What is Search For?

- Assumptions about the world: a single agent, deterministic actions, fully observed state, discrete state space

- Planning: sequences of actions
  - The path to the goal is the important thing
  - Paths have various costs, depths
  - Heuristics give problem-specific guidance

- Identification: assignments to variables
  - The goal itself is important, not the path
  - All paths at the same depth (for some formulations)
  - CSPs are a specialized class of identification problems
Constraint Satisfaction Problems
Constraint Satisfaction Problems

- Standard search problems:
  - State is a “black box”: arbitrary data structure
  - Goal test can be any function over states
  - Successor function can also be anything

- Constraint satisfaction problems (CSPs):
  - A special subset of search problems
  - State is defined by variables $X_i$ with values from a domain $D$ (sometimes $D$ depends on $i$)
  - Goal test is a set of constraints specifying allowable combinations of values for subsets of variables

- Simple example of a formal representation language

- Allows useful general-purpose algorithms with more power than standard search algorithms
CSP Examples

- Western Australia
- Northern Territory
- Queensland
- South Australia
- New South Wales
- Victoria
- Tasmania
Example: Map Coloring

- **Variables:** WA, NT, Q, NSW, V, SA, T
- **Domains:** \( D = \{\text{red, green, blue}\} \)
- **Constraints:** adjacent regions must have different colors
  - Implicit: \( WA \neq NT \)
  - Explicit: \( (WA, NT) \in \{(\text{red, green}), (\text{red, blue}), \ldots\} \)
- **Solutions are assignments satisfying all constraints, e.g.:**
  
  \( \{WA=\text{red, NT=green, Q=red, NSW=green, V=red, SA=blue, T=green}\} \)
Example: N-Queens

**Formulation 1:**
- Variables: $X_{ij}$
- Domains: $\{0, 1\}$
- Constraints

\[
\forall i, j, k \ (X_{ij}, X_{ik}) \in \{(0, 0), (0, 1), (1, 0)\}
\]
\[
\forall i, j, k \ (X_{ij}, X_{kj}) \in \{(0, 0), (0, 1), (1, 0)\}
\]
\[
\forall i, j, k \ (X_{ij}, X_{i+k,j+k}) \in \{(0, 0), (0, 1), (1, 0)\}
\]
\[
\forall i, j, k \ (X_{ij}, X_{i+k,j-k}) \in \{(0, 0), (0, 1), (1, 0)\}
\]
\[
\sum_{i,j} X_{ij} = N
\]
Example: N-Queens

- **Formulation 2:**
  - **Variables:** $Q_k$
  - **Domains:** $\{1, 2, 3, \ldots N\}$
  - **Constraints:**
    - Implicit: $\forall i, j \text{ non-threatening}(Q_i, Q_j)$
    - Explicit: $(Q_1, Q_2) \in \{(1, 3), (1, 4), \ldots\}$
Constraint Graphs
Constraint Graphs

- Binary CSP: each constraint relates (at most) two variables
- Binary constraint graph: nodes are variables, arcs show constraints
- General-purpose CSP algorithms use the graph structure to speed up search. E.g., Tasmania is an independent subproblem!

[Demo: CSP applet (made available by aispace.org) -- n-queens]
Example: Cryptarithmetic

- **Variables:**
  \[ F T U W R O X_1 X_2 X_3 \]

- **Domains:**
  \[ \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\} \]

- **Constraints:**
  \[ \text{alldiff}(F, T, U, W, R, O) \]
  \[ O + O = R + 10 \cdot X_1 \]
  \[ \ldots \]
Example: Sudoku

- **Variables:**
  - Each (open) square
- **Domains:**
  - \{1,2,…,9\}
- **Constraints:**
  - 9-way alldiff for each column
  - 9-way alldiff for each row
  - 9-way alldiff for each region
  (or can have a bunch of pairwise inequality constraints)
Example: The Waltz Algorithm

- The Waltz algorithm is for interpreting line drawings of solid polyhedra as 3D objects
- An early example of an AI computation posed as a CSP

Approach:
- Each intersection is a variable
- Adjacent intersections impose constraints on each other
- Solutions are physically realizable 3D interpretations
Varieties of CSPs and Constraints
Varieties of CSPs

