CS 188: Artificial Intelligence

Machine Learning: Naïve Bayes

University of California, Berkeley

[These slides were created by Dan Klein and Pieter Abbeel, Anca Dragan, Sergey Levine, with some materials from A. Farhadi. All CS188 materials are at http://ai.berkeley.edu.]

Machine Learning

- Up until now: how use a model to make optimal decisions

- **Machine learning:** how to acquire a model from data / experience
  - Learning parameters (e.g. probabilities)
  - Learning structure (e.g. BN graphs)
  - Learning features or hidden concepts (e.g. neural nets, clustering)

- What's our roadmap?

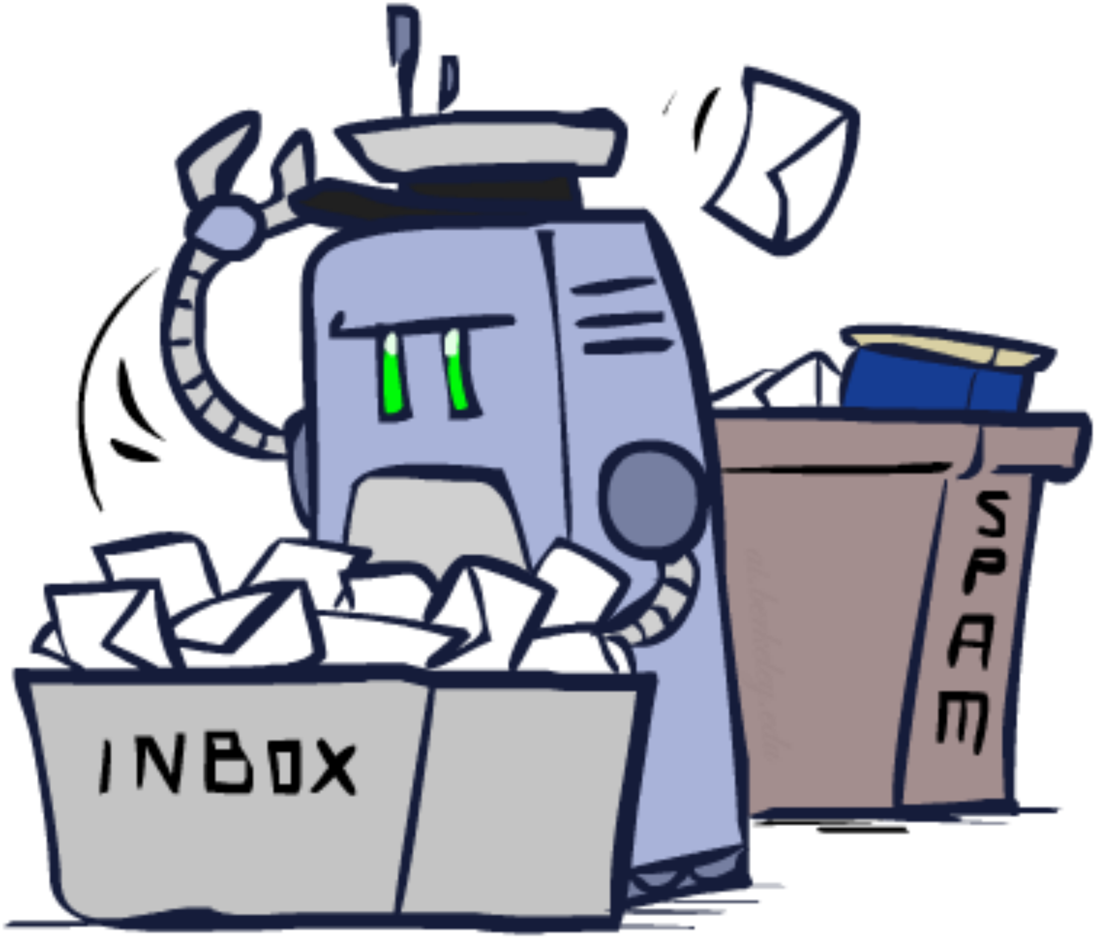
Our Machine Learning Roadmap

- Define the problem
  - Type of problem, domain (i.e. spam filtering, digit recognition)
- Look at several learning approaches / models
  - *Naïve Bayes* (today), *Perceptrons*, *Logistic Regression*, *Neural Networks* (next week)
- How to find model parameters: *Maximum Likelihood*
  - Special cases: solve analytically (today)
  - In general: *numerical optimization* (next week)
- Themes throughout
  - Working with data
  - Preventing overfitting
  - Evaluating performance

