CS 188 Fall 2022 | Introduction to Artificial Intelligence | Exam Prep 10 Solutions

Q1. Naive Bayes: Pacman or Ghost?

You are standing by an exit as either Pacmen or ghosts come out of it. Every time someone comes out, you get two observations: a visual one and an auditory one, denoted by the random variables X_v and X_a, respectively. The visual observation informs you that the individual is either a Pacman ($X_v = 1$) or a ghost ($X_v = 0$). The auditory observation X_a is defined analogously. Your observations are a noisy measurement of the individual's true type, which is denoted by Y. After the indiviual comes out, you find out what they really are: either a Pacman ($Y = 1$) or a ghost ($Y = 0$). You have logged your observations and the true types of the first 20 individuals:

individual i	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
first observation $X_v^{(i)}$	0	0	1	0	1	0	0	1	1	1	0	1	1	0	1	1	1	0	0	0
second observation $X_a^{(i)}$	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0
individual's type $Y^{(i)}$	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	0	0	0

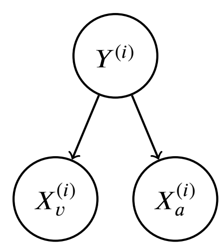
The superscript (i) denotes that the datum is the ith one. Now, the individual with $i = 20$ comes out, and you want to predict the individual's type $Y^{(20)}$ given that you observed $X_v^{(20)} = 1$ and $X_a^{(20)} = 1$.

(a) Assume that the types are independent, and that the observations are independent conditioned on the type. You can model this using naïve Bayes, with $X_v^{(i)}$ and $X_a^{(i)}$ as the features and $Y^{(i)}$ as the labels. Assume the probability distributions take on the following form:

$$P(X_v^{(i)} = x_v | Y^{(i)} = y) = \begin{cases} p_v & \text{if } x_v = y \\ 1 - p_v & \text{if } x_v \neq y \end{cases}$$

$$P(X_a^{(i)} = x_a | Y^{(i)} = y) = \begin{cases} p_a & \text{if } x_a = y \\ 1 - p_a & \text{if } x_a \neq y \end{cases}$$

$$P(Y^{(i)} = 1) = q$$



for $p_v, p_a, q \in [0, 1]$ and $i \in \mathbb{N}$.

(i) What's the maximum likelihood estimate of p_v, p_a and q?

$p_v =$ $\frac{4}{5}$ $\qquad p_a =$ $\frac{3}{5}$ $\qquad q =$ $\frac{1}{2}$

To estimate q, we count 10 $Y = 1$ and 10 $Y = 0$ in the data. For p_v, we have $p_v = 8/10$ cases where $X_v = 1$ given $Y = 1$ and $1 - p_v = 2/10$ cases where $X_v = 1$ given $Y = 0$. So $p_v = 4/5$. For p_a, we have $p_a = 2/10$ cases where $X_a = 1$ given $Y = 1$ and $1 - p_v = 0/10$ cases where $X_v = 1$ given $Y = 0$. The average of $2/10$ and 1 is $3/5$.

(ii) What is the probability that the next individual is Pacman given your observations? Express your answer in terms of the parameters p_v, p_a and q (you might not need all of them).

$$P(Y^{(20)} = 1 | X_v^{(20)} = 1, X_a^{(20)} = 1) = \frac{p_v p_a q}{p_v p_a q + (1 - p_v)(1 - p_a)(1 - q)}$$

The joint distribution $P(Y=1, X_v=1, X_a=1) = p_v p_a q$. For the denominator, we need to sum out over Y, that is, we need $P(Y=1, X_v=1, X_a=1) + P(Y=0, X_v=1, X_a=1)$.

Now, assume that you are given additional information: you are told that the individuals are actually coming out of a bus that just arrived, and each bus carries *exactly* 9 individuals. Unlike before, the types of every 9 consecutive individuals are *conditionally* independent given the bus type, which is denoted by Z. Only after all of the 9 individuals have walked out, you find out the bus type: one that carries mostly Pacmans ($Z=1$) or one that carries mostly ghosts ($Z=0$). Thus, you only know the bus type in which the first 18 individuals came in:

individual i	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
first observation $X_v^{(i)}$	0	0	1	0	1	0	0	1	1	1	0	1	1	0	1	1	1	0	0	0
second observation $X_a^{(i)}$	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0
individual's type $Y^{(i)}$	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	0	0	0

bus j	0	1	
bus type $Z^{(j)}$	0	1	

(b) You can model this using a variant of naïve bayes, where now 9 consecutive labels $Y^{(i)}, \dots, Y^{(i+8)}$ are *conditionally* independent given the bus type $Z^{(j)}$, for bus j and individual $i = 9j$. Assume the probability distributions take on the following form:

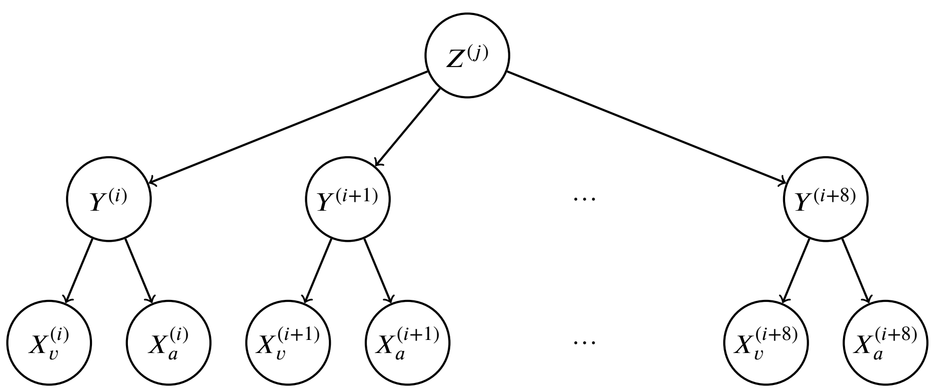
$$P(X_v^{(i)} = x_v | Y^{(i)} = y) = \begin{cases} p_v & \text{if } x_v = y \\ 1 - p_v & \text{if } x_v \neq y \end{cases}$$

