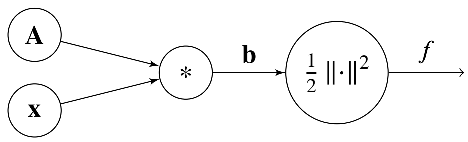
# CS 188 Fall 2024 — Introduction to Artificial Intelligence — Exam Prep 11 Solutions

## Q1. Backpropagation

In this question we will perform the backward pass algorithm on the formula

$$f = \frac{1}{2}\|\mathbf{A}\mathbf{x}\|^2$$

Here, $\mathbf{A} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\mathbf{b} = \mathbf{A}\mathbf{x} = \begin{bmatrix} A_{11}x_1 + A_{12}x_2 \\ A_{21}x_1 + A_{22}x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$, and $f = \frac{1}{2}\|\mathbf{b}\|^2 = \frac{1}{2}\left(b_1^2 + b_2^2\right)$ is a scalar.




**(a)** Calculate the following partial derivatives of $f$.

**(i)** Find $\frac{\partial f}{\partial \mathbf{b}} = \begin{bmatrix} \frac{\partial f}{\partial b_1} \\ \frac{\partial f}{\partial b_2} \end{bmatrix}$.

○ $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ ● $\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ ○ $\begin{bmatrix} b_2 \\ b_1 \end{bmatrix}$ ○ $\begin{bmatrix} f(b_1) \\ f(b_2) \end{bmatrix}$ ○ $\begin{bmatrix} A_{11} \\ A_{22} \end{bmatrix}$ ○ $\begin{bmatrix} b_1 + b_2 \\ b_1 - b_2 \end{bmatrix}$

**(b)** Calculate the following partial derivatives of $b_1$.

**(i)** $\left(\frac{\partial b_1}{\partial A_{11}}, \frac{\partial b_1}{\partial A_{12}}\right)$

○ $(A_{11}, A_{12})$ ○ $(0, 0)$ ○ $(x_2, x_1)$ ○ $(A_{11}x_1, A_{12}x_2)$ ● $(x_1, x_2)$

**(ii)** $\left(\frac{\partial b_1}{\partial A_{21}}, \frac{\partial b_1}{\partial A_{22}}\right)$

○ $(A_{21}, A_{22})$ ○ $(x_1, x_2)$ ○ $(1, 1)$ ● $(0, 0)$ ○ $(A_{21}x_1, A_{22}x_2)$

**(iii)** $\left(\frac{\partial b_1}{\partial x_1}, \frac{\partial b_1}{\partial x_2}\right)$

● $(A_{11}, A_{12})$ ○ $(A_{21}, A_{22})$ ○ $(0, 0)$ ○ $(b_1, b_2)$ ○ $(A_{21}x_1, A_{22}x_2)$

**(c)** Calculate the following partial derivatives of $f$.

**(i)** $\left(\frac{\partial f}{\partial A_{11}}, \frac{\partial f}{\partial A_{12}}\right)$

○ $(A_{11}, A_{12})$ ○ $(A_{11}b_1, A_{12}b_2)$ ○ $(A_{11}x_1, A_{12}x_2)$
● $(x_1b_1, x_2b_1)$ ○ $(x_1b_2, x_2b_2)$ ○ $(x_1b_1, x_2b_2)$

**(ii)** $\left(\frac{\partial f}{\partial A_{21}}, \frac{\partial f}{\partial A_{22}}\right)$

○ $(A_{21}, A_{22})$ ○ $(A_{21}b_1, A_{22}b_2)$ ○ $(A_{21}x_1, A_{22}x_2)$
○ $(x_1b_1, x_2b_1)$ ● $(x_1b_2, x_2b_2)$ ○ $(x_1b_1, x_2b_2)$

**(iii)** $\left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}\right)$

○ $(A_{11}b_1 + A_{12}b_2, A_{21}b_1 + A_{22}b_2)$ ● $(A_{11}b_1 + A_{21}b_2, A_{12}b_1 + A_{22}b_2)$
○ $(A_{11}b_1 + A_{12}b_1, A_{21}b_2 + A_{22}b_2)$ ○ $(A_{11}b_1 + A_{21}b_1, A_{12}b_2 + A_{22}b_2)$

**(d)** Now we consider the general case where $\mathbf{A}$ is an $n \times d$ matrix, and $\mathbf{x}$ is a $d \times 1$ vector. As before, $f = \frac{1}{2} \|\mathbf{A}\mathbf{x}\|^2$.