Multiple Types of Learning Problems

- **Supervised learning:** correct answers for each training example
  - **Classification:** learning predictor with *discrete* outputs
  - **Regression:** learning predictor with *real-valued* outputs
- **Reinforcement learning:** reward function, no correct answers
- **Unsupervised learning:** no correct answers, just find good representations / features of the data

Classification

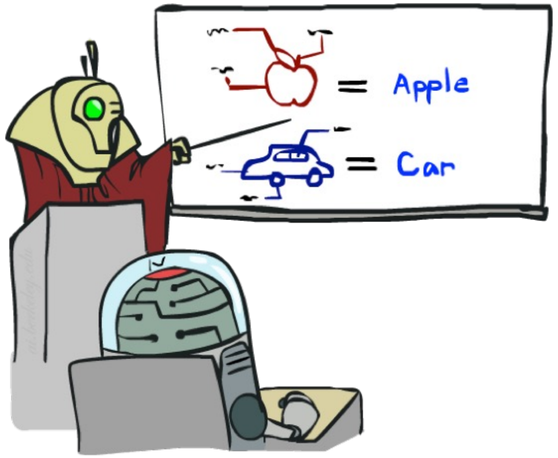


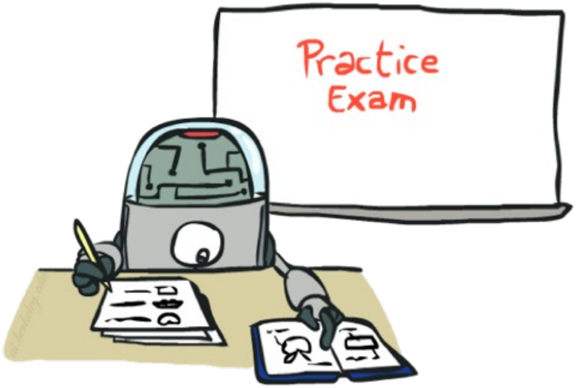
Classification and Machine Learning

- **Dataset:** each data point, **x**, is associated with some label (aka class), **y**
- **Goal of classification:** given inputs **x**, write an algorithm to predict labels **y**
- **Workflow** of classification process:
  - Input is provided to you
  - Extract **features** from the input: attributes of the input that characterize each **x** and hopefully help with classification
  - Run some machine learning algorithm on the features: today, *Naïve Bayes*
  - Output a predicted label **y**

Training and Machine Learning

- Big idea: ML algorithms learn patterns between features and labels from *data*
  - You don't have to reason about the data yourself
  - You're given **training data**: lots of example datapoints and their actual labels





Training: Learn patterns from labeled data, and periodically test how well you're doing

Eventually, use your algorithm to predict labels for unlabeled data

Example: Spam Filter

- Input: an email
- Output: spam/ham

- Setup:
  - Get a large collection of example emails, each labeled “spam” or “ham”
  - Note: someone has to hand label all this data!
  - Want to learn to predict labels of new, future emails

- Features: The attributes used to make the ham / spam decision
  - Words: FREE!
  - Text Patterns: $dd, CAPS
  - Non-text: SenderInContacts, WidelyBroadcast
  - …

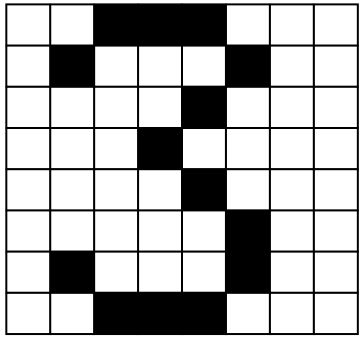
> Dear Sir.
>
> First, I must solicit your confidence in this transaction, this is by virture of its nature as being utterly confidencial and top secret. …