- **Discrete Variables**
  - Finite domains
    - Size $d$ means $O(d^n)$ complete assignments
    - E.g., Boolean CSPs, including Boolean satisfiability (NP-complete)
  - Infinite domains (integers, strings, etc.)
    - E.g., job scheduling, variables are start/end times for each job
    - Linear constraints solvable, nonlinear undecidable

- **Continuous variables**
  - E.g., start/end times for Hubble Telescope observations
  - Linear constraints solvable in polynomial time by LP methods (see cs170 for a bit of this theory)
Varieties of Constraints

- **Varieties of Constraints**
  - Unary constraints involve a single variable (equivalent to reducing domains), e.g.:
    \[ \text{SA} \neq \text{green} \]
  - Binary constraints involve pairs of variables, e.g.:
    \[ \text{SA} \neq \text{WA} \]
  - Higher-order constraints involve 3 or more variables: e.g., cryptarithmetic column constraints

- **Preferences (soft constraints):**
  - E.g., red is better than green
  - Often representable by a cost for each variable assignment
  - Gives constrained optimization problems
  - (We’ll ignore these until we get to Bayes’ nets)
Real-World CSPs

- Scheduling problems: e.g., when can we all meet?
- Timetabling problems: e.g., which class is offered when and where?
- Assignment problems: e.g., who teaches what class
- Hardware configuration
- Transportation scheduling
- Factory scheduling
- Circuit layout
- Fault diagnosis
- ... lots more!

- Many real-world problems involve real-valued variables...
Solving CSPs
Standard Search Formulation

- Standard search formulation of CSPs

- States defined by the values assigned so far (partial assignments)
  - Initial state: the empty assignment, {}  
  - Successor function: assign a value to an unassigned variable 
  - Goal test: the current assignment is complete and satisfies all constraints

- We’ll start with the straightforward, naïve approach, then improve it
Search Methods

- What would BFS do?
- What would DFS do?
- What problems does naïve search have?

[Demo: coloring -- dfs]
Backtracking Search
Backtracking Search

- Backtracking search is the basic uninformed algorithm for solving CSPs

- **Idea 1: One variable at a time**
  - Variable assignments are commutative, so fix ordering
  - I.e., [WA = red then NT = green] same as [NT = green then WA = red]
  - Only need to consider assignments to a single variable at each step

- **Idea 2: Check constraints as you go**
  - I.e. consider only values which do not conflict with previous assignments
  - Might have to do some computation to check the constraints
  - “Incremental goal test”

- Depth-first search with these two improvements is called *backtracking search* (not the best name)

- Can solve n-queens for $n \approx 25$
Backtracking Example
Backtracking Search

function BACKTRACKING-SEARCH(csp) returns solution/failure
  return RECURSIVE-BACKTRACKING({}, csp)

function RECURSIVE-BACKTRACKING(assignment, csp) returns soln/failure
  if assignment is complete then return assignment
  var ← SELECT-UNASSIGNED-VARIABLE(VARIABLES[csp], assignment, csp)
  for each value in ORDER-DOMAIN-VALUES(var, assignment, csp) do
    if value is consistent with assignment given CONSTRAINTS[csp] then
      add \{var = value\} to assignment
      result ← RECURSIVE-BACKTRACKING(assignment, csp)
      if result ≠ failure then return result
      remove \{var = value\} from assignment
  return failure

- Backtracking = DFS + variable-ordering + fail-on-violation
- What are the choice points?

[Demo: coloring -- backtracking]
Improving Backtracking

- General-purpose ideas give huge gains in speed

- Ordering:
  - Which variable should be assigned next?
  - In what order should its values be tried?

- Filtering: Can we detect inevitable failure early?

