are consistent:

- Important: If X loses a value, neighbors of X need to be rechecked!
- Arc consistency detects failure earlier than forward checking
- Can be run as a preprocessor or after each assignment
- What’s the downside of enforcing arc consistency?

![Diagram of Arc Consistency with states WA, NT, Q, NSW, V, SA]
Enforcing Arc Consistency in a CSP

function AC-3(csp) returns the CSP, possibly with reduced domains
inputs: csp, a binary CSP with variables \{X_1, X_2, \ldots, X_n\}
local variables queue, a queue of arcs, initially all the arcs in csp

while queue is not empty do
  \((X_i, X_j)\) ← REMOVE-FIRST(queue)
  if REMOVE-INCONSISTENT-VALUES\((X_i, X_j)\) then
    for each \(X_k\) in NEIGHBORS[\(X_i\)] do
      add \((X_k, X_j)\) to queue

function REMOVE-INCONSISTENT-VALUES\((X_i, X_j)\) returns true iff succeeds
removed ← false
for each \(x\) in DOMAIN[\(X_i\)] do
  if no value \(y\) in DOMAIN[\(X_j\)] allows \((x, y)\) to satisfy the constraint \(X_i \leftarrow X_j\)
    then delete \(x\) from DOMAIN[\(X_i\)]; removed ← true
return removed

- Runtime: \(O(n^2d^3)\), can be reduced to \(O(n^2d^2)\)
- … but detecting all possible future problems is NP-hard – why?
Forward Checking – how does it relate?

- Forward checking: Cross off values that violate a constraint when added to the existing assignment

![Diagram showing forward checking in Australia]
Forward checking is a special type of enforcing arc consistency, in which we only enforce the arcs pointing into the newly assigned variable.
Limitations of Arc Consistency

- After enforcing arc consistency:
  - Can have one solution left
  - Can have multiple solutions left
  - Can have no solutions left (and not know it)

- Arc consistency still runs inside a backtracking search!
Video of Demo Coloring – Backtracking with Forward Checking – Complex Graph
Video of Demo Coloring – Backtracking with Arc Consistency – Complex Graph
K-Consistency

- Increasing degrees of consistency
  - 1-Consistency (Node Consistency): Each single node’s domain has a value which meets that node’s unary constraints
  - 2-Consistency (Arc Consistency): For each pair of nodes, any consistent assignment to one can be extended to the other
  - K-Consistency: For each k nodes, any consistent assignment to k-1 can be extended to the k\textsuperscript{th} node.

- Higher k more expensive to compute
  - (You need to know the k=2 case: arc consistency)
Strong K-Consistency

- Strong k-consistency: also k-1, k-2, ... 1 consistent
- Claim: strong n-consistency means we can solve without backtracking!
- Why?
  - Choose any assignment to any variable
  - Choose a new variable
  - By 2-consistency, there is a choice consistent with the first
  - Choose a new variable
  - By 3-consistency, there is a choice consistent with the first 2
  - ...
- Lots of middle ground between arc consistency and n-consistency! (e.g. k=3, called path consistency)
Ordering
Variable Ordering: Minimum remaining values (MRV):
- Choose the variable with the fewest legal left values in its domain

Why min rather than max?
Also called “most constrained variable”
“Fail-fast” ordering
Ordering: Least Constraining Value

- Value Ordering: Least Constraining Value
  - Given a choice of variable, choose the least constraining value
  - I.e., the one that rules out the fewest values in the remaining variables
  - Note that it may take some computation to determine this! (E.g., rerunning filtering)

- Why least rather than most?

- Combining these ordering ideas makes 1000 queens feasible
Demo: Coloring -- Backtracking + Forward Checking + Ordering
Summary

- Work with your rubber duck to write down:
  - How we order variables and why
  - How we order values and why
Iterative Improvement
Iterative Algorithms for CSPs

- Local search methods typically work with “complete” states, i.e., all variables assigned

- To apply to CSPs:
  - Take an assignment with unsatisfied constraints
  - Operators *reassign* variable values
  - No fringe! Live on the edge.

- Algorithm: While not solved,
  - Variable selection: randomly select any conflicted variable
  - Value selection: min-conflicts heuristic:
    - Choose a value that violates the fewest constraints
    - I.e., hill climb with $h(x) = \text{total number of violated constraints}$
Example: 4-Queens

- States: 4 queens in 4 columns \((4^4 = 256)\) states
- Operators: move queen in column
- Goal test: no attacks
- Evaluation: \(c(n) = \) number of attacks

[Demo: n-queens – iterative improvement (L5D1)]
[Demo: coloring – iterative improvement]
Video of Demo Iterative Improvement – n Queens
Video of Demo Iterative Improvement – Coloring
Given random initial state, can solve $n$-queens in almost constant time for arbitrary $n$ with high probability (e.g., $n = 10,000,000$)!

The same appears to be true for any randomly-generated CSP except in a narrow range of the ratio 

$$R = \frac{\text{number of constraints}}{\text{number of variables}}$$
Summary: CSPs

- CSPs are a special kind of search problem:
  - States are partial assignments
  - Goal test defined by constraints

- Basic solution: backtracking search

- Speed-ups:
  - Ordering
  - Filtering
  - Structure – turns out trees are easy!