> TO BE REMOVED FROM FUTURE MAILINGS, SIMPLY REPLY TO THIS MESSAGE AND PUT "REMOVE" IN THE SUBJECT.
>
> 99 MILLION EMAIL ADDRESSES FOR ONLY $99

> Ok, Iknow this is blatantly OT but I'm beginning to go insane. Had an old Dell Dimension XPS sitting in the corner and decided to put it to use, I know it was working pre being stuck in the corner, but when I plugged it in, hit the power nothing happened.

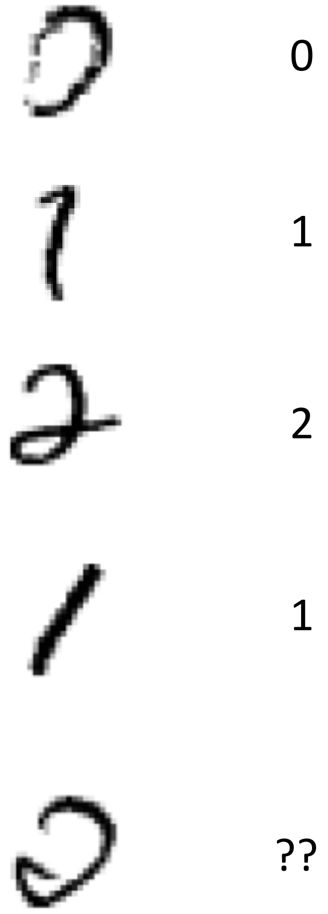
Example: Digit Recognition

- Input: images / pixel grids
- Output: a digit 0-9



- Setup:
  - Get a large collection of example images, each labeled with a digit
  - Note: someone has to hand label all this data!
  - Want to learn to predict labels of new, future digit images

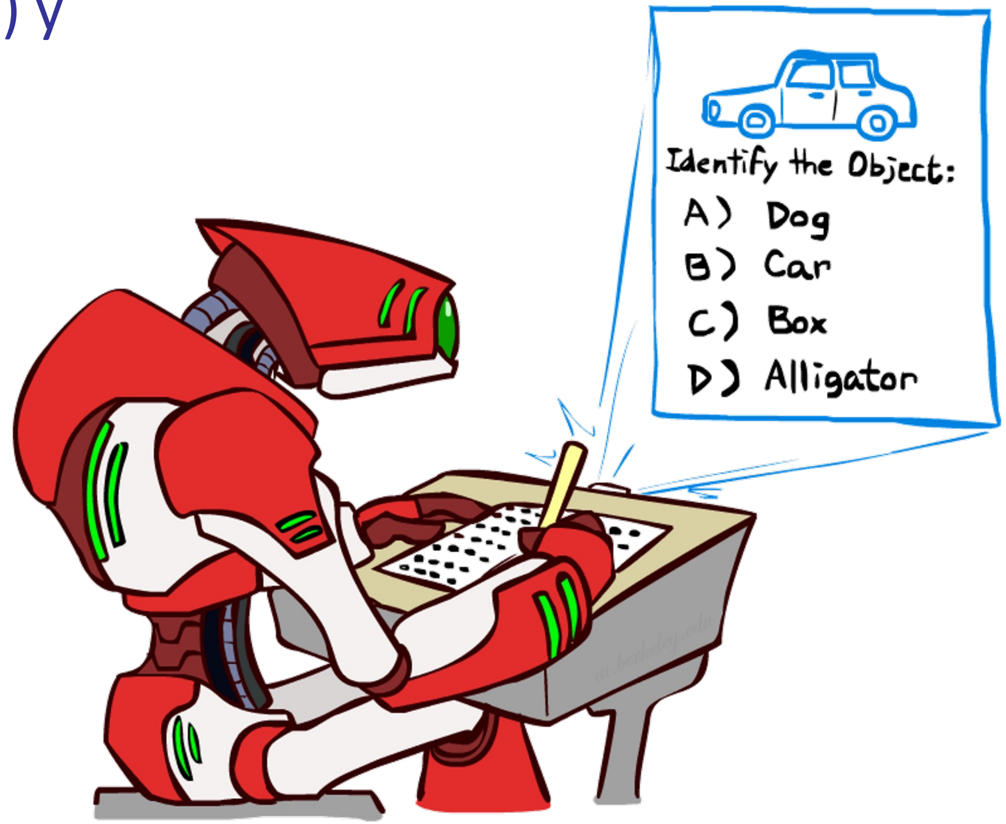
- Features: The attributes used to make the digit decision
  - Pixels: (6,8)=ON
  - Shape Patterns: NumComponents, AspectRatio, NumLoops
  - …



