Announcements

- **HW 6** due **today** (Oct 24) at 11:59pm PT

- **Project 4** released due **Monday, Nov 6** at 11:59pm PT

- **HW 4 & 5** solutions released, regrades available soon

- Midterm clobber policy finalized
  - See Ed announcement & policies page for details
CS 188: Artificial Intelligence

Hidden Markov Models

University of California, Berkeley

[These slides were created by Dan Klein and Pieter Abbeel for CS188 Intro to AI at UC Berkeley. All CS188 materials are available at http://ai.berkeley.edu.]
Often, we want to reason about a sequence of observations

- Speech recognition
- Robot localization
- User attention
- Medical monitoring
- Language understanding

Need to introduce time (or space) into our models and update beliefs based on:

- Getting more evidence (we did this with BNs)
- World changing over time/space (new this week)
Today’s Topics

- Quick probability recap
- **Markov Chains & their Stationary Distributions**
  - How beliefs about state change with passage of time
- **Hidden Markov Models (HMMs) formulation**
  - How beliefs change with passage of time and evidence
- **Filtering with HMMs**
  - How to infer beliefs from evidence
Probability Recap

- **Conditional probability**
  \[ P(x|y) = \frac{P(x, y)}{P(y)} \]

- **Marginal probability**
  \[ P(x) = \sum_y P(x, y) \]

- **Product rule**
  \[ P(x, y) = P(x|y)P(y) \]

- **Chain rule**
  \[ P(X_1, X_2, \ldots, X_n) = P(X_1)P(X_2|X_1)P(X_3|X_1, X_2)\ldots \]
  \[ = \prod_{i=1}^{n} P(X_i|X_1, \ldots, X_{i-1}) \]
Probability Recap

- **X, Y independent if and only if:** \( \forall x, y : P(x, y) = P(x)P(y) \)

- **X and Y are conditionally independent given Z if and only if:** \( X \perp Y \mid Z \)

  \[ \forall x, y, z : P(x, y \mid z) = P(x \mid z)P(y \mid z) \]

- **Proportionality:** \( P(X) \propto f(X) \) or \( P(X) \propto_x f(X) \) means \( P(X) = kf(X) \) (for some \( k \) that doesn’t depend on \( X \)). Equivalent to: \( P(X) = \frac{f(X)}{\sum_x f(x)} \)

  - **Example:**

    |   |   |   |
    |---|---|---|
    | \( x_1 \) | 0.4 | \( 0.4 / (0.4 + 0.2) \) |
    | \( x_2 \) | 0.2 | \( 0.2 / (0.4 + 0.2) \) |
Today’s Topics

- Quick probability recap
- Markov Chains & their Stationary Distributions
- Hidden Markov Models (HMMs) formulation
- Filtering with HMMs
Value of $X$ at a given time is called the state.

Parameters: called transition probabilities or dynamics, specify how the state evolves over time (also, initial state probabilities).

Stationarity assumption: transition probabilities the same at all times.

Same as MDP transition model, but no choice of action.

A "growable" BN (can always use BN methods if we truncate to fixed length).
Conditional Independence

- **Basic conditional independence:**
  - Past and future independent given the present
  - Each time step only depends on the previous
  - This is called the (first order) Markov property
Example Markov Chain: Weather

- States: $X = \{\text{rain, sun}\}$
- Initial distribution: 1.0 sun
- CPT $P(X_t | X_{t-1})$:

| $X_{t-1}$ | $X_t$ | $P(X_t | X_{t-1})$ |
|-----------|-------|-----------------|
| sun       | sun   | 0.9             |
| sun       | rain  | 0.1             |
| rain      | sun   | 0.3             |
| rain      | rain  | 0.7             |

Two new ways of representing the same CPT
Example Markov Chain: Weather

- **Initial distribution: 1.0 sun**
  - We know: \( P(X_1) \) \( P(X_t|X_{t-1}) \)

- **What is the probability distribution after one step?**

\[
P(X_2 = \text{sun}) = \sum_{x_1} P(x_1, X_2 = \text{sun}) = \sum_{x_1} P(X_2 = \text{sun}|x_1)P(x_1)
\]

\[
= P(X_2 = \text{sun}|X_1 = \text{sun})P(X_1 = \text{sun}) + P(X_2 = \text{sun}|X_1 = \text{rain})P(X_1 = \text{rain})
\]

\[
= 0.9 \cdot 1.0 + 0.3 \cdot 0.0 = 0.9
\]

| \( X_{t-1} \) | \( X_t \) | \( P(X_t|X_{t-1}) \) |
|----------------|--------|------------------|
| sun            | sun    | 0.9              |
| sun            | rain   | 0.1              |
| rain           | sun    | 0.3              |
| rain           | rain   | 0.7              |
Mini-Forward Algorithm

- Question: What’s $P(X)$ on some day $t$?

