Announcements

- Guest Lectures Announced!
  - **Tuesday, Nov 19:** Catherine Olsson (Anthropic) on LLM development & interpretability
  - **Thursday, Dec 3:** Miles Brundage (formerly OpenAI) on AI policy and social impacts

CS 188: Artificial Intelligence

Hidden Markov Models

University of California, Berkeley

Reasoning over Time or Space

- Often, we want to reason about a sequence of observations
  - Speech recognition
  - Robot localization
  - User attention
  - Medical monitoring
  - Language understanding
- Need to introduce time (or space) into our models and update beliefs based on:
  - Getting more evidence (we did this with BNs)
  - World changing over time/space (new this week)

Motivating Example: Pacman Sonar

Today's Topics

- Quick probability recap
- **Markov Chains** & their **Stationary Distributions**
  - How beliefs about state change with passage of time
- **Hidden Markov Models** (HMMs) formulation
  - How beliefs change with passage of time and evidence
- **Filtering** with HMMs
  - How to infer beliefs from evidence

Probability Recap

- Conditional probability $$P(x|y) = \frac{P(x,y)}{P(y)}$$
- Marginal probability $$P(x) = \sum_y P(x,y)$$
- Product rule $$P(x,y) = P(x|y)P(y)$$
- Chain rule $$\begin{aligned} P(X_1, X_2, \ldots X_n) &= P(X_1)P(X_2|X_1)P(X_3|X_1,X_2)\ldots \\ &= \prod_{i=1}^{n} P(X_i|X_1,\ldots,X_{i-1}) \end{aligned}$$

Probability Recap

- X, Y independent if and only if: $\forall x, y : P(x, y) = P(x)P(y)$

- X and Y are conditionally independent given Z if and only if: $X \perp\!\!\!\perp Y | Z$

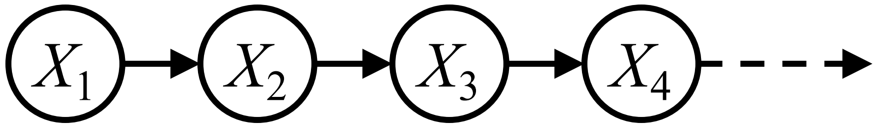
$$\forall x, y, z : P(x, y|z) = P(x|z)P(y|z)$$

- Proportionality: $P(X) \propto f(X)$ or $P(X) \propto_X f(X)$ means $P(X) = kf(X)$ (for some constant *k* that doesn't depend on *X*). Equivalent to: $P(X) = \frac{f(X)}{\sum_x f(x)}$
  - Example:

X	$\propto f(X)$	$P(X)$
x_1	0.4	0.4 / (0.4 + 0.2)
x_2	0.2	0.2 / (0.4 + 0.2)

Markov Models

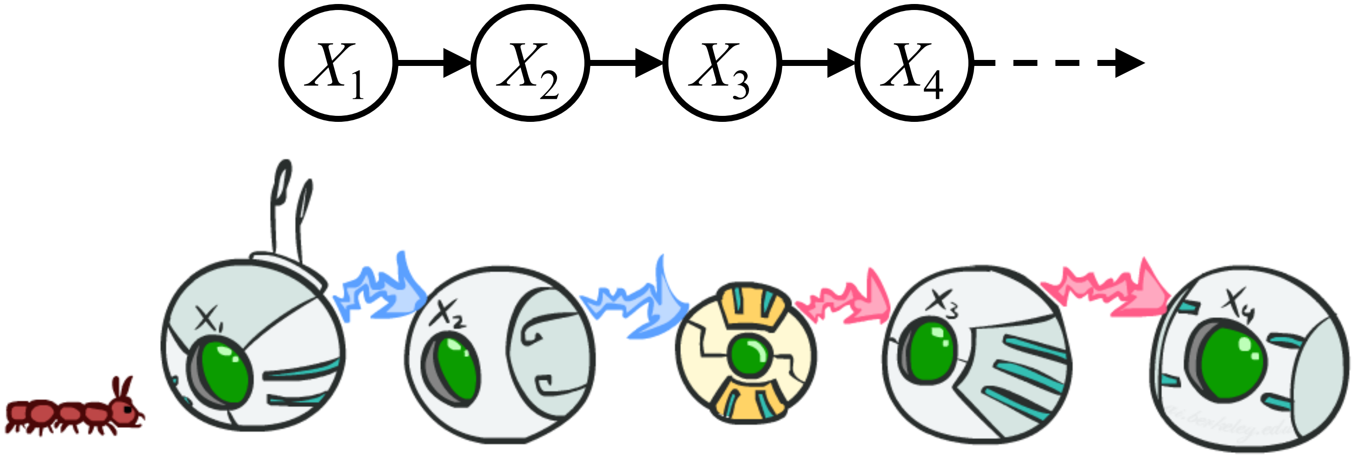
- Value of X at a given time is called the state



$$P(X_1) \qquad P(X_t|X_{t-1})$$

- Parameters: called transition probabilities or dynamics, specify how the state evolves over time (also, initial state probabilities)
- Stationarity assumption: transition probabilities the same at all times
- Same as MDP transition model, but no choice of action
- A "growable” BN (can always use BN methods if we truncate to fixed length)

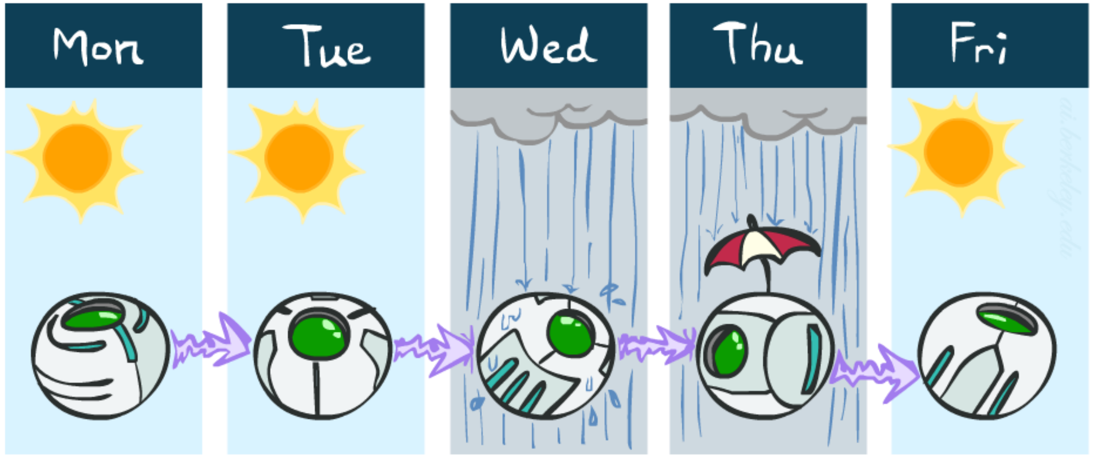
Conditional Independence



- Basic conditional independence:
  - Past and future independent given the present
  - Each time step only depends on the previous
  - This is called the (first order) Markov property

Example Markov Chain: Weather

- States: X = {rain, sun}

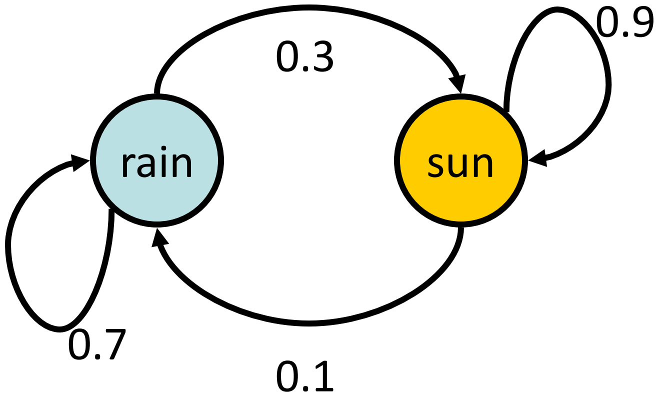


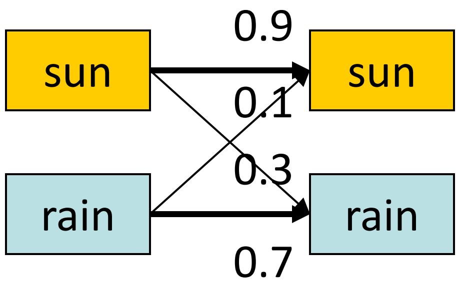
- Initial distribution: 1.0 sun

- CPT $P(X_t \mid X_{t-1})$:

X_{t-1}	X_t	$P(X_t \mid X_{t-1})$
sun	sun	0.9
sun	rain	0.1
rain	sun	0.3
rain	rain	0.7

Two new ways of representing the same CPT





Example Markov Chain: Weather

- Initial distribution: 1.0 sun
  - We know: $P(X_1)$ $P(X_t|X_{t-1})$

X_{t-1}	X_t	$P(X_t \mid X_{t-1})$
sun	sun	0.9
sun	rain	0.1
rain	sun	0.3
rain	rain	0.7

- What is the probability distribution after one step?

$$P(X_2 = sun) = \sum_{x_1} P(x_1, X_2 = sun) = \sum_{x_1} P(X_2 = sun|x_1)P(x_1)$$

$$= \quad P(X_2 = \text{sun}|X_1 = \text{sun})P(X_1 = \text{sun}) + P(X_2 = \text{sun}|X_1 = \text{rain})P(X_1 = \text{rain})$$

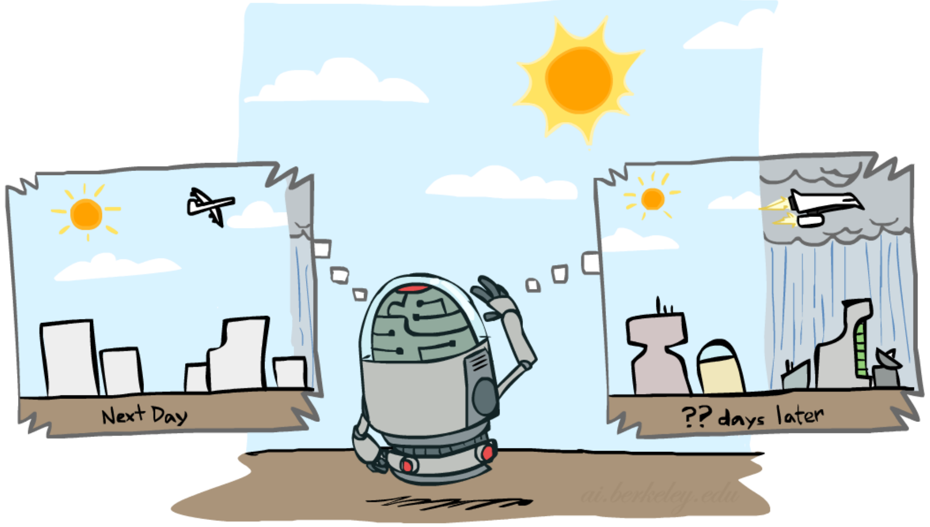
$$0.9 \cdot 1.0 + 0.3 \cdot 0.0 = 0.9$$

Mini-Forward Algorithm

- Question: What's P(X) on some day t?

$$X_1 \rightarrow X_2 \rightarrow X_3 \rightarrow X_4 \dashrightarrow \qquad X_t \ ?$$