Other Classification Tasks

- Classification: given inputs x, predict labels (classes) y

- Examples:
  - Object recognition
  - Input: images; classes: object type
  - Medical diagnosis
    input: symptoms; classes: diseases
  - Automatic essay grading
    input: document; classes: grades
  - Fraud detection
    input: account activity; classes: fraud / no fraud
  - Customer service email routing
  - … many more



- Classification is an important commercial technology!

Model-Based Classification

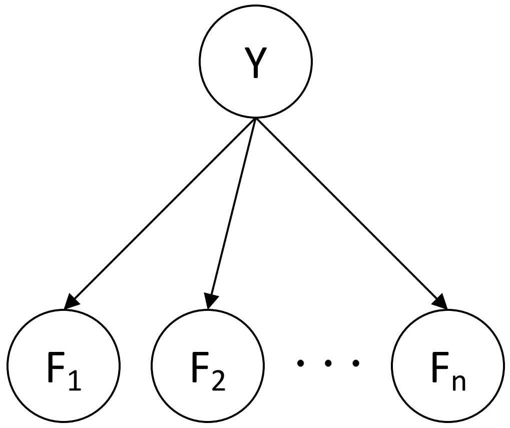
Model-Based Classification

- Model-based approach
  - Build a model (e.g. Bayes' net) where both the label and features are random variables
  - Instantiate any observed features
  - Query for the distribution of the label conditioned on the features

- Challenges
  - What structure should the BN have?
  - How should we learn its parameters?

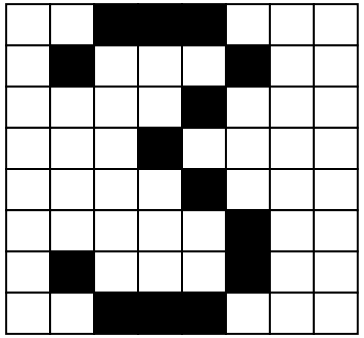
Naïve Bayes Model

- Naïve Bayes: Assume all features are independent effects of the label
- Random variables in this Bayes' net:
  - Y = The label
  - $F_1, F_2, \ldots, F_n$ = The n features
- Probability tables in this Bayes' net:
  - $P(Y)$ = Probability of each label, given no information about the features.
    - Sometimes called the *prior*.
  - $P(F_i|Y)$ = One table per feature. Probability of feature, given the label.

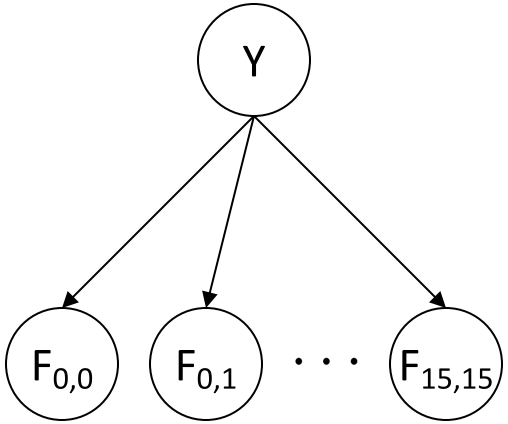


Naïve Bayes for Digits

- Simple digit recognition version:
  - One feature (variable) F_{ij} for each grid position <i,j>
  - Feature values are on / off, based on whether intensity is more or less than 0.5 in underlying image
  - Each input maps to a feature vector, e.g.