**(i)** Find $\frac{\partial f}{\partial \mathbf{A}}$ in terms of $\mathbf{A}$ and $\mathbf{x}$ only.

○ $\mathbf{x}^\top \mathbf{A}^\top \mathbf{A}\mathbf{x}$ ● $\mathbf{A}\mathbf{x}\mathbf{x}^\top$ ○ $\mathbf{A}\left(\mathbf{A}^\top \mathbf{A}\right)^{-1}$ ○ $\mathbf{A}\mathbf{A}^\top \mathbf{A}\mathbf{x}$ ○ $\mathbf{A}$

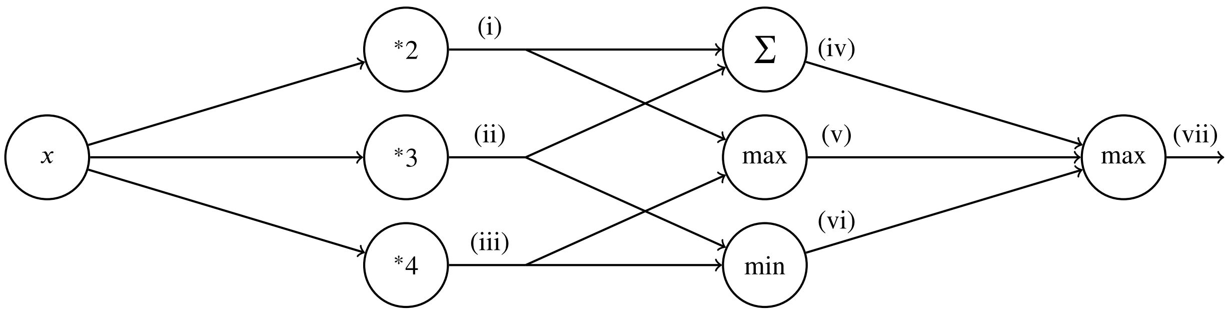
**(ii)** Find $\frac{\partial f}{\partial \mathbf{x}}$ in terms of $\mathbf{A}$ and $\mathbf{x}$ only.

○ $\mathbf{x}$ ○ $\left(\mathbf{A}^\top \mathbf{A}\right)^{-1} \mathbf{x}$ ○ $\mathbf{x}\mathbf{x}^\top \mathbf{x}$ ○ $\mathbf{x}^\top \mathbf{A}^\top \mathbf{A}\mathbf{x}$ ● $\mathbf{A}^\top \mathbf{A}\mathbf{x}$

# Q2. Deep Learning

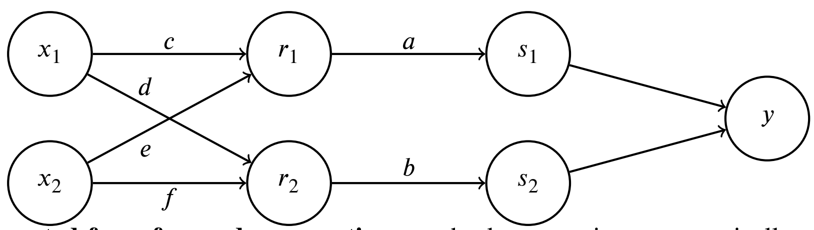
**(a)** Perform forward propagation on the neural network below for $x = 1$ by filling in the values in the table. Note that (i), ..., (vii) are outputs after performing the appropriate operation as indicated in the node.

| (i) | (ii) | (iii) | (iv) | (v) | (vi) | (vii) |
|---|---|---|---|---|---|---|
| 2 | 3 | 4 | 5 | 4 | 3 | 5 |




**(b)** **[Optional]** Below is a neural network with weights $a, b, c, d, e, f$. The inputs are $x_1$ and $x_2$.
The first hidden layer computes $r_1 = \max(c \cdot x_1 + e \cdot x_2, 0)$ and $r_2 = \max(d \cdot x_1 + f \cdot x_2, 0)$.
The second hidden layer computes $s_1 = \frac{1}{1+\exp(-a \cdot r_1)}$ and $s_2 = \frac{1}{1+\exp(-b \cdot r_2)}$.
The output layer computes $y = s_1 + s_2$. Note that the weights $a, b, c, d, e, f$ are indicated along the edges of the neural network here.