- Iterative min-conflicts is often effective in practice
Local Search
Local Search

- Tree search keeps unexplored alternatives on the fringe (ensures completeness)

- Local search: improve a single option until you can’t make it better (no fringe!)

- New successor function: local changes

- Generally much faster and more memory efficient (but incomplete and suboptimal)
Hill Climbing

- Simple, general idea:
  - Start wherever
  - Repeat: move to the best neighboring state
  - If no neighbors better than current, quit

- What’s bad about this approach?
- What’s good about it?
Hill Climbing Quiz

Starting from X, where do you end up?

Starting from Y, where do you end up?

Starting from Z, where do you end up?
Simulated Annealing

- Idea: Escape local maxima by allowing downhill moves
  - But make them rarer as time goes on

```plaintext
function SIMULATED-ANNEALING(problem, schedule) returns a solution state
    inputs: problem, a problem
             schedule, a mapping from time to "temperature"
    local variables: current, a node
                     next, a node
                     T, a "temperature" controlling prob. of downward steps

    current ← MAKE-NODE(Initial-State[problem])
    for t ← 1 to ∞ do
        T ← schedule[t]
        if T = 0 then return current
        next ← a randomly selected successor of current
        ΔE ← VALUE[next] - VALUE[current]
        if ΔE > 0 then current ← next
        else current ← next only with probability $e^{ΔE/T}$
```
Simulated Annealing

- Theoretical guarantee:
  - Stationary distribution: \( p(x) \propto e^{\frac{E(x)}{kT}} \)
  - If \( T \) decreased slowly enough, will converge to optimal state!

- Is this an interesting guarantee?
  - Sounds like magic, but reality is reality:
    - The more downhill steps you need to escape a local optimum, the less likely you are to ever make them all in a row
    - People think hard about \textit{ridge operators} which let you jump around the space in better ways
Genetic algorithms use a natural selection metaphor

- Keep best N hypotheses at each step (selection) based on a fitness function
- Also have pairwise crossover operators, with optional mutation to give variety

Possibly the most misunderstood, misapplied (and even maligned) technique around
Example: N-Queens

- Why does crossover make sense here?
- When wouldn’t it make sense?
- What would mutation be?
- What would a good fitness function be?
Bonus (time permitting): Structure
Problem Structure

- Extreme case: independent subproblems
  - Example: Tasmania and mainland do not interact

- Independent subproblems are identifiable as connected components of constraint graph

- Suppose a graph of $n$ variables can be broken into subproblems of only $c$ variables:
  - Worst-case solution cost is $O((n/c)(d^c))$, linear in $n$
  - E.g., $n = 80$, $d = 2$, $c = 20$
  - $2^{80} = 4$ billion years at 10 million nodes/sec
  - $(4)(2^{20}) = 0.4$ seconds at 10 million nodes/sec
Tree-Structured CSPs

- Theorem: if the constraint graph has no loops, the CSP can be solved in $O(n d^2)$ time
  - Compare to general CSPs, where worst-case time is $O(d^n)$

- This property also applies to probabilistic reasoning (later): an example of the relation between syntactic restrictions and the complexity of reasoning
Tree-Structured CSPs

- Algorithm for tree-structured CSPs:
  - Order: Choose a root variable, order variables so that parents precede children
  - Remove backward: For \( i = n : 2 \), apply \( \text{RemoveInconsistent}(\text{Parent}(X_i), X_i) \)
  - Assign forward: For \( i = 1 : n \), assign \( X_i \) consistently with \( \text{Parent}(X_i) \)

- Runtime: \( O(n d^2) \) (why?)
Claim 1: After backward pass, all root-to-leaf arcs are consistent
Proof: Each $X \rightarrow Y$ was made consistent at one point and $Y$’s domain could not have been reduced thereafter (because $Y$’s children were processed before $Y$)

Claim 2: If root-to-leaf arcs are consistent, forward assignment will not backtrack
Proof: Induction on position

Why doesn’t this algorithm work with cycles in the constraint graph?

Note: we’ll see this basic idea again with Bayes’ nets
Improving Structure
Nearly Tree-Structured CSPs

- Conditioning: instantiate a variable, prune its neighbors' domains
- Cutset conditioning: instantiate (in all ways) a set of variables such that the remaining constraint graph is a tree
- Cutset size \( c \) gives runtime \( O( (d^c)(n-c) d^2) \), very fast for small \( c \)
Cutset Conditioning

Choose a cutset

Instantiate the cutset (all possible ways)

Compute residual CSP for each assignment

Solve the residual CSPs (tree structured)
Tree Decomposition*

- Idea: create a tree-structured graph of mega-variables
- Each mega-variable encodes part of the original CSP
- Subproblems overlap to ensure consistent solutions

\(\{(WA=r,SA=g,NT=b), (WA=b,SA=r,NT=g), \ldots\}\)

\(\{(NT=r,SA=g,Q=b), (NT=b,SA=g,Q=r), \ldots\}\)

Agree: \((M1, M2) \in \{(WA=g,SA=g,NT=g), (NT=g,SA=g,Q=g)\}, \ldots\)
Next Time:
Search when you’re not the only agent!