- We know $P(X_1)$ and $P(X_t|X_{t-1})$



$$P(X_1) = \text{known}$$

$$P(x_t) = \sum_{x_{t-1}} P(x_{t-1}, x_t)$$

$$= \sum_{x_{t-1}} P(x_t \mid x_{t-1}) P(x_{t-1})$$

Forward simulation

Example Run of Mini-Forward Algorithm

- From initial observation of sun

$$\underset{P(X_1)}{\left\langle \begin{matrix} 1.0 \\ 0.0 \end{matrix} \right\rangle} \quad \underset{P(X_2)}{\left\langle \begin{matrix} 0.9 \\ 0.1 \end{matrix} \right\rangle} \quad \underset{P(X_3)}{\left\langle \begin{matrix} 0.84 \\ 0.16 \end{matrix} \right\rangle} \quad \underset{P(X_4)}{\left\langle \begin{matrix} 0.804 \\ 0.196 \end{matrix} \right\rangle} \Rightarrow \underset{P(X_\infty)}{\left\langle \begin{matrix} 0.75 \\ 0.25 \end{matrix} \right\rangle}$$

- From initial observation of rain

$$\underset{P(X_1)}{\left\langle \begin{matrix} 0.0 \\ 1.0 \end{matrix} \right\rangle} \quad \underset{P(X_2)}{\left\langle \begin{matrix} 0.3 \\ 0.7 \end{matrix} \right\rangle} \quad \underset{P(X_3)}{\left\langle \begin{matrix} 0.48 \\ 0.52 \end{matrix} \right\rangle} \quad \underset{P(X_4)}{\left\langle \begin{matrix} 0.588 \\ 0.412 \end{matrix} \right\rangle} \Rightarrow \underset{P(X_\infty)}{\left\langle \begin{matrix} 0.75 \\ 0.25 \end{matrix} \right\rangle}$$

- From yet another initial distribution $P(X_1)$:

$$\underset{P(X_1)}{\left\langle \begin{matrix} p \\ 1-p \end{matrix} \right\rangle} \quad \ldots \Rightarrow \underset{P(X_\infty)}{\left\langle \begin{matrix} 0.75 \\ 0.25 \end{matrix} \right\rangle}$$

[Demo: L13D1,2,3]

Video of Demo Ghostbusters Basic Dynamics

Video of Demo Ghostbusters Circular Dynamics

Video of Demo Ghostbusters Whirlpool Dynamics

Stationary Distributions

- For most chains:
  - Influence of the initial distribution gets less and less over time.
  - The distribution we end up in is independent of the initial distribution
- Stationary distribution:
  - The distribution we end up with is called the stationary distribution P_∞ of the chain
  - It satisfies

$$P_\infty(X) = P_{\infty+1}(X) = \sum_x P(X|x)P_\infty(x)$$

Example: Stationary Distributions

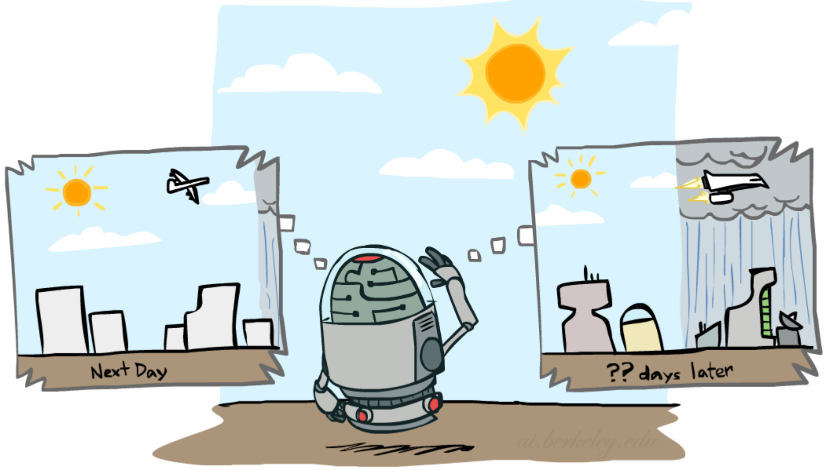
- Question: What's P(X) at time t = infinity?

$$X_1 \rightarrow X_2 \rightarrow X_3 \rightarrow X_4 \dashrightarrow \quad X_\infty \ ?$$

$$P_\infty(X) = P_{\infty+1}(X) = \sum_x P(X|x)P_\infty(x)$$

$$P_\infty(sun) = P(sun|sun)P_\infty(sun) + P(sun|rain)P_\infty(rain)$$
$$P_\infty(rain) = P(rain|sun)P_\infty(sun) + P(rain|rain)P_\infty(rain)$$

$$P_\infty(sun) = 0.9P_\infty(sun) + 0.3P_\infty(rain)$$
$$P_\infty(rain) = 0.1P_\infty(sun) + 0.7P_\infty(rain)$$



X_{t-1}	X_t	$P(X_t \mid X_{t-1})$
sun	sun	0.9
sun	rain	0.1
rain	sun	0.3
rain	rain	0.7

$$P_\infty(sun) = 3P_\infty(rain)$$

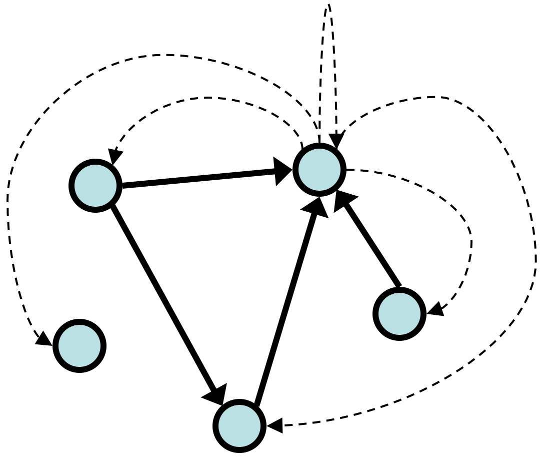
Also: $P_\infty(sun) + P_\infty(rain) = 1$ $\Rightarrow$

$$P_\infty(sun) = 3/4$$
$$P_\infty(rain) = 1/4$$

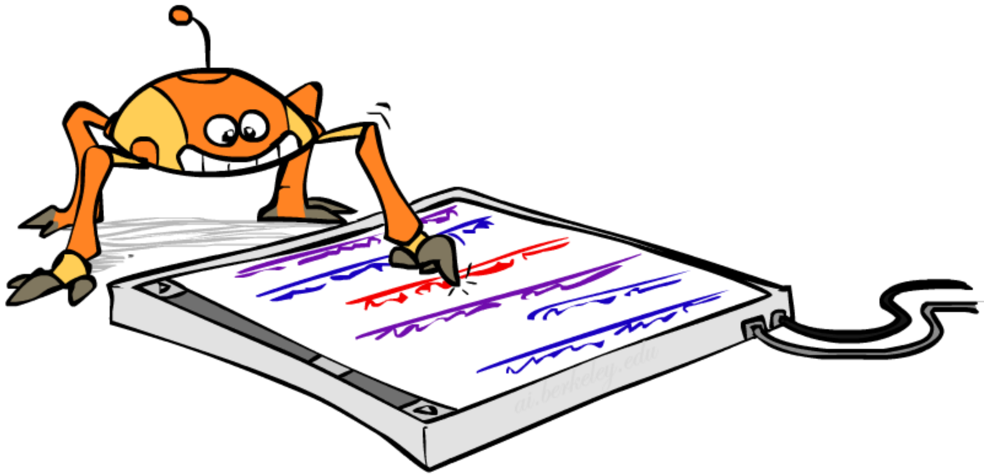
- Alternatively: run simulation for a long (ideally infinite) time

Application of Stationary Distribution: Web Link Analysis

- PageRank over a web graph
  - Each web page is a state
  - Initial distribution: uniform over pages
  - Transitions:
    - With prob. c, uniform jump to a random page (dotted lines, not all shown)
    - With prob. 1-c, follow a random outlink (solid lines)

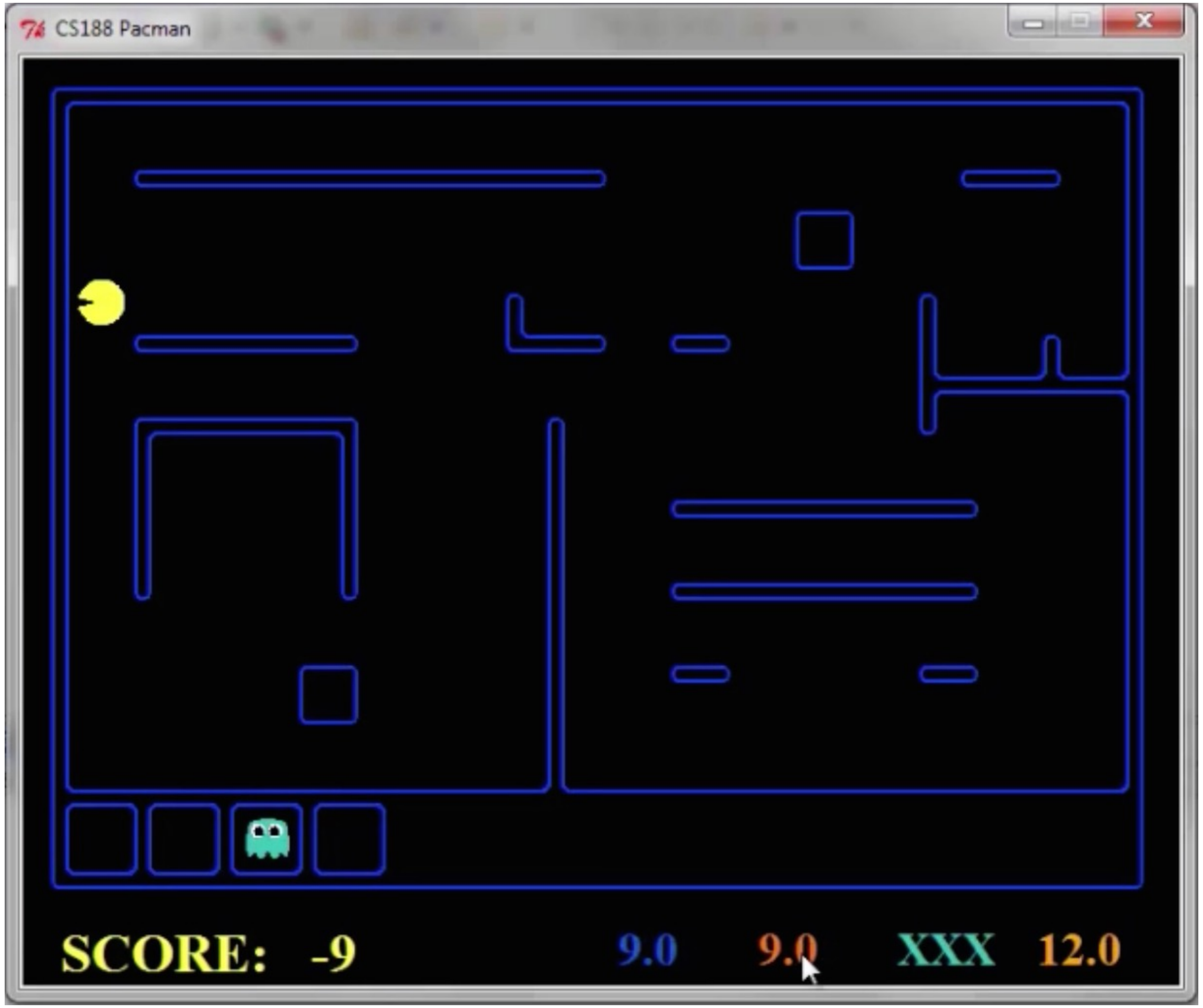


- Stationary distribution
  - Will spend more time on highly reachable pages
  - E.g. many ways to get to the Acrobat Reader download page
  - Somewhat robust to link spam
  - Google 1.0 returned the set of pages containing all your keywords in decreasing rank, now all search engines use link analysis along with many other factors (rank actually getting less important over time)



