Q1. Games

Alice is playing a two-player game with Bob, in which they move alternately. Alice is a maximizer. Although Bob is also a maximizer, Alice believes Bob is a minimizer with probability 0.5, and a maximizer with probability 0.5. Bob is aware of Alice’s assumption.

In the game tree below, square nodes are the outcomes, triangular nodes are Alice’s moves, and round nodes are Bob’s moves. Each node for Alice/Bob contains a tuple, the left value being Alice’s expectation of the outcome, and the right value being Bob’s expectation of the outcome.

Tie-breaking: choose the left branch.

The left values are Alice’s expectations, and are the only thing Alice can refer to when making decisions. The right values are Bob’s expectations, and they also accurately track the expected outcome of the game according to each choice of branching (regardless of it is Alice’s or Bob’s decision, since Bob has all the information). Hence the right values are accurate information about the game, and would be what Bob looks at when making his decisions. However, when it is Alice’s turn to make decisions, Bob will think about how Alice would maximize the outcome w.r.t to what she believes, and he will update his expectations accordingly.

(a) In the blanks below, fill in the tuple values for tuples $(B_a, B_b)$ and $(E_a, E_b)$ from the above game tree.

$(B_a, B_b) = (5, 9)$

$(E_a, E_b) = (7, 13)$

For a square node, its value v means the same to Alice and Bob, i.e., we can think of it as a tuple (v,v).

The left value of Alice’s nodes is the maximum of the left values of it’s children nodes, since Alice believes that the values of the nodes are given by left values, and it’s her turn of action, so she will choose the largest value.

The right value of Alice’s nodes is the right value from the child node that attains the maximum left value since Bob’s expectation is consistent with how Alice will act.

So for a triangular node, its tuple is the same as its child that has the maximum left value.

The left value of Bob’s nodes is the average of the maximum and minimum of the left values of it’s children nodes since Alice believes Bob is 50% possible to be adversarial and 50% possible to be friendly.

The right value of Bob’s nodes is the maximum of the right values of the immediate children nodes since Bob would choose the branch that gives the maximum outcome during his turn.

So for a round node, left = 0.5(max(children.left) + min(children.left)), and right = max(children.right).

(b) In this part, we will determine the values for tuple $(D_a, D_b)$.

(i) $D_a = 8 \quad X \quad 8+X \quad \bullet \quad 4+0.5X \quad \text{O} \quad \text{min}(8,X) \quad \text{O} \quad \text{max}(8,X)$

(ii) $D_b = 8 \quad X \quad 8+X \quad \text{O} \quad 4+0.5X \quad \text{O} \quad \text{min}(8,X) \quad \bullet \text{max}(8,X)$
It’s a round node, so left = 0.5(max(children.left) + min(children.left)), and right = max(children.right).
Its children: (8,8) and (X,X). So left = 0.5(8+X) = 4+0.5X, and right = max(8, X).

(c) Fill in the values for tuple \((C_a, C_b)\) below. For the bounds of X, you may write scalars, \(\infty\) or \(-\infty\).
If your answer contains a fraction, please write down the corresponding simplified decimal value in its place. (i.e., 4 instead of \(\frac{8}{2}\), and 0.5 instead of \(\frac{1}{2}\)).

1. If \(-\infty < X < 6\), \((C_a, C_b) = (\begin{array}{c} 7 \\ 13 \end{array})\)
2. Else, \((C_a, C_b) = (\begin{array}{c} 4+0.5X \\ \max(8, X) \end{array})\)

It’s a triangular node, so its tuple is the same as its child that has the maximum left value.
Its children: \((4+0.5X, \max(8,X))\) and \((7, 13)\).
So if \(4+0.5X < 7\), i.e. \(-\infty < X < 6\), it’s the same as child node \((7, 13)\), and otherwise it’s \((4+0.5X, \max(8,X))\).

(d) Fill in the values for tuple \((A_a, A_b)\) below. For the bounds of X, you may write scalars, \(\infty\) or \(-\infty\).
If your answer contains a fraction, please write down the corresponding simplified decimal value in its place. (i.e., 4 instead of \(\frac{8}{2}\), and 0.5 instead of \(\frac{1}{2}\)).