$$\rightarrow \langle F_{0,0} = 0 \;\; F_{0,1} = 0 \;\; F_{0,2} = 1 \;\; F_{0,3} = 1 \;\; F_{0,4} = 0 \;\; \ldots F_{15,15} = 0 \rangle$$



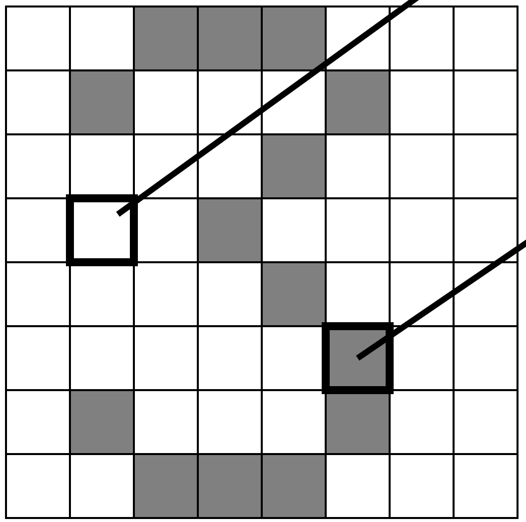
  - Here: lots of features, each is binary valued

- What are the parameters of this model?

Naïve Bayes for Digits: Parameters

$P(Y)$

1	0.1
2	0.1
3	0.1
4	0.1
5	0.1
6	0.1
7	0.1
8	0.1
9	0.1
0	0.1



$P(F_{3,1} = on|Y)$

1	0.01
2	0.05
3	0.05
4	0.30
5	0.80
6	0.90
7	0.05
8	0.60
9	0.50
0	0.80

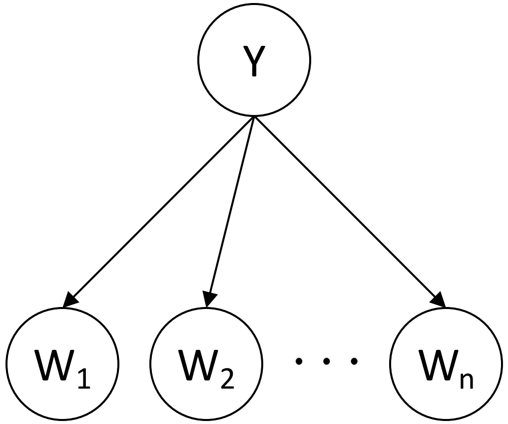
$P(F_{5,5} = on|Y)$

1	0.05
2	0.01
3	0.90
4	0.80
5	0.90
6	0.90
7	0.25
8	0.85
9	0.60
0	0.80

Naïve Bayes for Text

- Bag-of-words Naïve Bayes:
  - Features: W_i is the word at positon i
  - As before: predict label conditioned on feature variables (spam vs. ham)
  - As before: assume features are conditionally independent given label
  - New: each W_i is identically distributed, so $P(W_1|Y) = P(W_2|Y) = \ldots$

how many features are there?
how many values?



- "Tied" distributions and bag-of-words
  - Usually, each variable gets its own conditional probability distribution $P(F|Y)$
  - In a bag-of-words model
    - Each position is identically distributed
    - All positions share the same conditional probs $P(W|Y)$
    - Why make this assumption?
  - Called "bag-of-words" because model is insensitive to word order or reordering

free our offer try please

please try our free offer

Naïve Bayes for Text: Parameters

- What are the parameters?

$P(Y)$

ham	0.66
spam	0.33

$P(W|\text{spam})$

the	0.0156
to	0.0153
and	0.0115
of	0.0095
you	0.0093
a	0.0086
with	0.0080
from	0.0075
...	

$P(W|\text{ham})$

the	0.0210
to	0.0133
of	0.0119
2002	0.0110
with	0.0108
from	0.0107
and	0.0105
a	0.0100
...	