Hidden Markov Models

Pacman – Sonar



[Demo: Pacman – Sonar – No Beliefs(L14D1)]

Video of Demo Pacman – Sonar (no beliefs)

Video of Demo Pacman – Sonar (with beliefs)

Hidden Markov Models

- Markov chains not so useful for most agents

$$X_1 \rightarrow X_2 \rightarrow X_3 \rightarrow X_4 \dashrightarrow$$

Hidden Markov Models

- Markov chains not so useful for most agents

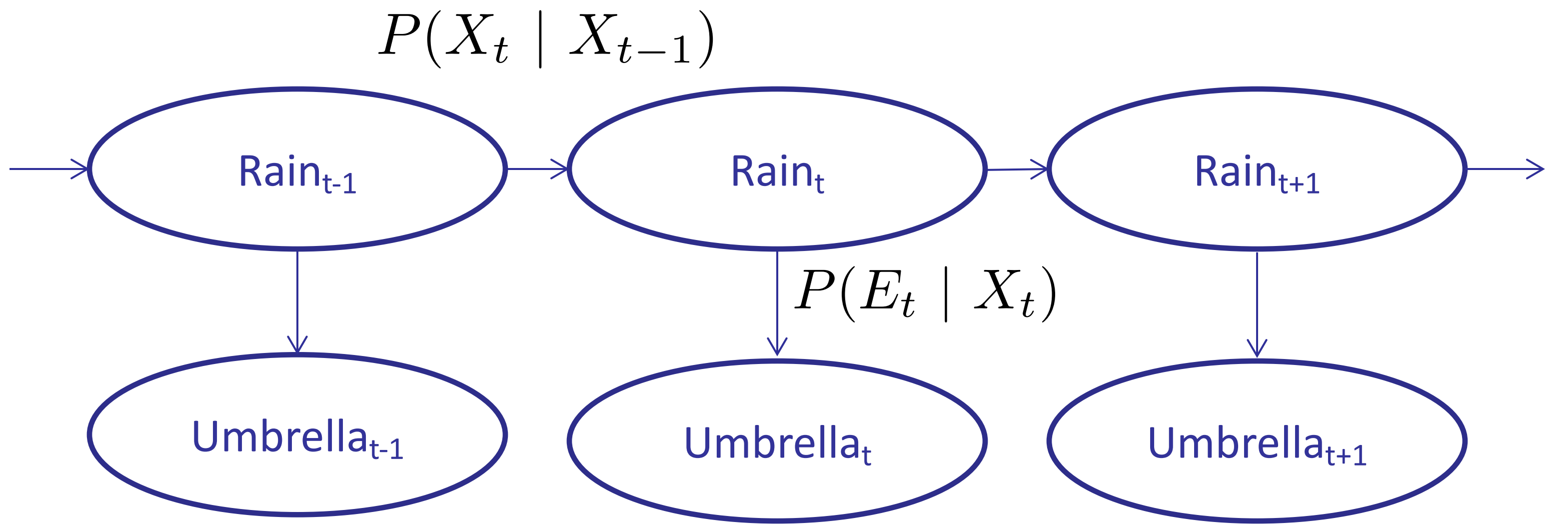
$$X_1 \rightarrow X_2 \rightarrow X_3 \rightarrow X_4 \dashrightarrow$$

  - Need observations to update your beliefs

- Hidden Markov models (HMMs)
  - Underlying Markov chain over states X
  - You observe outputs (effects) at each time step

$$\begin{tikzcd} X_1 \arrow[r] \arrow[d] & X_2 \arrow[r] \arrow[d] & X_3 \arrow[r] \arrow[d] & X_4 \arrow[r, dashed] \arrow[d] & {} \\ E_1 & E_2 & E_3 & E_4 & \end{tikzcd}$$

Example: Weather HMM



- An HMM is defined by:
  - Initial distribution: $P(X_1)$
  - Transitions: $P(X_t \mid X_{t-1})$
  - Emissions: $P(E_t \mid X_t)$

Transitions

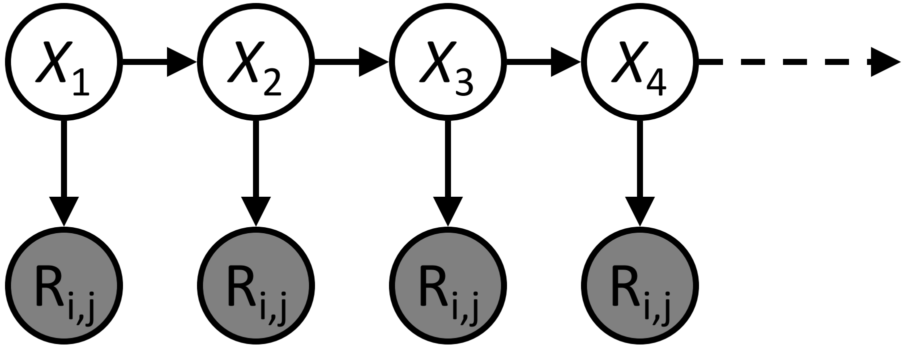
R_{t-1}	R_t	$P(R_t \mid R_{t-1})$
+r	+r	0.7
+r	-r	0.3
-r	+r	0.3
-r	-r	0.7

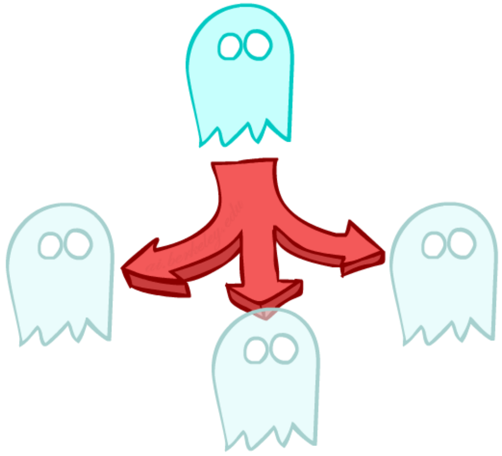
Emissions

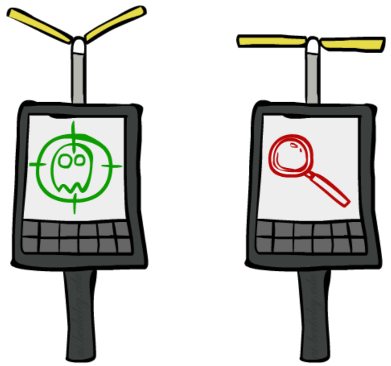
R_t	U_t	$P(U_t \mid R_t)$
+r	+u	0.9
+r	-u	0.1
-r	+u	0.2
-r	-u	0.8

Example: Ghostbusters HMM

- **$P(X_1)$** = uniform
- **$P(X'|X)$** = usually move clockwise, but sometimes move in a random direction or stay in place
- **$P(R_{ij}|X)$** = same sensor model as before: red means close, green means far away.







1/9	1/9	1/9
1/9	1/9	1/9
1/9	1/9	1/9

$P(X_1)$

1/6	1/6	1/2
0	1/6	0
0	0	0

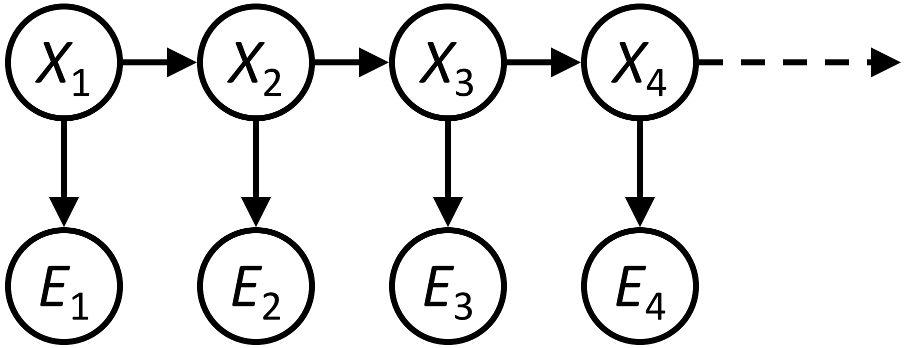
$P(X'|X = \langle 1,2 \rangle)$

[Demo: Ghostbusters – Circular Dynamics – HMM (L14D2)]

Video of Demo Ghostbusters – Circular Dynamics -- HMM

Conditional Independence

- HMMs have two important independence properties:
  - Markov hidden process: future depends on past via the present
  - Current observation independent of all else given current state



- Does this mean that evidence variables are guaranteed to be independent?

Conditional Independence

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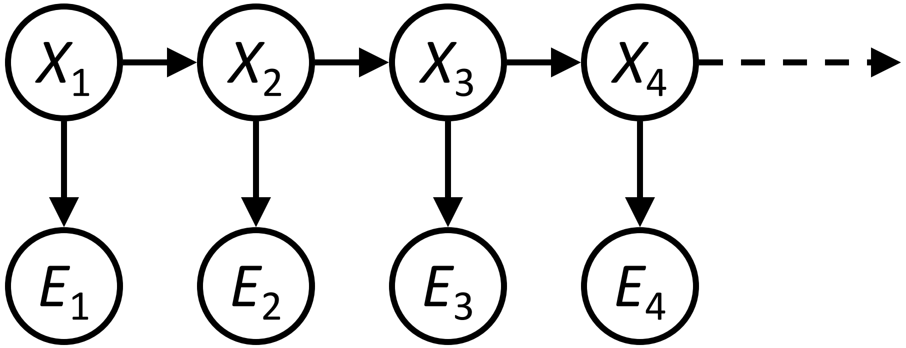
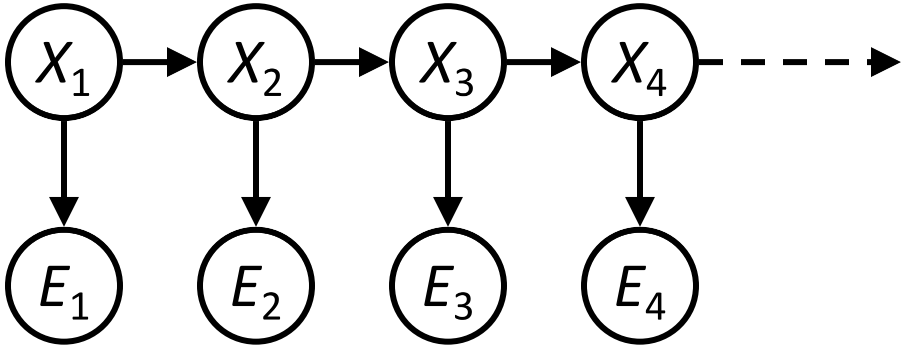