1. If \(-\infty < X < 6\), \((A_a, A_b) = (\begin{array}{c} 6 \\ 13 \end{array})\)
2. Else, \((A_a, A_b) = (\begin{array}{c} 4.5+0.25X \\ \max(9, X) \end{array})\)

It’s a round node, so left = 0.5(max(children.left) + min(children.left)), and right = max(children.right).
Its children: (5,9) and node "Part (c)".
If \(-\infty < X < 6\), these children are (5,9) and (7, 13).
left = 0.5(max(children.left) + min(children.left)) = 0.5(5+7) = 6
right = max(children.right) = max(9, 13) = 13.
Otherwise (6 < X < +\infty), these children are (5,9) and (4+0.5X, \max(8,X)).
left = 0.5(max(children.left) + min(children.left)) = 0.5(5+4+0.5X) = 4.5 + 0.25X
right = max(children.right) = max(9, \max(8,X)) = max(9, X).
Q2. MedianMiniMax

You’re living in utopia! Despite living in utopia, you still believe that you need to maximize your utility in life, other people want to minimize your utility, and the world is a 0 sum game. But because you live in utopia, a benevolent social planner occasionally steps in and chooses an option that is a compromise. Essentially, the social planner (represented as the pentagon) is a median node that chooses the successor with median utility. Your struggle with your fellow citizens can be modelled as follows:

![Diagram of MedianMiniMax](image)

There are some nodes that we are sometimes able to prune. In each part, mark all of the terminal nodes such that there exists a possible situation for which the node can be pruned. In other words, you must consider all possible pruning situations. Assume that evaluation order is left to right and all $V_i$’s are distinct.

Note that as long as there exists ANY pruning situation (does not have to be the same situation for every node), you should mark the node as prunable. Also, alpha-beta pruning does not apply here, simply prune a sub-tree when you can reason that its value will not affect your final utility.

(a) $\square V_1$  
\$ V_2 \checkmark$  
\$ V_3 \checkmark$  
\$ V_4 \checkmark$  
\$ None \checkmark$

(b) $\square V_5 \checkmark$  
\$ V_6 \checkmark$  
\$ V_7 \checkmark$  
\$ V_8 \checkmark$  
\$ None \checkmark$

(c) $\square V_9 \checkmark$  
\$ V_10 \checkmark$  
\$ V_11 \checkmark$  
\$ V_12 \checkmark$  
\$ None \checkmark$

(d) $\square V_{13} \checkmark$  
\$ V_{14} \checkmark$  
\$ V_{15} \checkmark$  
\$ V_{16} \checkmark$  
\$ None \checkmark$
**Part a:**
For the left median node with three children, at least two of the children’s values must be known since one of them will be guaranteed to be the value of the median node passed up to the final maximizer. For this reason, none of the nodes in part a can be pruned.

**Part b (pruning $V_7, V_8$):**
Let $min_1, min_2, min_3$ be the values of the three minimizer nodes in this subtree.

In this case, we may not need to know the final value $min_3$. The reason for this is that we may be able to put a bound on its value after exploring only partially, and determine the value of the median node as either $min_1$ or $min_2$ if $min_3 \leq \min(min_1, min_2)$ or $min_3 \geq \max(min_1, min_2)$.

We can put an upper bound on $min_3$ by exploring the left subtree $V_5, V_6$ and if $\max(V_5, V_6)$ is lower than both $min_1$ and $min_2$, the median node’s value is set as the smaller of $min_1, min_2$ and we don’t have to explore $V_7, V_8$ in Figure 1.

**Part b (pruning $V_6$):**
It’s possible for us to put a lower bound on $min_3$. If $V_5$ is larger than both $min_1$ and $min_2$, we do not need to explore $V_6$.

The reason for this is subtle, but if the minimizer chooses the left subtree, we know that $min_3 \geq V_5 \geq \max(min_1, min_2)$ and we don’t need $V_6$ to get the correct value for the median node which will be the larger of $min_1, min_2$.

If the minimizer chooses the value of the right subtree, the value at $V_6$ is unnecessary again since the minimizer never chose its subtree.
Part c (pruning $V_{11}, V_{12}$):
Assume the highest maximizer node has a current value $max_1 \geq Z$ set by the left subtree and the three minimizers on this right subtree have value $min_1, min_2, min_3$.

In this part, if $min_1 \leq \max (V_9, V_{10}) \leq Z$, we do not have to explore $V_{11}, V_{12}$. Once again, the reasoning is subtle, but we can now realize if either $min_2 \leq Z$ or $min_3 \leq Z$ then the value of the right median node is for sure $\leq Z$ and is useless.

Only if both $min_2, min_3 \geq Z$ will the whole right subtree have an effect on the highest maximizer, but in this case the exact value of $min_1$ is not needed, just the information that it is $\leq Z$. Clearly in both cases, $V_{11}, V_{12}$ are not needed since an exact value of $min_1$ is not needed.

We will also take the time to note that if $V_9 \geq Z$ we do have to continue the exploring as $V_{10}$ could be even greater and the final value of the top maximizer, so $V_{10}$ can’t really be pruned.

Part d (pruning $V_{14}, V_{15}, V_{16}$):
Continuing from part c, if we find that $min_1 \leq Z$ and $min_2 \leq Z$ we can stop.

We can realize this as soon we explore $V_{13}$. Once we figure this out, we know that our median node’s value must be one of these two values, and neither will replace $Z$ so we can stop.