$$P(X_a^{(i)} = x_a | Y^{(i)} = y) = \begin{cases} p_a & \text{if } x_a = y \\ 1 - p_a & \text{if } x_a \neq y \end{cases}$$

$$P(Y^{(i)} = 1 | Z^{(j)} = z) = \begin{cases} q_0 & \text{if } z = 0 \\ q_1 & \text{if } z = 1 \end{cases}$$

$$P(Z^{(j)} = 1) = r$$

for $p, q_0, q_1, r \in [0,1]$ and $i, j \in \mathbb{N}$.



(i) What's the maximum likelihood estimate of q_0, q_1 and r?

$q_0 =$ $\frac{2}{9}$ $\qquad q_1 =$ $\frac{8}{9}$ $\qquad r =$ $\frac{1}{2}$

For r, we've seen one ghost bus and one pacman bus, so $r = 1/2$. For q_0, we're finding $P(Y=1|Z=0)$, which is $2/9$. For q_1, we're finding $P(Y=1|Z=1)$, which is $8/9$.

(ii) Compute the following joint probability. Simplify your answer as much as possible and express it in terms of the parameters p_v, p_a, q_0, q_1 and r (you might not need all of them).

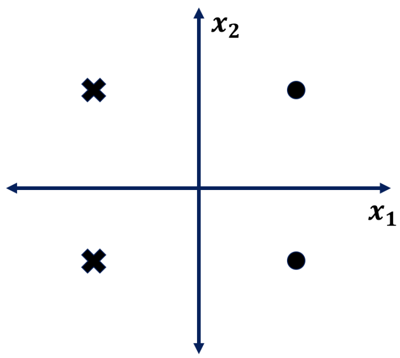
$P(Y^{(20)} = 1, X_v^{(20)} = 1, X_a^{(20)} = 1, Y^{(19)} = 1, Y^{(18)} = 1) =$ $p_a p_v [q_0^3 (1-r) + q_1^3 r]$

$$
\begin{aligned}
&P(Y^{(20)} = 1, X_v^{(20)} = 1, X_a^{(20)} = 1, Y^{(19)} = 1, Y^{(18)} = 1) \\
&= \sum_z P(Y^{(20)} = 1 | Z^{(2)} = z) P(Z^{(2)} = z) P(X_v^{(20)} = 1 | Y^{(20)} = 1) P(X_a^{(20)} = 1 | Y^{(20)} = 1) \\
&\qquad P(Y^{(19)} = 1 | Z^{(2)} = z) P(Y^{(18)} = 1 | Z^{(2)} = z) \\
&= q_0 (1 - r) p_a p_v q_0 q_0 + q_1 r p_a p_v q_1 q_1 \\
&= p_a p_v [q_0^3 (1 - r) + q_1^3 r]
\end{aligned}
$$

Q2. Linear Separability

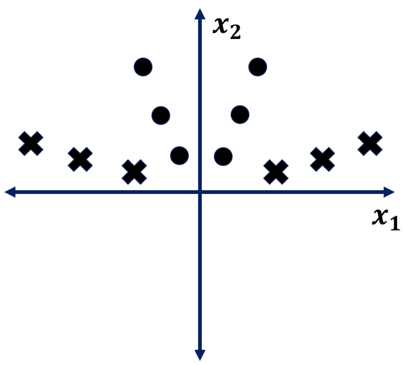
(a) For each of the datasets represented by the graphs below, please select the feature maps for which the perceptron algorithm can perfectly classify the data.

Each data point is in the form (x_1, x_2), and has some label Y, which is either a 1 (dot) or -1 (cross).



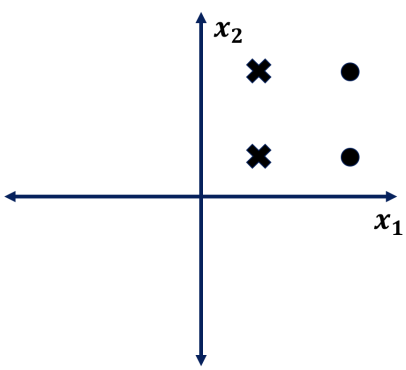
(i)

- [x] $[x_1 \quad x_2 \quad 1]$
- [x] $[x_1 \quad x_2 \quad x_1^2]$
- [x] $[x_1 \quad x_2 \quad |x_1|]$
- [x] $[x_1 \quad x_2 \quad Y]$
- [x] $[x_1 \quad x_2]$



(ii)

- [ ] $[x_1 \quad x_2 \quad 1]$
- [x] $[x_1 \quad x_2 \quad x_1^2]$
- [x] $[x_1 \quad x_2 \quad |x_1|]$
- [x] $[x_1 \quad x_2 \quad Y]$
- [ ] $[x_1 \quad x_2]$



(iii)

- [x] $[x_1 \quad x_2 \quad 1]$
- [x] $[x_1 \quad x_2 \quad x_1^2]$
- [ ] $[x_1 \quad x_2 \quad |x_1|]$
- [x] $[x_1 \quad x_2 \quad Y]$
- [ ] $[x_1 \quad x_2]$