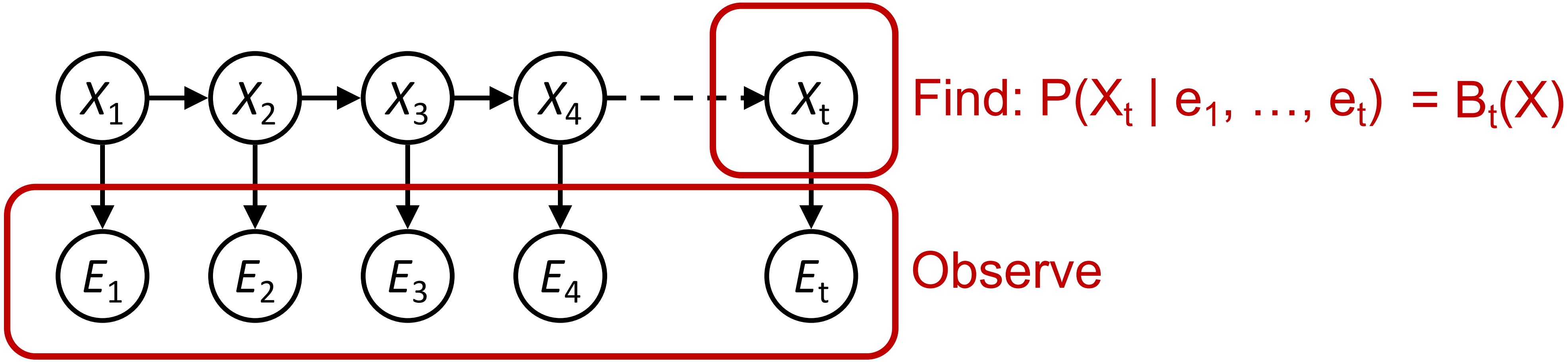
- Does this mean that evidence variables are guaranteed to be independent?
  - No, they are correlated by the hidden state

Real HMM Examples

- Speech recognition HMMs:
  - Observations are acoustic signals (continuous valued)
  - States are specific positions in specific words (so, tens of thousands)
- Machine translation HMMs:
  - Observations are words (tens of thousands)
  - States are translation options
- Robot tracking:
  - Observations are range readings (continuous)
  - States are positions on a map (continuous)



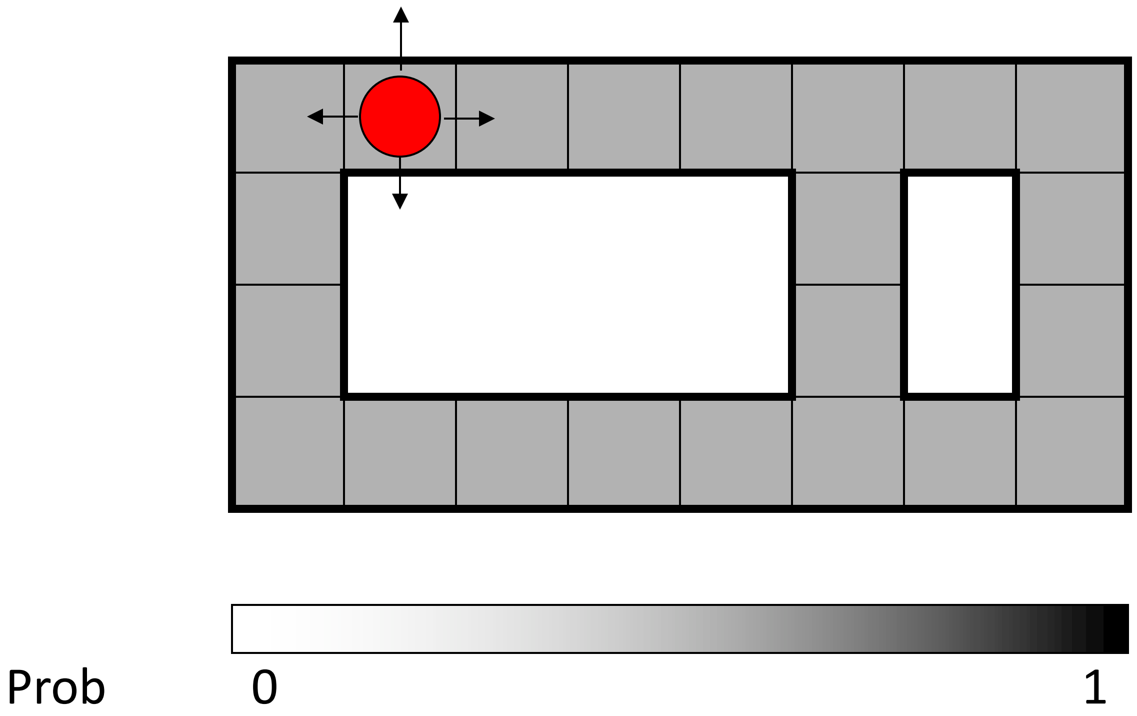
Filtering / Monitoring

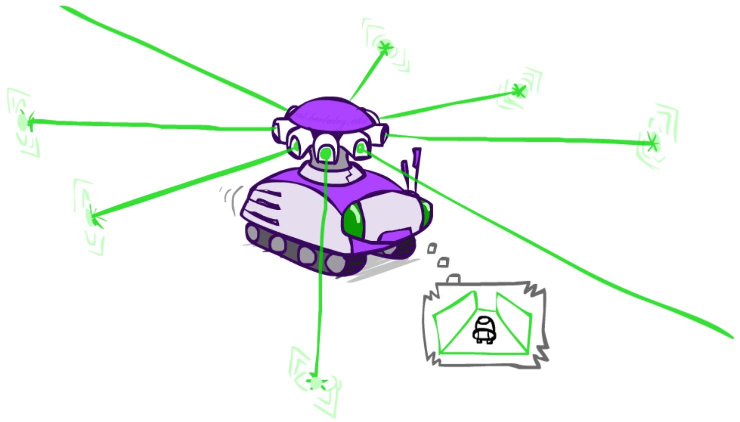


- Filtering, or monitoring, is the task of tracking the distribution $B_t(X) = P_t(X_t \mid e_1, \ldots, e_t)$ (the belief state) over time
- We start with $B_1(X)$ in an initial setting, usually uniform
- As time passes, or we get observations, we update $B(X)$
- The Kalman filter was invented in the 60's and first implemented as a method of trajectory estimation for the Apollo program

Example: Robot Localization

Example from
Michael Pfeiffer



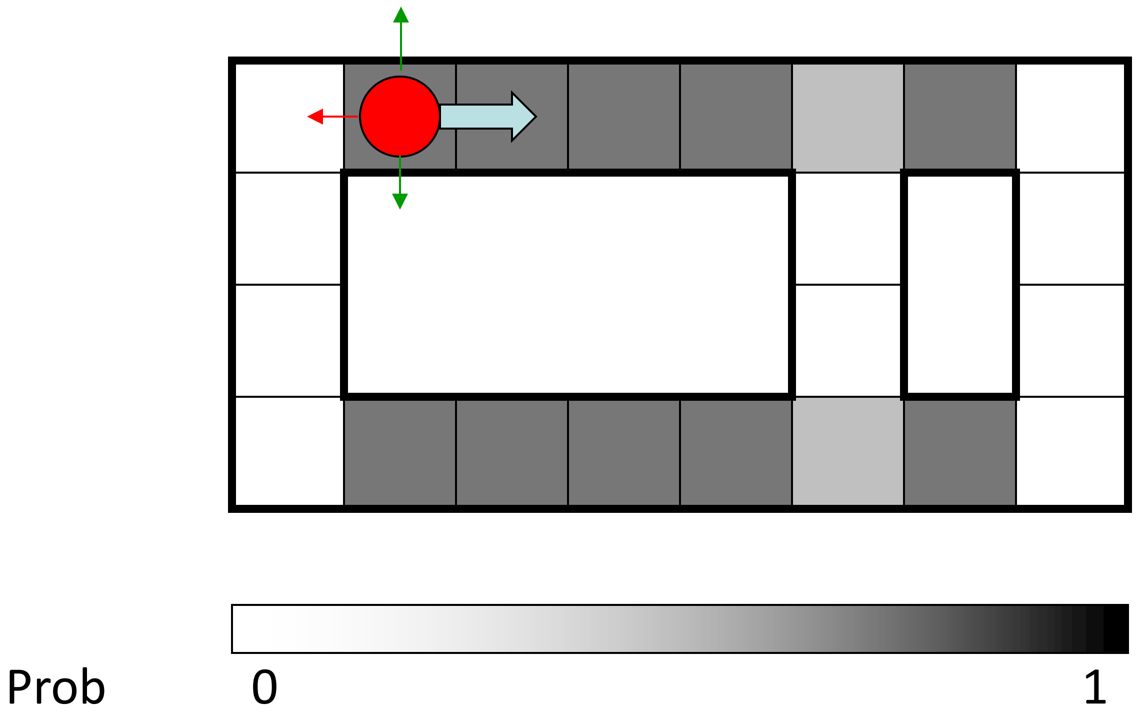


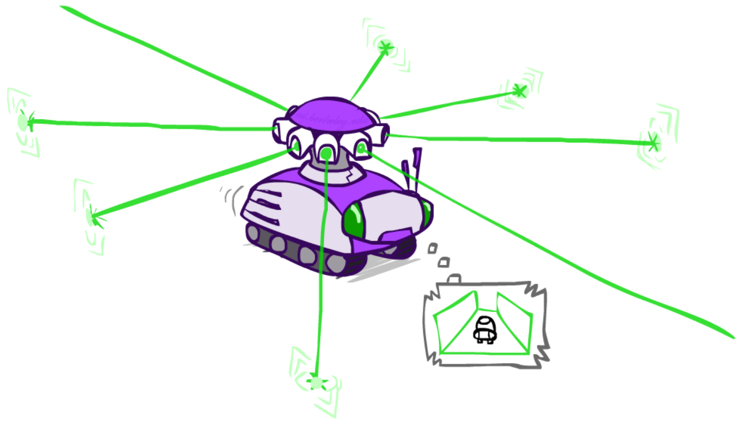
t=0

Sensor model: can read in which directions there is a wall, never more than 1 mistake

Motion model: may not execute action with small prob.

Example: Robot Localization

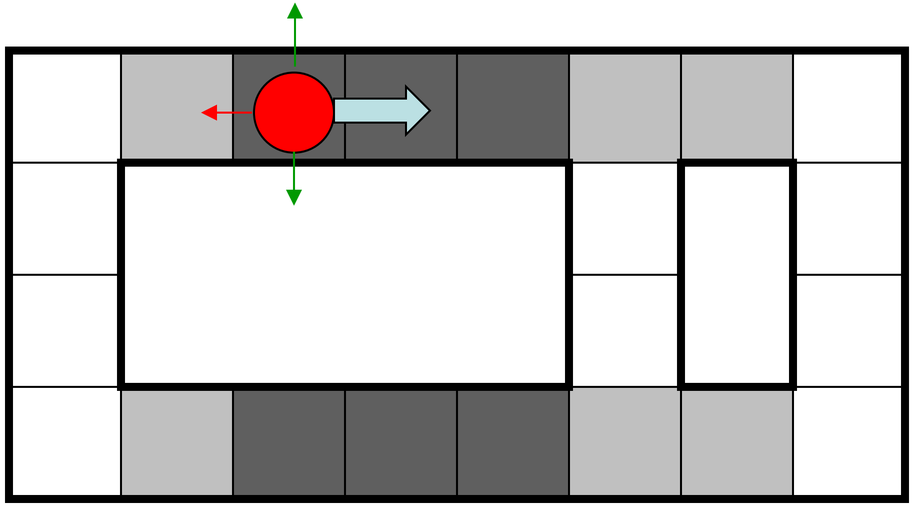


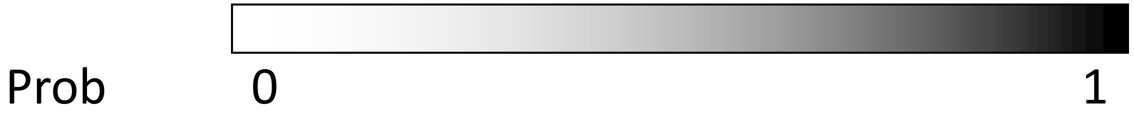


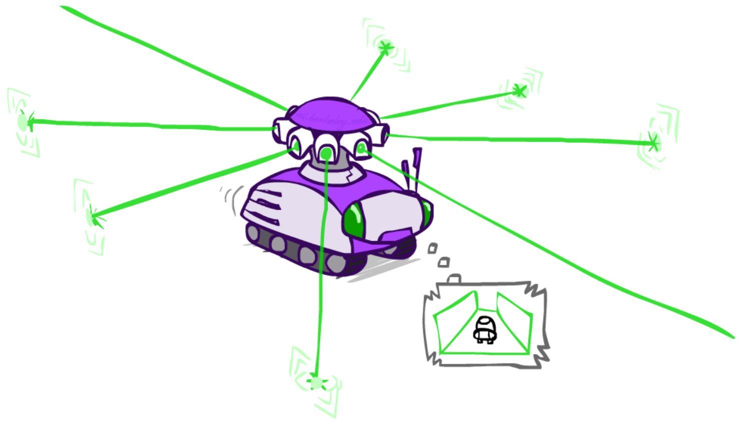
t=1

Lighter grey: was possible to get the reading, but less likely b/c required 1 mistake