Naïve Bayes Model

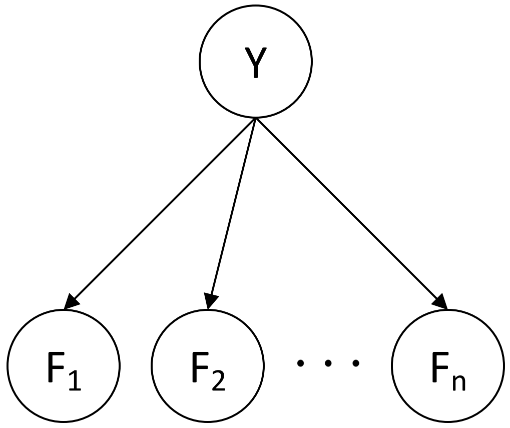
- In general, the joint probability in Naïve Bayes model is:

|Y| parameters

$$P(\mathsf{Y}, \mathsf{F}_1 \dots \mathsf{F}_n) = \quad P(\mathsf{Y}) \prod_i P(\mathsf{F}_i | \mathsf{Y})$$

$|Y| \times |F|^n$ values

$n \times |F| \times |Y|$ parameters



- We only have to specify how each feature depends on the class
- Total number of parameters is *linear* in n
- Model is very simplistic, but often works anyway

Inference for Naïve Bayes

- Goal: compute posterior distribution over label variable Y
  - Step 1: get joint probability of label and evidence for each label

$$P(Y, f_1 \dots f_n) = \begin{bmatrix} P(y_1, f_1 \dots f_n) \\ P(y_2, f_1 \dots f_n) \\ \vdots \\ P(y_k, f_1 \dots f_n) \end{bmatrix} \Rightarrow \begin{bmatrix} P(y_1) \prod_i P(f_i|y_1) \\ P(y_2) \prod_i P(f_i|y_2) \\ \vdots \\ P(y_k) \prod_i P(f_i|y_k) \end{bmatrix}$$

$$\xrightarrow{+} P(f_1 \dots f_n)$$

  - Step 2: sum to get probability of evidence

  - Step 3: normalize by dividing Step 1 by Step 2

$$P(Y|f_1 \dots f_n)$$

Example: Spam Filtering

- Model: $P(Y, W_1 \dots W_n) = P(Y) \prod_i P(W_i|Y)$
- Parameters:

$P(Y)$

ham	0.66
spam	0.33

$P(W|\text{spam})$

the	0.0156
to	0.0153
and	0.0115
of	0.0095
you	0.0093
a	0.0086
with	0.0080
from	0.0075
...	

$P(W|\text{ham})$

the	0.0210
to	0.0133
of	0.0119
2002	0.0110
with	0.0108
from	0.0107
and	0.0105
a	0.0100
...	