- We know $P(X_1)$ and $P(X_t | X_{t-1})$

\[
P(X_1) = \text{known}
\]

\[
P(x_t) = \sum_{x_{t-1}} P(x_{t-1}, x_t)
\]

\[
= \sum_{x_{t-1}} P(x_t | x_{t-1}) P(x_{t-1})
\]

Forward simulation
Example Run of Mini-Forward Algorithm

- From initial observation of sun

\[
\begin{align*}
\langle 1.0 \rangle & \quad \langle 0.9 \rangle & \quad \langle 0.16 \rangle & \quad \langle 0.804 \rangle \\
0.0 & \quad 0.1 & \quad 0.196 & \quad P(X_\infty)
\end{align*}
\]

- From initial observation of rain

\[
\begin{align*}
\langle 0.0 \rangle & \quad \langle 0.3 \rangle & \quad \langle 0.52 \rangle & \quad \langle 0.588 \rangle \\
1.0 & \quad 0.7 & \quad 0.412 & \quad P(X_\infty)
\end{align*}
\]

- From yet another initial distribution \( P(X_1) \):

\[
\begin{align*}
\langle p \rangle & \quad \langle 1 - p \rangle \\
\text{...} & \quad \text{...} \\
0.75 & \quad 0.25
\end{align*}
\]
Video of Demo Ghostbusters Basic Dynamics
Video of Demo Ghostbusters Circular Dynamics
Video of Demo Ghostbusters Whirlpool Dynamics
Stationary Distributions

- For most chains:
  - Influence of the initial distribution gets less and less over time.
  - The distribution we end up in is independent of the initial distribution.

- Stationary distribution:
  - The distribution we end up with is called the stationary distribution $P_\infty$ of the chain.
  - It satisfies

$$P_\infty(X) = P_{\infty+1}(X) = \sum_x P(X|x)P_\infty(x)$$
Example: Stationary Distributions

- Question: What’s \( P(X) \) at time \( t = \infty \)?

\[
P_\infty(X) = P_{\infty+1}(X) = \sum_x P(X|x)P_\infty(x)
\]

\[
P_\infty(\text{sun}) = P(\text{sun}|\text{sun})P_\infty(\text{sun}) + P(\text{sun}|\text{rain})P_\infty(\text{rain})
\]

\[
P_\infty(\text{rain}) = P(\text{rain}|\text{sun})P_\infty(\text{sun}) + P(\text{rain}|\text{rain})P_\infty(\text{rain})
\]

\[
P_\infty(\text{sun}) = 0.9P_\infty(\text{sun}) + 0.3P_\infty(\text{rain})
\]

\[
P_\infty(\text{rain}) = 0.1P_\infty(\text{sun}) + 0.7P_\infty(\text{rain})
\]

\[
P_\infty(\text{sun}) = 3P_\infty(\text{rain})
\]

Also: \( P_\infty(\text{sun}) + P_\infty(\text{rain}) = 1 \)

\[
P_\infty(\text{sun}) = 3/4
\]

\[
P_\infty(\text{rain}) = 1/4
\]

- Alternatively: run simulation for a long (ideally infinite) time

| \( X_{t-1} \) | \( X_t \) | \( P(X_t|X_{t-1}) \) |
|------------|--------|------------------|
| sun        | sun    | 0.9              |
| sun        | rain   | 0.1              |
| rain       | sun    | 0.3              |
| rain       | rain   | 0.7              |
Application of Stationary Distribution: Web Link Analysis

- **PageRank over a web graph**
  - Each web page is a state
  - Initial distribution: uniform over pages
  - Transitions:
    - With prob. $c$, uniform jump to a random page (dotted lines, not all shown)
    - With prob. $1-c$, follow a random outlink (solid lines)