Example: Robot Localization

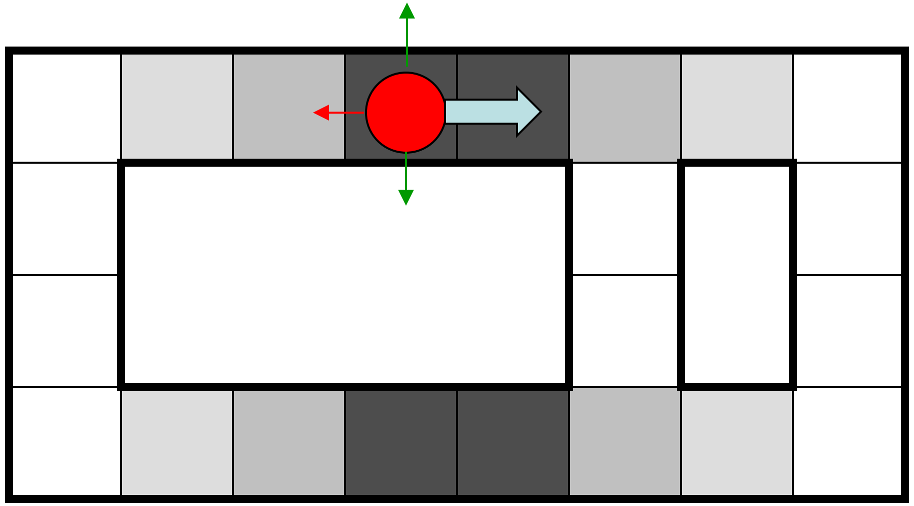


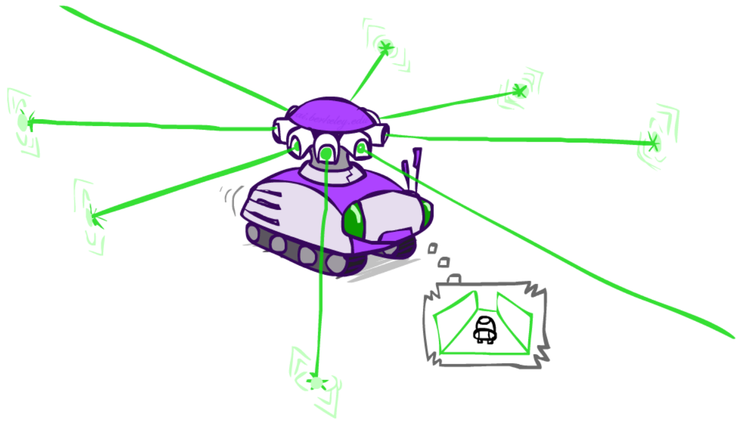




t=2

Example: Robot Localization

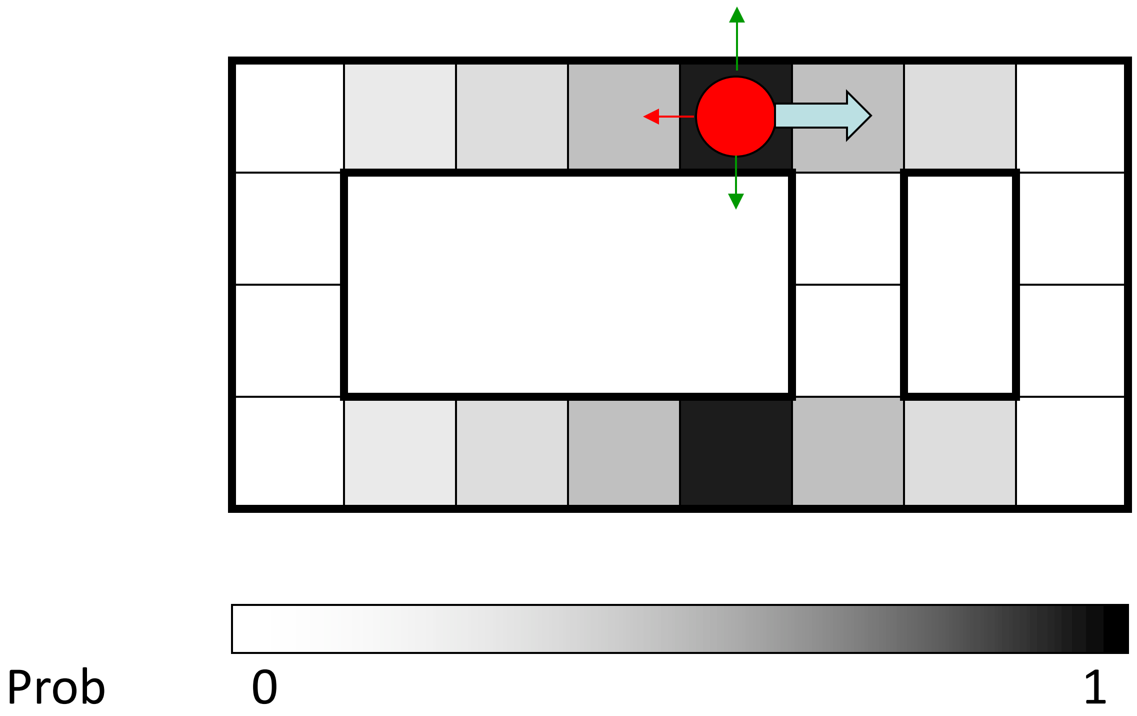




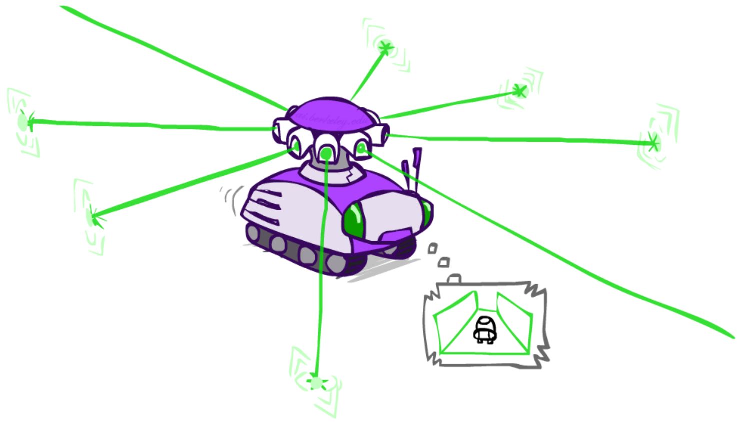
Prob 0 1

t=3

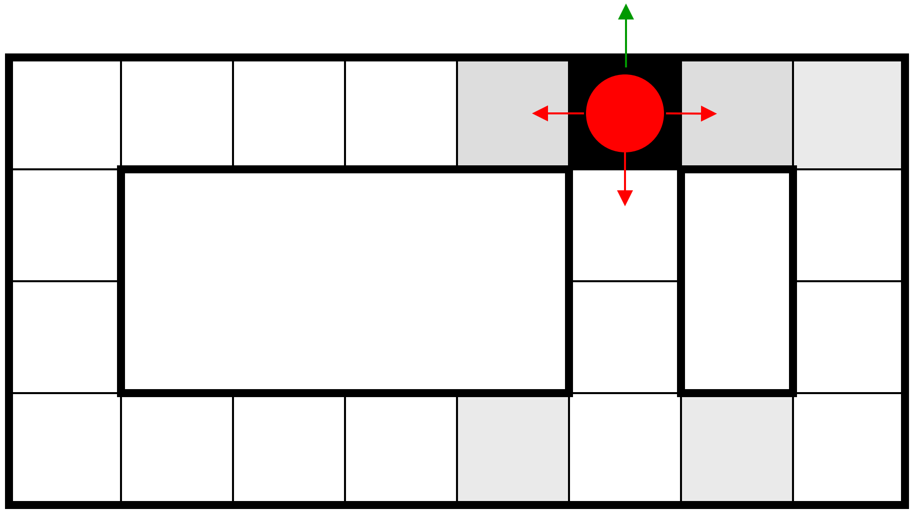
Example: Robot Localization

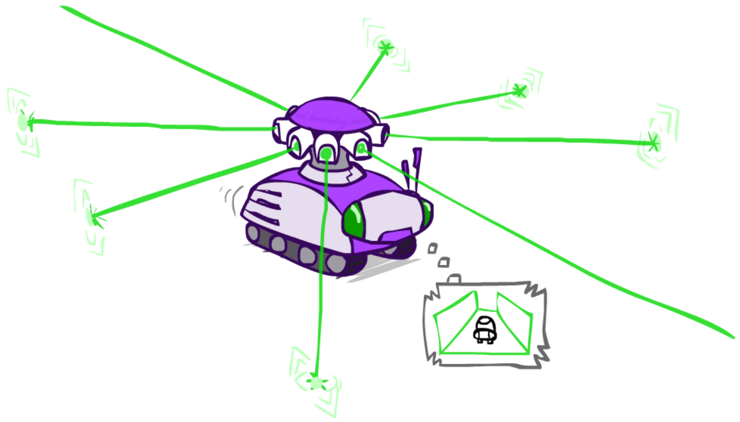


t=4



Example: Robot Localization





Prob 0 1

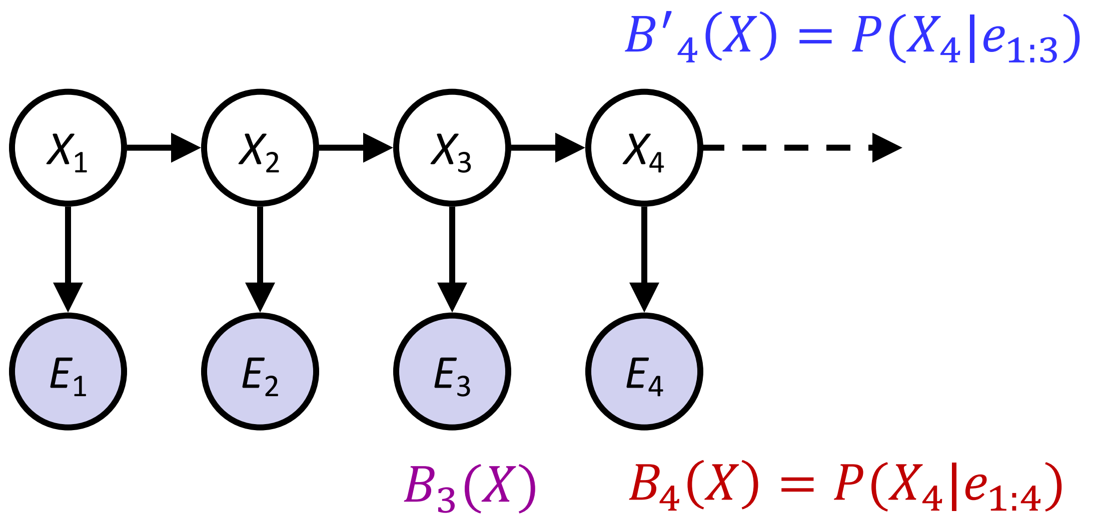
t=5

HMM Inference: Find State Given Evidence

- We are given evidence at each time and want to know

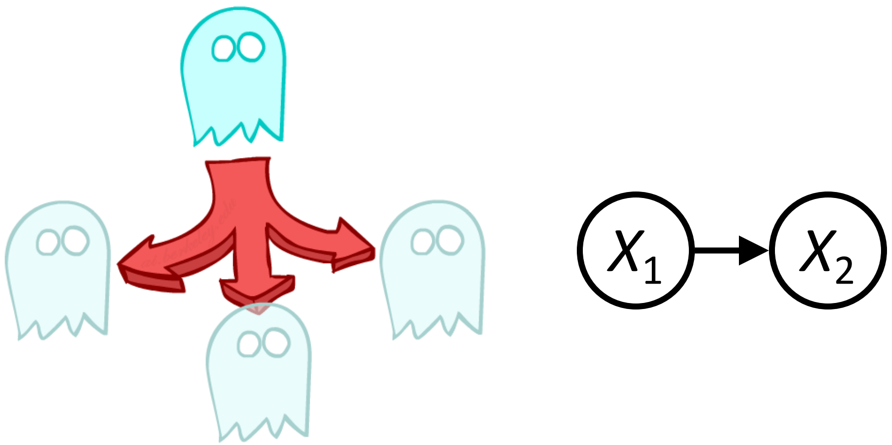
$$B_t(X) = P(X_t | e_{1:t})$$

- Idea: start with $P(X_1)$ and derive $B_t(X)$ in terms of $B_{t-1}(X)$
  - Two steps: **Passage of Time** & **Observation**



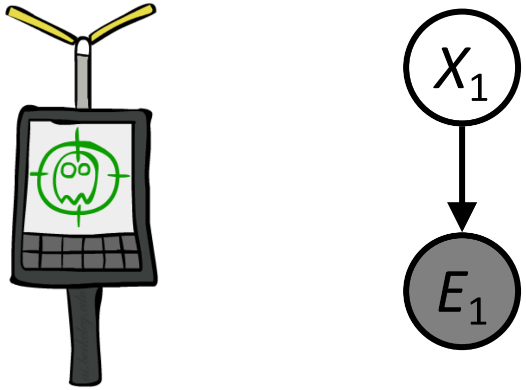
Inference: Base Cases

Passage of Time:



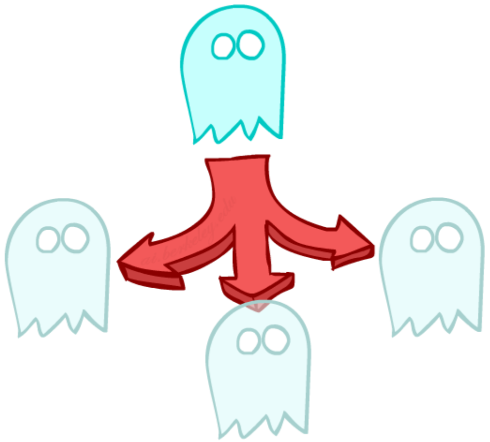
$$P(X_2)$$

Observation:



$$P(X_1|e_1)$$

Passage of Time: Base Case



$$X_1 \rightarrow X_2$$

Have: $P(X_1)$ $P(X_2|X_1)$

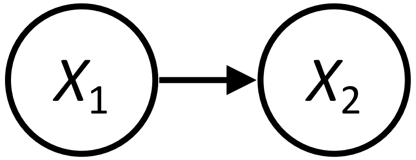
Want: $P(X_2)$

$$\begin{aligned} P(x_2) &= \sum_{x_1} P(x_1, x_2) \\ &= \sum_{x_1} P(x_1) P(x_2|x_1) \end{aligned}$$

Passage of Time: General Case

- Assume we have current belief P(X | evidence to date)

$$B(X_t) = P(X_t|e_{1:t})$$



- Then, after one time step passes:

$$P(X_{t+1}|e_{1:t}) = \sum_{x_t} P(X_{t+1}, x_t|e_{1:t})$$

$$= \sum_{x_t} P(X_{t+1}|x_t, e_{1:t}) P(x_t|e_{1:t})$$

$$= \sum_{x_t} P(X_{t+1}|x_t) P(x_t|e_{1:t})$$

- Or compactly:

$$B'(X_{t+1}) = \sum_{x_t} P(X'|x_t) B(x_t)$$

- Basic idea: beliefs get “pushed” through the transitions
  - With the “B” notation, we have to be careful about what time step t the belief is about, and what evidence it includes