Suppose the network has inputs $x_1 = 1, x_2 = -1$.
The weight values are $a = 1, b = 1, c = 4, d = 1, e = 2, f = 2$.
Forward propagation then computes $r_1 = 2, r_2 = 0, s_1 = 0.9, s_2 = 0.5, y = 1.4$. Note: some values are rounded.



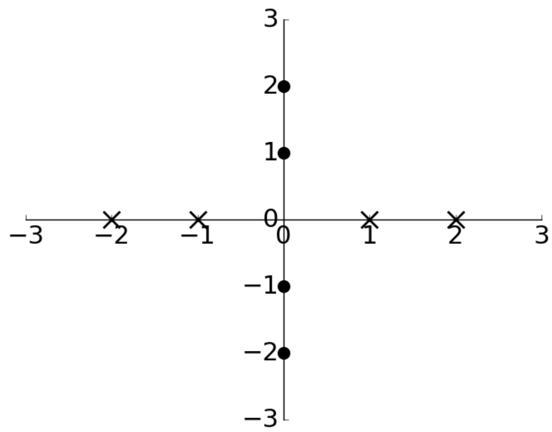

**Using the values computed from forward propagation**, use backpropagation to numerically calculate the following partial derivatives. Write your answers as a single number (not an expression). You do not need a calculator. Use scratch paper if needed.
*Hint:* For $g(z) = \frac{1}{1+\exp(-z)}$, the derivative is $\frac{\partial g}{\partial z} = g(z)(1 - g(z))$.

| $\frac{\partial y}{\partial a}$ | $\frac{\partial y}{\partial b}$ | $\frac{\partial y}{\partial c}$ | $\frac{\partial y}{\partial d}$ | $\frac{\partial y}{\partial e}$ | $\frac{\partial y}{\partial f}$ |
|---|---|---|---|---|---|
| 0.18 | 0 | 0.09 | 0 | −0.09 | 0 |

$$
\begin{aligned}
\frac{\partial y}{\partial a} &= \frac{\partial y}{\partial s_1}\frac{\partial s_1}{\partial a} \\
&= 1 \cdot \frac{\partial g(a \cdot r_1)}{\partial a} \\
&= r_1 \cdot g(a \cdot r_1)(1 - g(a \cdot r_1)) \\
&= r_1 \cdot s_1(1 - s_1) \\
&= 2 \cdot 0.9 \cdot (1 - 0.9) \\
&= 0.18 \\
\frac{\partial y}{\partial b} &= \frac{\partial y}{\partial s_2}\frac{\partial s_2}{\partial b} \\
&= 1 \cdot \frac{\partial g(b \cdot r_2)}{\partial b} \\
&= r_2 \cdot g(b \cdot r_2)(1 - g(b \cdot r_2)) \\
&= r_2 \cdot s_2(1 - s_2) \\
&= 0 \cdot 0.5(1 - 0.5) \\
&= 0 \\
\frac{\partial y}{\partial c} &= \frac{\partial y}{\partial s_1}\frac{\partial s_1}{\partial r_1}\frac{\partial r_1}{\partial c} \\
&= 1 \cdot [a \cdot g(a \cdot r_1)(1 - g(a \cdot r_1))] \cdot x_1 \\
&= [a \cdot s_1(1 - s_1)] \cdot x_1 \\
&= [1 \cdot 0.9(1 - 0.9)] \cdot 1 \\
&= 0.09 \\
\frac{\partial y}{\partial d} &= \frac{\partial y}{\partial s_2}\frac{\partial s_2}{\partial r_2}\frac{\partial r_2}{\partial d} \\
&= \frac{\partial y}{\partial s_2}\frac{\partial s_2}{\partial r_2} \cdot 0 \\
&= 0 \\
\frac{\partial y}{\partial e} &= \frac{\partial y}{\partial s_1}\frac{\partial s_1}{\partial r_1}\frac{\partial r_1}{\partial e} \\
&= 1 \cdot [a \cdot g(a \cdot r_1)(1 - g(a \cdot r_1))] \cdot x_2 \\
&= [a \cdot s_1(1 - s_1)] \cdot x_2 \\
&= [1 \cdot 0.9(1 - 0.9)] \cdot -1 \\
&= -0.09 \\
\frac{\partial y}{\partial f} &= \frac{\partial y}{\partial s_2}\frac{\partial s_2}{\partial r_2}\frac{\partial r_2}{\partial f} \\
&= \frac{\partial y}{\partial s_2}\frac{\partial s_2}{\partial r_2} \cdot 0 \\
&= 0
\end{aligned}
$$