Example: Spam Filtering

$P(Y)$

$P(W_1|Y)$

$P(W_2|Y)$

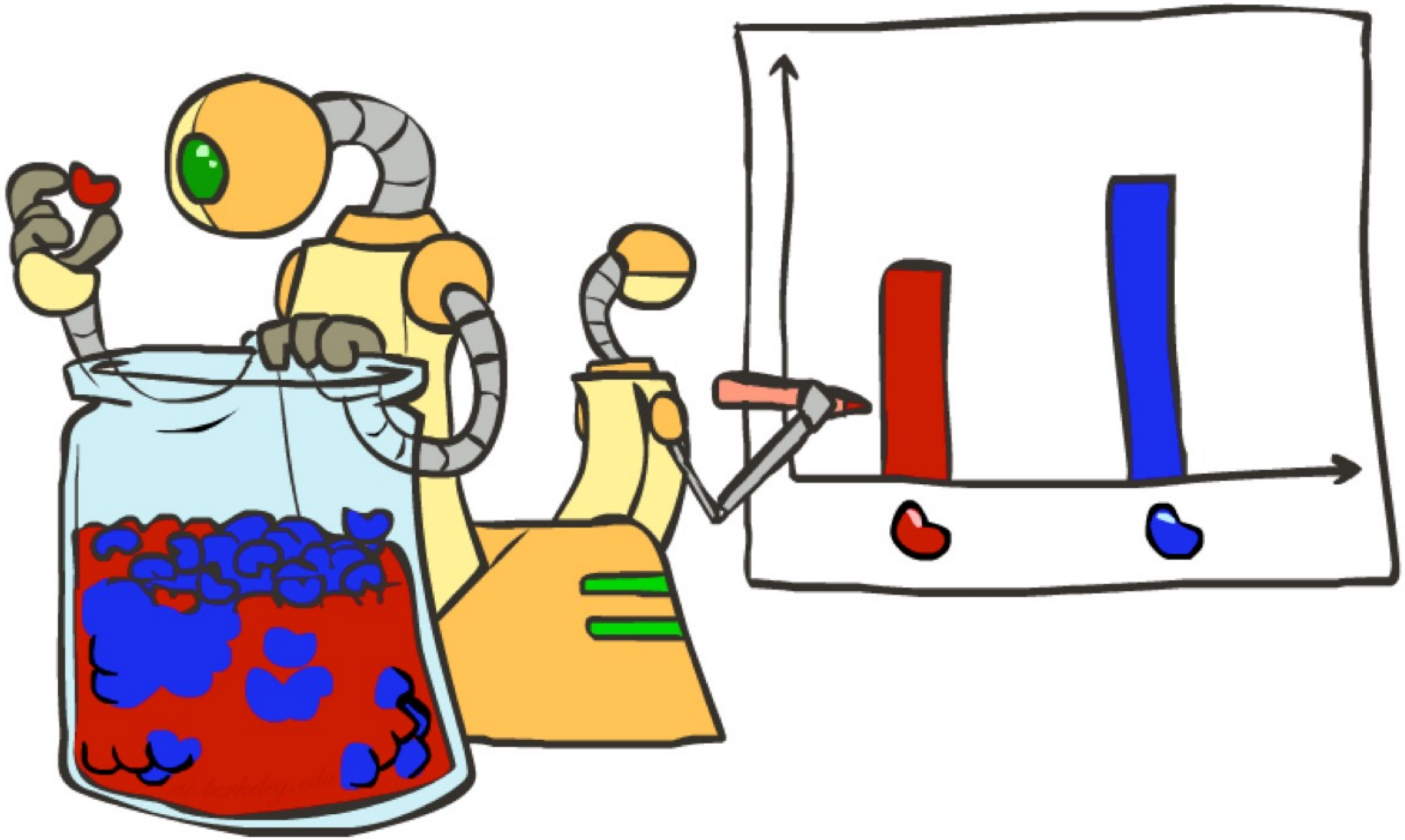
⋮

Word	P(w|spam)	P(w|ham)	Tot Spam	Tot Ham
(prior)	0.33333	0.66666	-1.1	-0.4

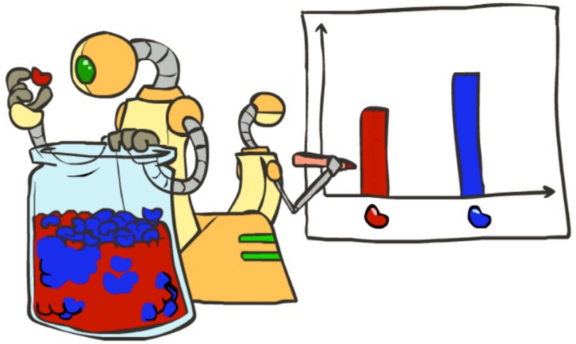
General Naïve Bayes

- What do we need in order to use Naïve Bayes?
  - Inference method (we just saw this part)
    - Start with a bunch of probabilities: P(Y) and the $P(F_i|Y)$ tables
    - Use standard inference to compute $P(Y|F_1 \dots F_n)$
    - Nothing new here
  - Estimates of local conditional probability tables
    - P(Y), the prior over labels
    - $P(F_i|Y)$ for each feature (evidence variable)
    - These probabilities are collectively called the *parameters* of the model and denoted by θ
    - Up until now, we assumed these appeared by magic, but they typically come from training data counts

Parameter Estimation



Parameter Estimation



- Estimating the distribution of a random variable
- *Elicitation:* ask a human (why is this hard?)
- *Empirically:* use training data (learning!)
  - Example: The parameter θ is the true fraction of red beans in the jar. You don't know θ but would like to estimate it.
  - Collecting training data: You randomly pull out 3 beans:

    r r b

  - Estimating θ using counts, you guess 2/3 of beans in the jar are red.
  - Can we mathematically show that using counts is the "right" way to estimate θ?