Example: Passage of Time

- As time passes, uncertainty “accumulates”

(Transition model: ghosts usually go clockwise)

<0.01	<0.01	<0.01	<0.01	<0.01	<0.01
<0.01	<0.01	<0.01	<0.01	<0.01	<0.01
<0.01	<0.01	1.00	<0.01	<0.01	<0.01
<0.01	<0.01	<0.01	<0.01	<0.01	<0.01

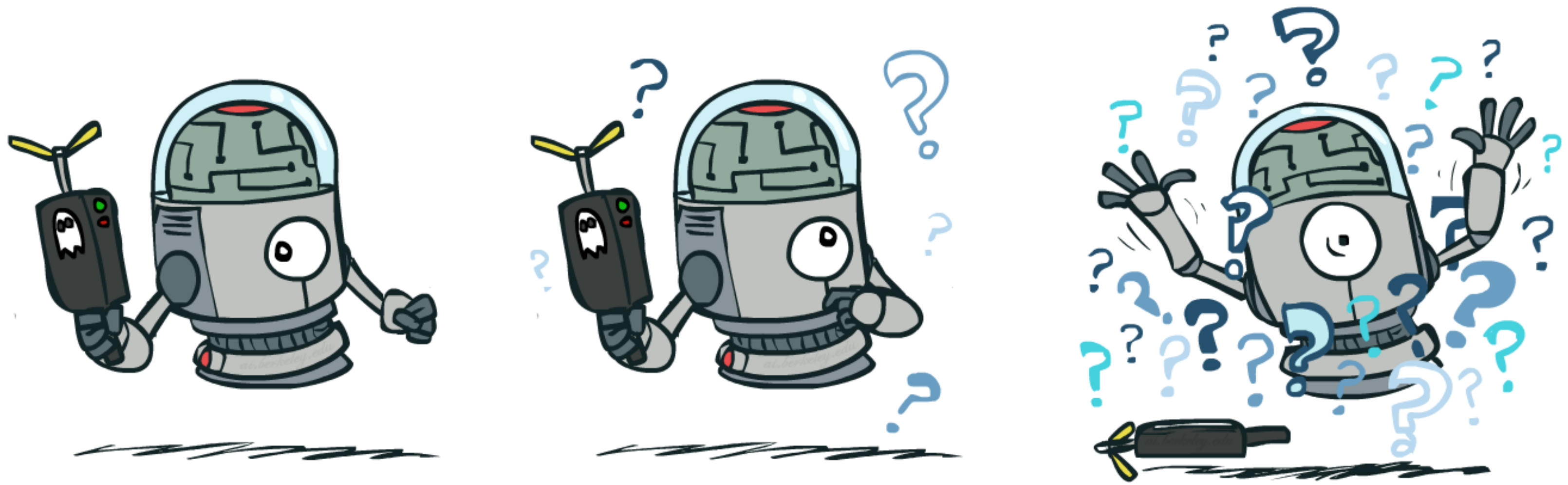
T = 1

<0.01	<0.01	<0.01	<0.01	<0.01	<0.01
<0.01	<0.01	0.06	<0.01	<0.01	<0.01
<0.01	0.76	0.06	0.06	<0.01	<0.01
<0.01	<0.01	0.06	<0.01	<0.01	<0.01

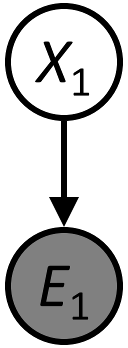
T = 2

0.05	0.01	0.05	<0.01	<0.01	<0.01
0.02	0.14	0.11	0.35	<0.01	<0.01
0.07	0.03	0.05	<0.01	0.03	<0.01
0.03	0.03	<0.01	<0.01	<0.01	<0.01

T = 5



Observation: Base Case



Have: $P(X_1)$ $\quad$ $P(E_1|X_1)$

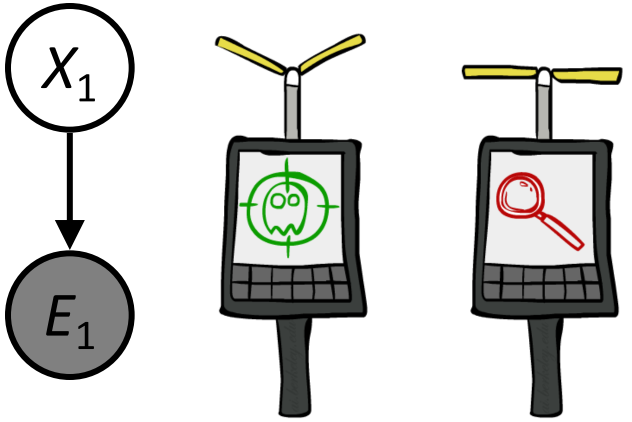
Want: $P(X_1|e_1)$

$$\begin{aligned} P(x_1|e_1) &= P(x_1, e_1)/P(e_1) \\ &\propto_{X_1} P(x_1, e_1) \\ &= P(x_1)P(e_1|x_1) \end{aligned}$$

Also can write as:

$$P(x_1|e_1) = \frac{P(x_1)P(e_1|x_1)}{\sum_{x'} P(x')P(e_1|x')}$$

Observation: General Case



- Assume we have current belief P(X | previous evidence):

$$B'(X_{t+1}) = P(X_{t+1}|e_{1:t})$$

- Then, after evidence comes in:

$$P(X_{t+1}|e_{1:t+1}) = P(X_{t+1}, e_{t+1}|e_{1:t})/P(e_{t+1}|e_{1:t})$$

$$\propto_{X_{t+1}} P(X_{t+1}, e_{t+1}|e_{1:t})$$

$$= P(e_{t+1}|e_{1:t}, X_{t+1})P(X_{t+1}|e_{1:t})$$

$$= P(e_{t+1}|X_{t+1})P(X_{t+1}|e_{1:t})$$

- Or, compactly:

$$B(X_{t+1}) \propto_{X_{t+1}} P(e_{t+1}|X_{t+1})B'(X_{t+1})$$

- Basic idea: beliefs “reweighted” by likelihood of evidence
- Unlike passage of time, we have to renormalize

Example: Observation

- As we get observations, beliefs get reweighted, uncertainty "decreases"