**(c)** Below are two plots with horizontal axis $x_1$ and vertical axis $x_2$ containing data labelled $\times$ and $\bullet$. For each plot, we wish to find a function $f(x_1, x_2)$ such that $f(x_1, x_2) \geq 0$ for all data labelled $\times$ and $f(x_1, x_2) < 0$ for all data labelled $\bullet$.

Below each plot is the function $f(x_1, x_2)$ for that specific plot. Complete the expressions such that all the data is labelled correctly. If not possible, mark "No valid combination".



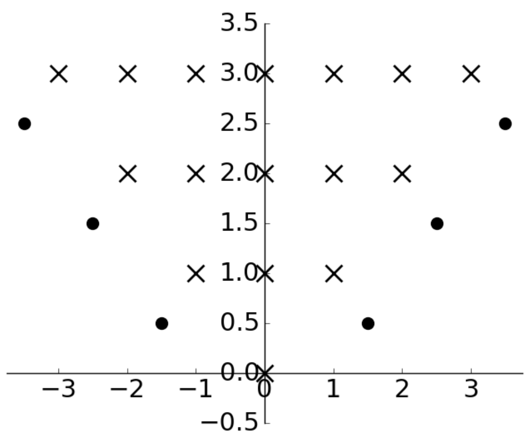
$f(x_1, x_2) = \max(\underline{\textbf{(i)}} + \underline{\textbf{(ii)}}, \underline{\textbf{(iii)}} + \underline{\textbf{(iv)}}) + \underline{\textbf{(v)}}$

| | | | |
|---|---|---|---|
| **(i)** | ● $x_1$ | ○ $-x_1$ | ○ 0 |
| **(ii)** | ○ $x_2$ | ○ $-x_2$ | ● 0 |
| **(iii)** | ○ $x_1$ | ● $-x_1$ | ○ 0 |
| **(iv)** | ○ $x_2$ | ○ $-x_2$ | ● 0 |
| **(v)** | ○ 1 | ● $-1$ | ○ 0 |

○ No valid combination

There are two possible solutions:

$$f(x_1, x_2) = \max(x_1, -x_1) - 1$$
$$f(x_1, x_2) = \max(-x_1, x_1) - 1$$



$f(x_1, x_2) = \underline{\textbf{(vi)}} - \max(\underline{\textbf{(vii)}} + \underline{\textbf{(viii)}}, \underline{\textbf{(ix)}} + \underline{\textbf{(x)}})$

| | | | |
|---|---|---|---|
| **(vi)** | ● $x_2$ | ○ $-x_2$ | ○ 0 |
| **(vii)** | ● $x_1$ | ○ $-x_1$ | ○ 0 |
| **(viii)** | ○ $x_2$ | ○ $-x_2$ | ● 0 |
| **(ix)** | ○ $x_1$ | ● $-x_1$ | ○ 0 |
| **(x)** | ○ $x_2$ | ○ $-x_2$ | ● 0 |

○ No valid combination

There are four possible solutions:

$$f(x_1, x_2) = x_2 - \max(x_1, -x_1)$$
$$f(x_1, x_2) = x_2 - \max(-x_1, x_1)$$
$$f(x_1, x_2) = -\max(x_1 - x_2, -x_1 - x_2)$$
$$f(x_2, x_2) = -\max(-x_1 - x_2, x_1 - x_2)$$