- Structure: Can we exploit the problem structure?
Filtering
Filtering: Forward Checking

- Filtering: Keep track of domains for unassigned variables and cross off bad options
- Forward checking: Cross off values that violate a constraint when added to the existing assignment
Filtering: Constraint Propagation

- Forward checking propagates information from assigned to unassigned variables, but doesn't provide early detection for all failures:

- NT and SA cannot both be blue!
- Why didn’t we detect this yet?
- *Constraint propagation:* reason from constraint to constraint
Consistency of A Single Arc

- An arc \( X \rightarrow Y \) is **consistent** iff for every \( x \) in the tail there is some \( y \) in the head which could be assigned without violating a constraint

- Tail = NT, head = WA
  - If NT = blue: we could assign WA = red
  - If NT = green: we could assign WA = red
  - If NT = red: there is no remaining assignment to WA that we can use
  - Deleting NT = red from the tail makes this arc consistent

- Forward checking: Enforcing consistency of arcs pointing to each new assignment
A simple form of propagation makes sure all arcs are consistent:

- Arc V to NSW is consistent: for every x in the tail there is some y in the head which could be assigned without violating a constraint.
A simple form of propagation makes sure all arcs are consistent:

- Arc SA to NSW is consistent: for every \( x \) in the tail there is some \( y \) in the head which could be assigned without violating a constraint.
Arc Consistency of an Entire CSP (3/6)

- A simple form of propagation makes sure all arcs are consistent:
  - Arc NSW to SA is not consistent: if we assign NSW = blue, there is no valid assignment left for SA
  - To make this arc consistent, we delete NSW = blue from the tail
A simple form of propagation makes sure all arcs are consistent:

- Remember that arc V to NSW was consistent, when NSW had red and blue in its domain.
- After removing blue from NSW, this arc might not be consistent anymore! We need to recheck this arc.
- Important: If X loses a value, neighbors of X need to be rechecked!
A simple form of propagation makes sure all arcs are consistent:

- Arc SA to NT is inconsistent. We make it consistent by deleting from the tail (SA = blue).
A simple form of propagation makes sure all arcs are consistent:

- SA has an empty domain, so we detect failure. There is no way to solve this CSP with WA = red and Q = green, so we backtrack.
- Arc consistency detects failure earlier than forward checking
- Can be run as a preprocessor or after each assignment
Enforcing Arc Consistency in a CSP

function \text{AC-3}(csp) \text{ returns} the CSP, possibly with reduced domains

\textbf{inputs:} \(csp\), a binary CSP with variables \(\{X_1, X_2, \ldots, X_n\}\)

\textbf{local variables:} \textit{queue}, a queue of arcs, initially all the arcs in \(csp\)

\hspace{1em} \textbf{while} \textit{queue} is not empty do
\hspace{2em} \((X_i, X_j) \leftarrow \text{REMOVE-FIRST}(\textit{queue})\)
\hspace{2em} \textbf{if} \text{REMOVE-INCONSISTENT-VALUES}(X_i, X_j) \textbf{then}
\hspace{3em} \textbf{for each} \(X_k\) in \text{NEIGHBORS}[X_i] \textbf{do}
\hspace{4em} \text{add} \((X_k, X_i)\) to \textit{queue}

\textbf{function} \text{REMOVE-INCONSISTENT-VALUES}(X_i, X_j) \text{ returns} \text{true iff succeeds}
\hspace{1.5em} \textit{removed} \leftarrow \text{false}
\hspace{2em} \textbf{for each} \(x\) in \text{DOMAIN}[X_i] \textbf{do}
\hspace{3em} \textbf{if} no value \(y\) in \text{DOMAIN}[X_j] allows \((x, y)\) to satisfy the constraint \(X_i \leftarrow X_j\)
\hspace{4em} \textbf{then} delete \(x\) from \text{DOMAIN}[X_i]; \textit{removed} \leftarrow \text{true}
\hspace{1em} \textbf{return} \textit{removed}

- Runtime: \(O(n^2d^3)\), can be reduced to \(O(n^2d^2)\)
- ... but detecting all possible future problems is NP-hard – why?

[Demo: CSP applet (made available by aispace.org) -- n-queens]
Limitations of Arc Consistency

- After enforcing arc consistency:
  - Can have one solution left
  - Can have multiple solutions left
  - Can have no solutions left (and not know it)

- Arc consistency still runs inside a backtracking search!

What went wrong here?

[Demo: coloring -- forward checking]
[Demo: coloring -- arc consistency]