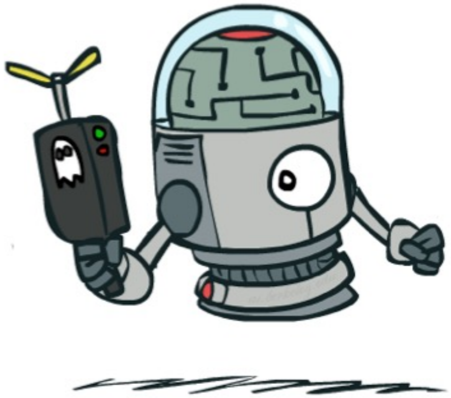
0.05	0.01	0.05	<0.01	<0.01	<0.01
0.02	0.14	0.11	0.35	<0.01	<0.01
0.07	0.03	0.05	<0.01	0.03	<0.01
0.03	0.03	<0.01	<0.01	<0.01	<0.01

Before observation

<0.01	<0.01	<0.01	<0.01	0.02	<0.01
<0.01	<0.01	<0.01	0.83	0.02	<0.01
<0.01	<0.01	0.11	<0.01	<0.01	<0.01
<0.01	<0.01	<0.01	<0.01	<0.01	<0.01

After observation

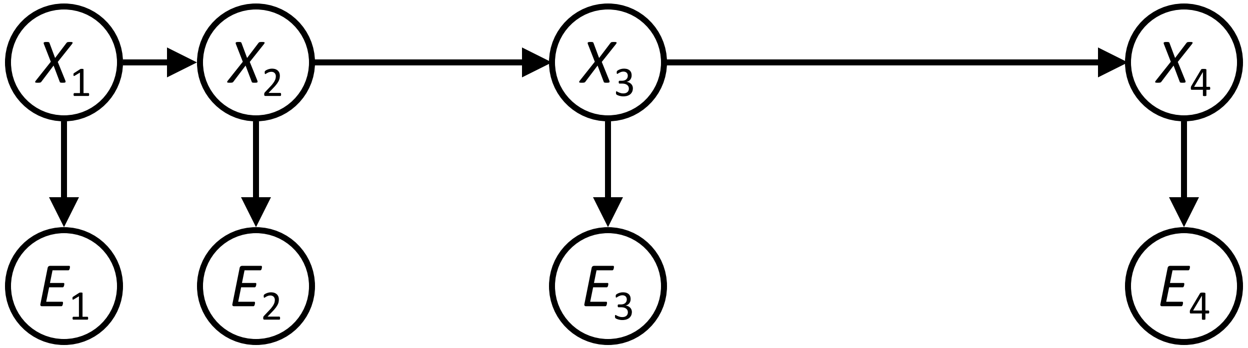
$$B(X) \propto P(e|X)B'(X)$$



Two Steps: Passage of Time + Observation

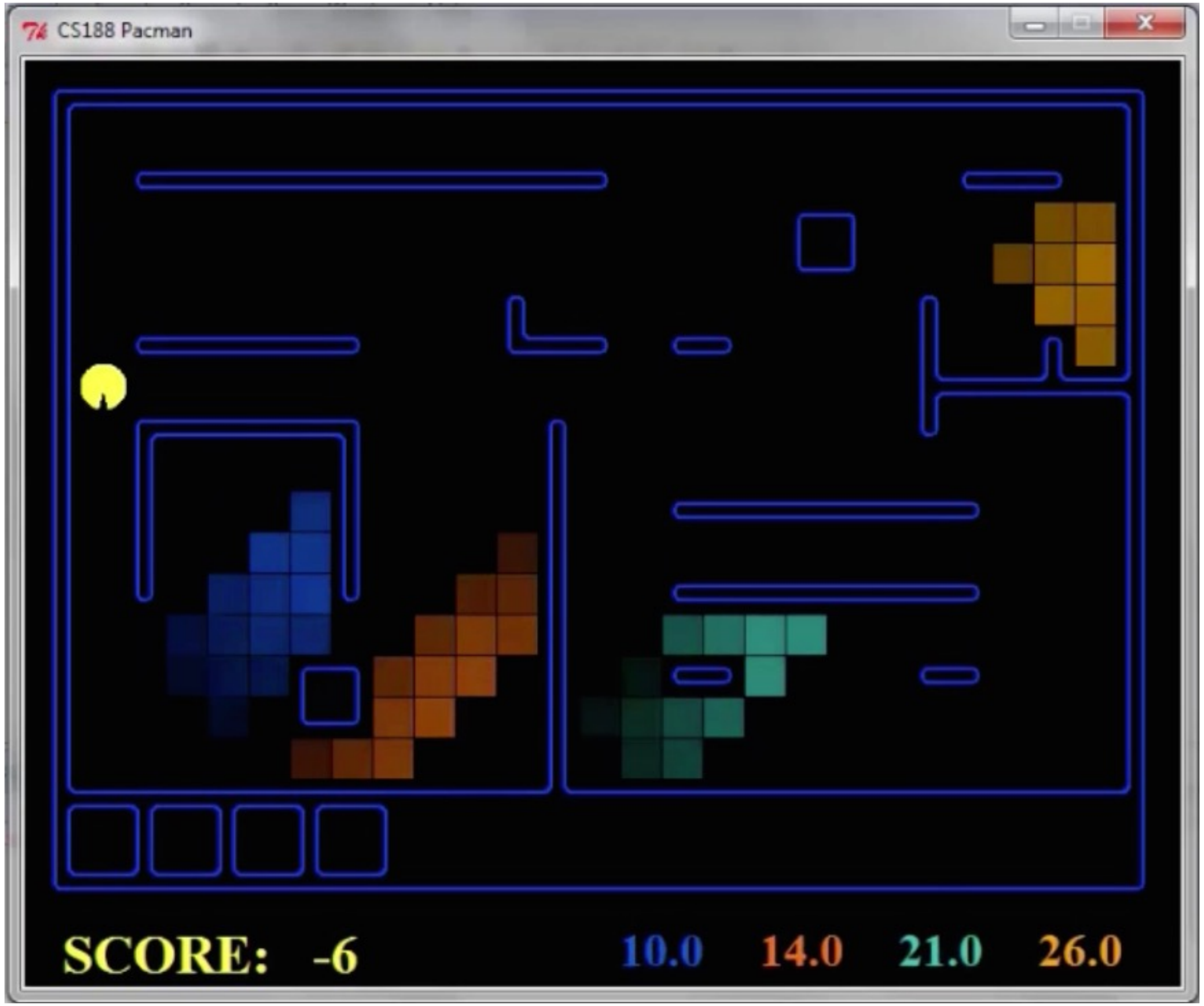
$$B(X_t) = P(X_t|e_{1:t})$$

$$B'(X_{t+1}) = \sum_{x_t} P(X'|x_t)B(x_t)$$



$$B(X_{t+1}) \propto_{X_{t+1}} P(e_{t+1}|X_{t+1})B'(X_{t+1})$$

Pacman – Sonar



[Demo: Pacman – Sonar – No Beliefs(L14D1)]

Video of Demo Pacman – Sonar (with beliefs)

Example: Weather HMM

B(+r) = 0.5
B(-r) = 0.5

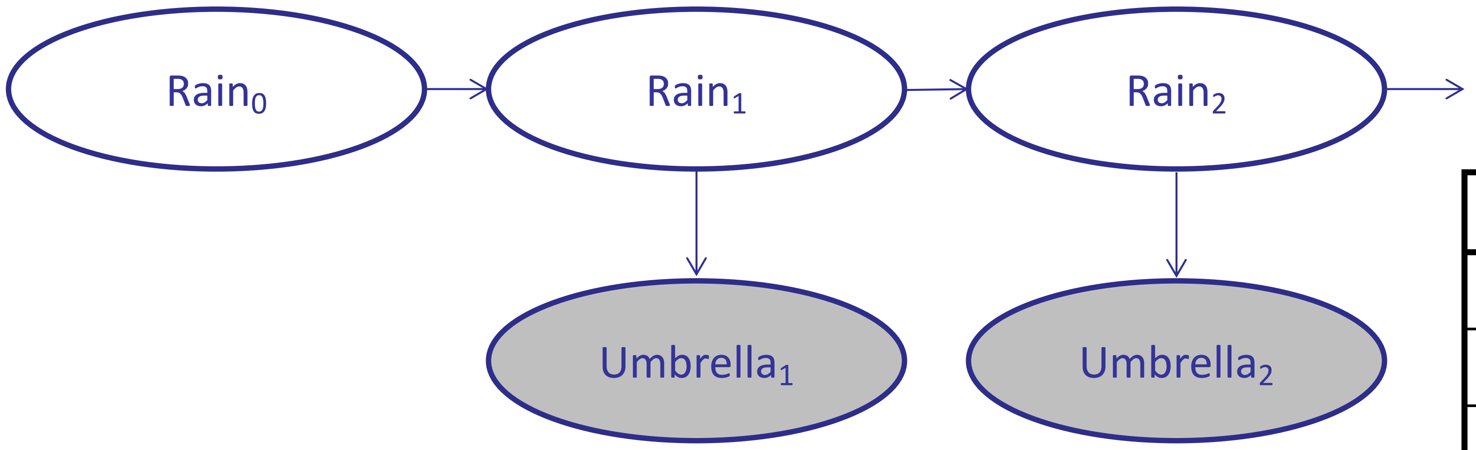
B'(+r) = ?
B'(-r) = ?

Passage of Time:

$$\color{blue}{B'(X_{t+1})} = \sum_{x_t} P(X_{t+1}|x_t)\color{red}{B(x_t)}$$

Observation:

$$\color{red}{B(X_{t+1})} \propto_{X_{t+1}} P(e_{t+1}|X_{t+1})\color{blue}{B'(X_{t+1})}$$



$P(X_{t+1}|X_t)$

R_t	R_{t+1}	$P(R_{t+1}\|R_t)$
+r	+r	0.7
+r	-r	0.3
-r	+r	0.3
-r	-r	0.7

$P(E_t|X_t)$

R_t	U_t	$P(U_t\|R_t)$
+r	+u	0.9
+r	-u	0.1
-r	+u	0.2
-r	-u	0.8

Example: Weather HMM

Passage of Time:

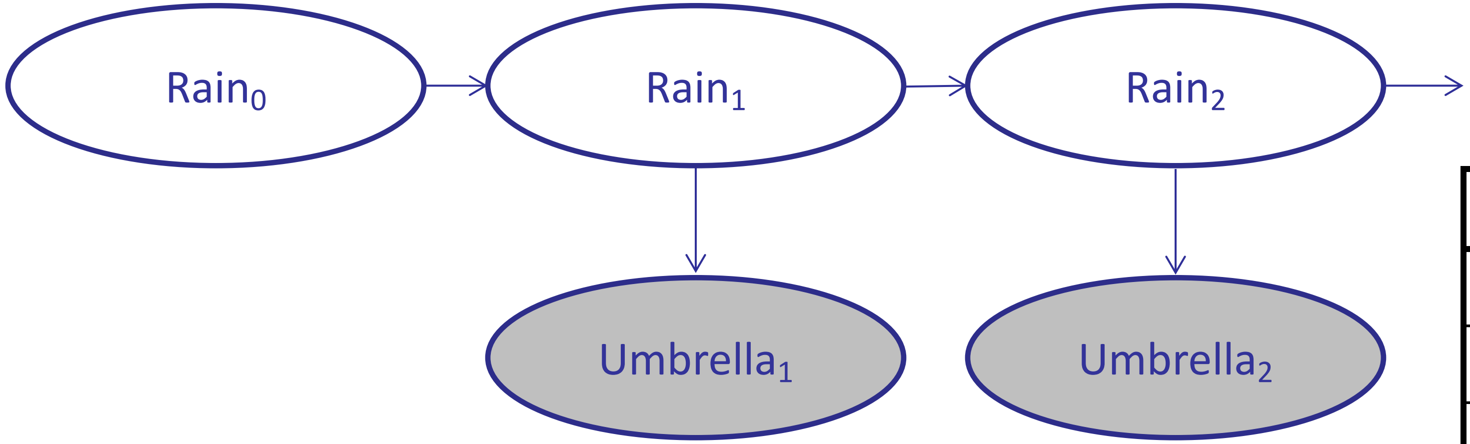
$$B'(X_{t+1}) = \sum_{x_t} P(X_{t+1}|x_t)B(x_t)$$

