1 HMMs

Consider the following Hidden Markov Model. $O_1$ and $O_2$ are supposed to be shaded.

Suppose that we observe $O_1 = a$ and $O_2 = b$. Using the forward algorithm, compute the probability distribution $P(W_2|O_1 = a, O_2 = b)$ one step at a time.

(a) Compute $P(W_1, O_1 = a)$.

\[
P(W_1, O_1 = a) = P(W_1)P(O_1 = a|W_1) \\
P(W_1 = 0, O_1 = a) = (0.3)(0.9) = 0.27 \\
P(W_1 = 1, O_1 = a) = (0.7)(0.5) = 0.35
\]

(b) Using the previous calculation, compute $P(W_2, O_1 = a)$.

\[
P(W_2, O_1 = a) = \sum_{w_1} P(w_1, O_1 = a)P(W_2|w_1) \\
P(W_2 = 0, O_1 = a) = (0.27)(0.4) + (0.35)(0.8) = 0.388 \\
P(W_2 = 1, O_1 = a) = (0.27)(0.6) + (0.35)(0.2) = 0.232
\]

(c) Using the previous calculation, compute $P(W_2, O_1 = a, O_2 = b)$.

\[
P(W_2, O_1 = a, O_2 = b) = P(W_2, O_1 = a)P(O_2 = b|W_2) \\
P(W_2 = 0, O_1 = a, O_2 = b) = (0.388)(0.1) = 0.0388 \\
P(W_2 = 1, O_1 = a, O_2 = b) = (0.232)(0.5) = 0.116
\]

(d) Finally, compute $P(W_2|O_1 = a, O_2 = b)$.

Renormalizing the distribution above, we have

\[
P(W_2 = 0|O_1 = a, O_2 = b) = 0.0388/(0.0388 + 0.116) \approx 0.25 \\
P(W_2 = 1|O_1 = a, O_2 = b) = 0.116/(0.0388 + 0.116) \approx 0.75
\]
2 Particle Filtering

Let’s use Particle Filtering to estimate the distribution of \( P(W_2|O_1 = a, O_2 = b) \). Here’s the HMM again. \( O_1 \) and \( O_2 \) are supposed to be shaded.

We start with two particles representing our distribution for \( W_1 \).

\[
\begin{align*}
P_1 &: W_1 = 0 \\
P_2 &: W_1 = 1
\end{align*}
\]

Use the following random numbers to run particle filtering:

\[
[0.22, 0.05, 0.33, 0.20, 0.84, 0.54, 0.79, 0.66, 0.14, 0.96]
\]

(a) **Observe**: Compute the weight of the two particles after evidence \( O_1 = a \).

\[
\begin{align*}
w(P_1) &= P(O_t = a|W_t = 0) = 0.9 \\
w(P_2) &= P(O_t = a|W_t = 1) = 0.5
\end{align*}
\]

(b) **Resample**: Using the random numbers, resample \( P_1 \) and \( P_2 \) based on the weights.

We now sample from the weighted distribution we found above. Using the first two random samples, we find:

\[
\begin{align*}
P_1 &= \text{sample}(\text{weights}, 0.22) = 0 \\
P_2 &= \text{sample}(\text{weights}, 0.05) = 0
\end{align*}
\]

(c) **Predict**: Sample \( P_1 \) and \( P_2 \) from applying the time update.

\[
\begin{align*}
P_1 &= \text{sample}(P(W_{t+1}|W_t = 0), 0.33) = 0 \\
P_2 &= \text{sample}(P(W_{t+1}|W_t = 0), 0.20) = 0
\end{align*}
\]

(d) **Update**: Compute the weight of the two particles after evidence \( O_2 = b \).

\[
\begin{align*}
w(P_1) &= P(O_t = b|W_t = 0) = 0.1 \\
w(P_2) &= P(O_t = b|W_t = 0) = 0.1
\end{align*}
\]

(e) **Resample**: Using the random numbers, resample \( P_1 \) and \( P_2 \) based on the weights.

Because both of our particles have \( X = 0 \), resampling will still leave us with two particles with \( X = 0 \).

\[
\begin{align*}
P_1 &= 0 \\
P_2 &= 0
\end{align*}
\]
(f) What is our estimated distribution for $P(W_2|O_1 = a, O_2 = b)$?

$$P(W_2 = 0|O_1 = a, O_2 = b) = 2/2 = 1$$
$$P(W_2 = 1|O_1 = a, O_2 = b) = 0/2 = 0$$