- **Stationary distribution**
  - Will spend more time on highly reachable pages
  - E.g. many ways to get to the Acrobat Reader download page
  - Somewhat robust to link spam
  - Google 1.0 returned the set of pages containing all your keywords in decreasing rank, now all search engines use link analysis along with many other factors (rank actually getting less important over time)
Today’s Topics

- Quick probability recap
- Markov Chains & their Stationary Distributions
- Hidden Markov Models (HMMs) formulation
- Filtering with HMMs
Hidden Markov Models
Pacman – Sonar
Video of Demo Pacman – Sonar (no beliefs)
Hidden Markov Models

- Markov chains not so useful for most agents
Hidden Markov Models

- Markov chains not so useful for most agents

  \[
  X_1 \rightarrow X_2 \rightarrow X_3 \rightarrow X_4 \rightarrow \ldots
  \]

  - Need observations to update your beliefs

- Hidden Markov models (HMMs)
  - Underlying Markov chain over states X
  - You observe outputs (effects) at each time step

\[
X_1 \rightarrow X_2 \rightarrow X_3 \rightarrow X_4 \rightarrow \ldots
\]

\[
E_1 \rightarrow E_2 \rightarrow E_3 \rightarrow E_4 \rightarrow \ldots
\]
Example: Weather HMM

- An HMM is defined by:
  - Initial distribution: \( P(X_1) \)
  - Transitions: \( P(X_t | X_{t-1}) \)
  - Emissions: \( P(E_t | X_t) \)

\[
\begin{align*}
P(X_t | X_{t-1}) \\
&\text{Rain}_{t-1} \quad \text{Rain}_t \quad \text{Rain}_{t+1} \\
&\text{Umbrella}_{t-1} \quad \text{Umbrella}_t \quad \text{Umbrella}_{t+1}
\end{align*}
\]

\[
\begin{array}{c|c|c|c|c}
\text{Transitions} & \text{Rain}_t & \text{Rain}_{t+1} & \text{Umbrella}_t & \text{Umbrella}_{t+1} \\
\hline
\text{R}_{t-1} & \text{R}_t & P(R_t | R_{t-1}) & \text{U}_t & P(U_t | R_t) \\
+\text{r} & +\text{r} & 0.7 & +\text{u} & 0.9 \\
+\text{r} & -\text{r} & 0.3 & -\text{u} & 0.1 \\
-\text{r} & +\text{r} & 0.3 & +\text{u} & 0.2 \\
-\text{r} & -\text{r} & 0.7 & -\text{u} & 0.8 \\
\end{array}
\]
Example: Ghostbusters HMM

- \( P(X_1) = \text{uniform} \)

- \( P(X' | X) = \) usually move clockwise, but sometimes move in a random direction or stay in place

- \( P(R_{ij} | X) = \) same sensor model as before: red means close, green means far away.

\[
\begin{array}{ccc}
1/9 & 1/9 & 1/9 \\
1/9 & 1/9 & 1/9 \\
1/9 & 1/9 & 1/9 \\
\end{array}
\]

\( P(X_1) \)

\[
\begin{array}{ccc}
1/6 & 1/6 & 1/2 \\
0 & 1/6 & 0 \\
0 & 0 & 0 \\
\end{array}
\]

\( P(X' | X = <1,2>) \)
Video of Demo Ghostbusters – Circular Dynamics -- HMM
HMMs have two important independence properties:
- Markov hidden process: future depends on past via the present
- Current observation independent of all else given current state

Does this mean that evidence variables are guaranteed to be independent?
HMMs have two important independence properties:

- Markov hidden process: future depends on past via the present
- Current observation independent of all else given current state

Does this mean that evidence variables are guaranteed to be independent?

- No, they are correlated by the hidden state
Real HMM Examples

- **Speech recognition HMMs:**
  - Observations are acoustic signals (continuous valued)
  - States are specific positions in specific words (so, tens of thousands)

- **Machine translation HMMs:**
  - Observations are words (tens of thousands)
  - States are translation options

- **Robot tracking:**
  - Observations are range readings (continuous)
  - States are positions on a map (continuous)
Today’s Topics

- Quick probability recap
- **Markov Chains** & their **Stationary Distributions**
- **Hidden Markov Models** (HMMs) formulation
- **Filtering** with HMMs
Filtering, or monitoring, is the task of tracking the distribution 
\[ B_t(X) = P_t(X_t | e_1, ..., e_t) \] (the belief state) over time

We start with \( B_1(X) \) in an initial setting, usually uniform

As time passes, or we get observations, we update \( B(X) \)