Observation:

$$B(X_{t+1}) \propto_{X_{t+1}} P(e_{t+1}|X_{t+1})B'(X_{t+1})$$

B'(+r) = 0.5*0.7 + 0.5*0.3 = 0.5
B'(-r) = 0.5*0.3 + 0.5*0.7 = 0.5

B(+r) = 0.5
B(-r) = 0.5



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+r	+r	0.7
+r	-r	0.3
-r	+r	0.3
-r	-r	0.7

$P(E_t|X_t)$

R_t	U_t	$P(U_t\|R_t)$
+r	+u	0.9
+r	-u	0.1
-r	+u	0.2
-r	-u	0.8

Example: Weather HMM

Passage of Time:

$$B'(X_{t+1}) = \sum_{x_t} P(X_{t+1}|x_t)B(x_t)$$

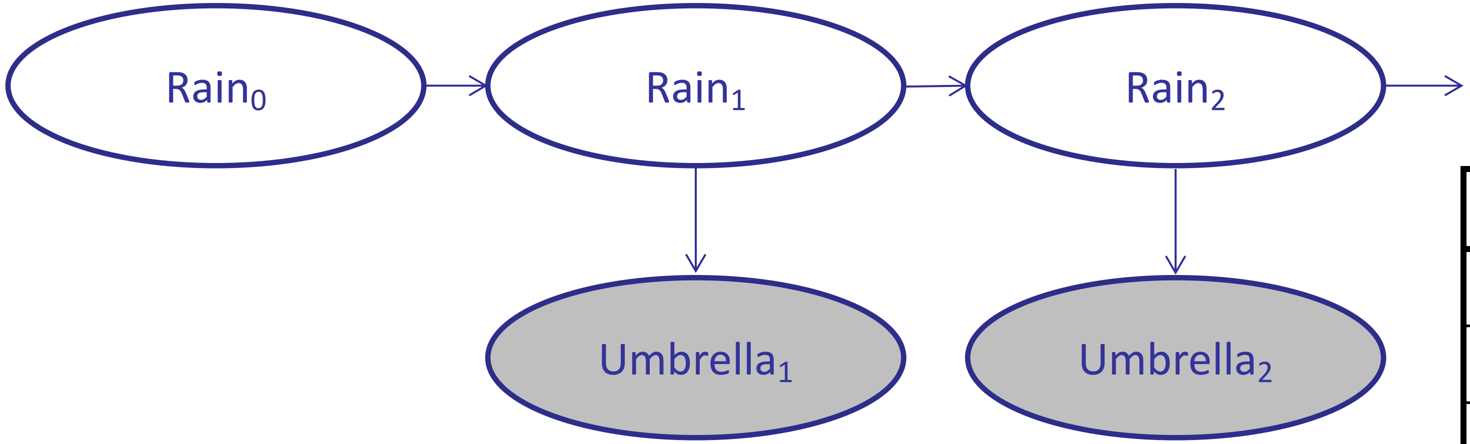
Observation:

$$B(X_{t+1}) \propto_{X_{t+1}} P(e_{t+1}|X_{t+1})B'(X_{t+1})$$

B(+r) = 0.5
B(-r) = 0.5

B'(+r) = 0.5
B'(-r) = 0.5

B(+r) = ?
B(-r) = ?



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-r	-r	0.7

$P(E_t|X_t)$

R_t	U_t	$P(U_t\|R_t)$
+r	+u	0.9
+r	-u	0.1
-r	+u	0.2
-r	-u	0.8

Example: Weather HMM

Passage of Time:

$$B'(X_{t+1}) = \sum_{x_t} P(X_{t+1}|x_t)B(x_t)$$

Observation:

$$B(X_{t+1}) \propto_{X_{t+1}} P(e_{t+1}|X_{t+1})B'(X_{t+1})$$

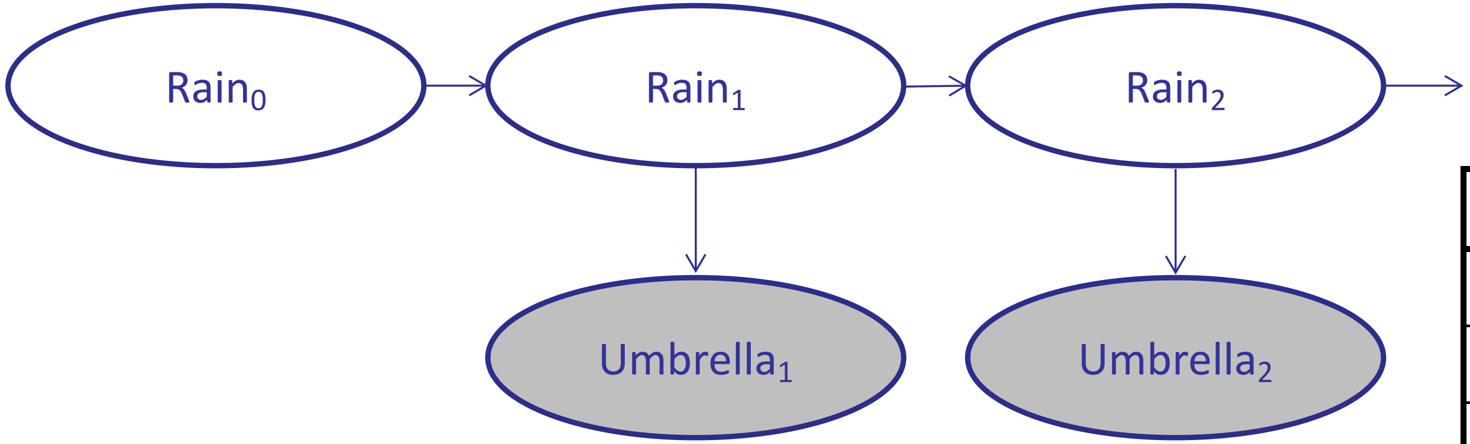
B’(+r) = 0.5
B’(-r) = 0.5

B(+r) = 0.5
B(-r) = 0.5

B(+r) = 0.9*0.5 = 0.45
B(-r) = 0.2*0.5 = 0.10

normalize

0.818
0.182



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R_t	R_{t+1}	$P(R_{t+1}\|R_t)$
+r	+r	0.7
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-r	+r	0.3
-r	-r	0.7

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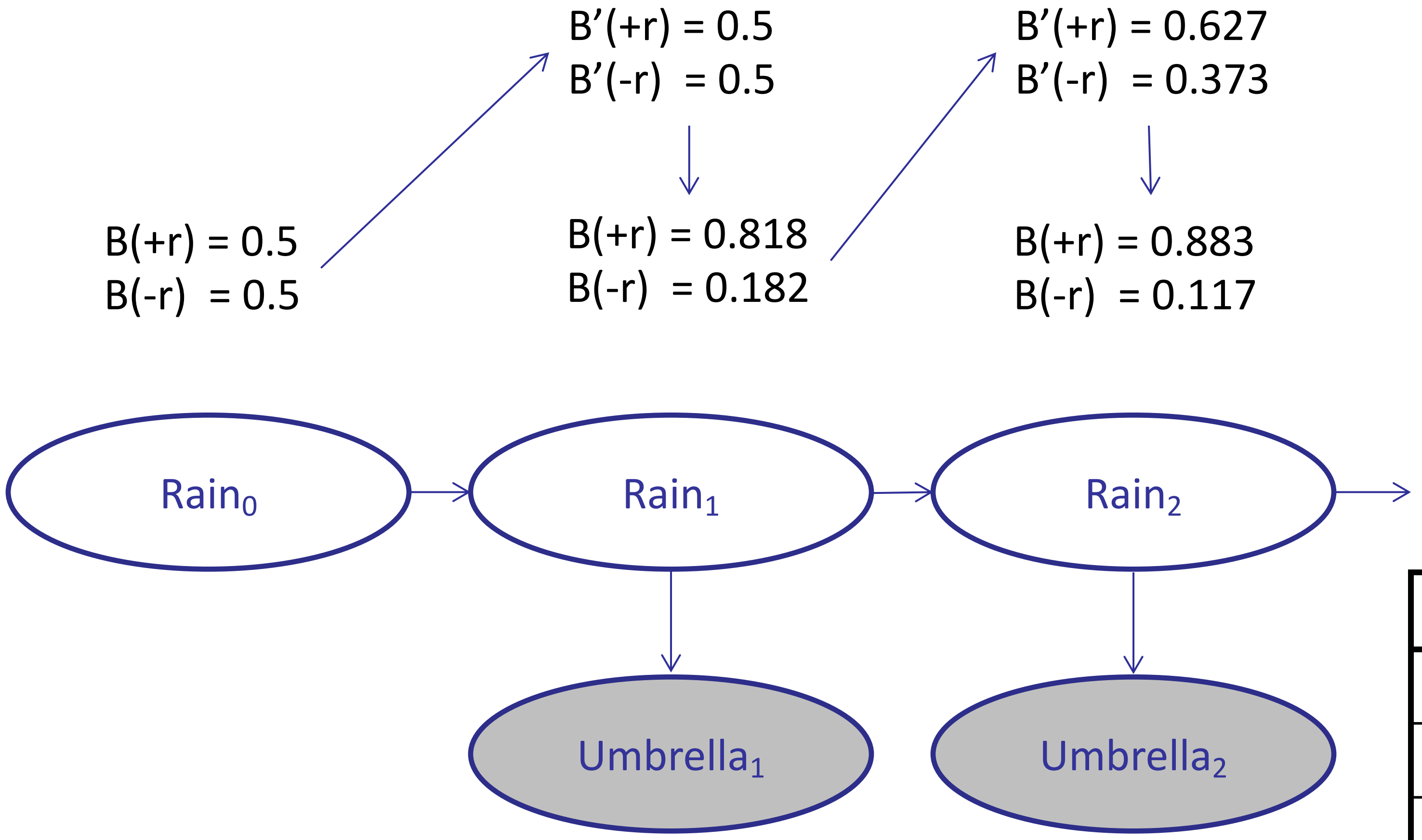
R_t	U_t	$P(U_t\|R_t)$
+r	+u	0.9
+r	-u	0.1
-r	+u	0.2
-r	-u	0.8

Example: Weather HMM



Passage of Time:

$$B'(X_{t+1}) = \sum_{x_t} P(X_{t+1}|x_t)B(x_t)$$

Observation:

$$B(X_{t+1}) \propto_{X_{t+1}} P(e_{t+1}|X_{t+1})B'(X_{t+1})$$

$P(X_{t+1}|X_t)$

R_t	R_{t+1}	$P(R_{t+1}\|R_t)$
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R_t	U_t	$P(U_t\|R_t)$
+r	+u	0.9
+r	-u	0.1
-r	+u	0.2
-r	-u	0.8

What we did today

- **Markov Chains** & their **Stationary Distributions**
  - How beliefs about state change with passage of time
- **Hidden Markov Models** (HMMs) formulation
  - How beliefs change with passage of time and evidence
- **Filtering** with HMMs
  - How to infer beliefs from evidence

Next Time: More Filtering!