The Kalman filter was invented in the 60’s and first implemented as a method of trajectory estimation for the Apollo program
Example: Robot Localization

Example from Michael Pfeiffer

\[ t=0 \]

Sensor model: can read in which directions there is a wall, never more than 1 mistake

Motion model: may not execute action with small prob.
Example: Robot Localization

Lighter grey: was possible to get the reading, but less likely b/c required 1 mistake
Example: Robot Localization

Prob

0

1

t=2
Example: Robot Localization

\[ t=3 \]
Example: Robot Localization

$\text{t=4}$
Example: Robot Localization
Inference: Find State Given Evidence

- We are given evidence at each time and want to know

\[ B_t(X) = P(X_t|e_{1:t}) \]

- Idea: start with \( P(X_1) \) and derive \( B_t \) in terms of \( B_{t-1} \)
  - equivalently, derive \( B_{t+1} \) in terms of \( B_t \)
Inference: Base Cases

\[ P(X_1|e_1) \]

\[ P(X_2) \]
Inference: Base Cases

\[ P(X_2) \]

\[ P(x_2) = \sum_{x_1} P(x_1, x_2) \]

\[ = \sum_{x_1} P(x_1)P(x_2|x_1) \]
Passage of Time

- Assume we have current belief $P(X \mid \text{evidence to date})$

$$B(X_t) = P(X_t | e_{1:t})$$

- Then, after one time step passes:

$$P(X_{t+1} | e_{1:t}) = \sum_{x_t} P(X_{t+1}, x_t | e_{1:t})$$

$$= \sum_{x_t} P(X_{t+1} | x_t, e_{1:t}) P(x_t | e_{1:t})$$

$$= \sum_{x_t} P(X_{t+1} | x_t) P(x_t | e_{1:t})$$

- Basic idea: beliefs get “pushed” through the transitions
  - With the “B” notation, we have to be careful about what time step $t$ the belief is about, and what evidence it includes

- Or compactly:

$$B'(X_{t+1}) = \sum_{x_t} P(X' | x_t) B(x_t)$$
As time passes, uncertainty “accumulates”

(Transition model: ghosts usually go clockwise)
Inference: Base Cases

\[ P(X_1|e_1) \]

\[ P(x_1|e_1) = \frac{P(x_1, e_1)}{P(e_1)} \]
\[ \propto_{X_1} P(x_1, e_1) \]
\[ = P(x_1)P(e_1|x_1) \]

Also can write as:

\[ P(x_1|e_1) = \frac{P(x_1)P(e_1|x_1)}{\sum_{x'} P(x')P(e_1|x')} \]
Assume we have current belief $P(X \mid \text{previous evidence})$:

$$B'(X_{t+1}) = P(X_{t+1} \mid e_{1:t})$$

Then, after evidence comes in:

$$P(X_{t+1} \mid e_{1:t+1}) = \frac{P(X_{t+1}, e_{t+1} \mid e_{1:t})}{P(e_{t+1} \mid e_{1:t})}$$

$$\propto_{X_{t+1}} P(X_{t+1}, e_{t+1} \mid e_{1:t})$$

$$= P(e_{t+1} \mid e_{1:t}, X_{t+1}) P(X_{t+1} \mid e_{1:t})$$

$$= P(e_{t+1} \mid X_{t+1}) P(X_{t+1} \mid e_{1:t})$$

Or, compactly:

$$B(X_{t+1}) \propto_{X_{t+1}} P(e_{t+1} \mid X_{t+1}) B'(X_{t+1})$$

Basic idea: beliefs “reweighted” by likelihood of evidence

Unlike passage of time, we have to renormalize.
Example: Observation

- As we get observations, beliefs get reweighted, uncertainty “decreases”

\[ B(X) \propto P(e|X)B'(X) \]
Two Steps: Passage of Time + Observation

\[ B(X_t) = P(X_t|e_{1:t}) \]
\[ B'(X_{t+1}) = \sum_{x_t} P(X'|x_t)B(x_t) \]
\[ B(X_{t+1}) \propto_{X_{t+1}} P(e_{t+1}|X_{t+1})B'(X_{t+1}) \]
Pacman – Sonar

[Demo: Pacman – Sonar – No Beliefs(L14D1)]
Video of Demo Pacman – Sonar (with beliefs)
Next Time: More Filtering!