Parameter Estimation with Maximum Likelihood

- θ is the true fraction of red beans in the jar (i.e. P(red | θ) = θ)
- Can we mathematically show that using counts is the "right" way to estimate θ?
- **Maximum likelihood estimation:** Choose the θ value that maximizes the probability of the observation
  - In other words, choose the θ value that maximizes P(observation | θ)
  - For our problem:

    P(observation | θ)

    = P(randomly selected 2 red and 1 blue | θ of beans are red)

    = P(red | θ) * P(red | θ) * P(blue | θ)

    = $\theta^2 (1 - \theta)$

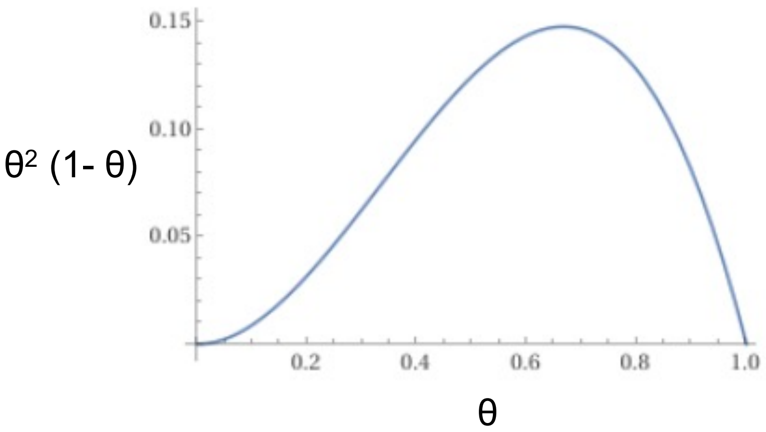
  - We want to compute:

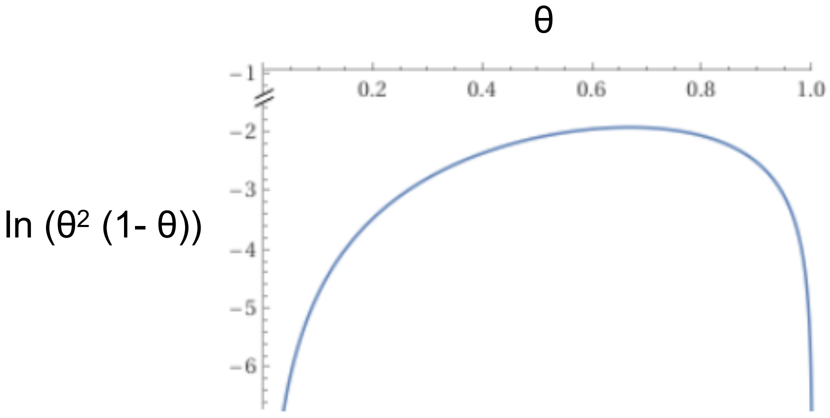
    $$\underset{\theta}{\operatorname{argmax}}\ \theta^2 (1 - \theta)$$

Parameter Estimation with Maximum Likelihood

- We want to compute:

  $$\underset{\theta}{\text{argmax}}\ \theta^2 (1-\theta)$$





- Set derivative to 0, and solve!
  - Common issue: The likelihood (expression we're maxing) is the product of a lot of probabilities. This can lead to complicated derivatives.
  - Solution: Maximize the log-likelihood instead. Useful fact:

    $$\underset{\theta}{\text{argmax}}\ f(\theta) = \underset{\theta}{\text{argmax}}\ \ln f(\theta)$$

Parameter Estimation with Maximum Likelihood

$$\underset{\theta}{\operatorname{argmax}}\, \theta^2(1-\theta)$$

Find θ that maximizes likelihood

$$= \underset{\theta}{\operatorname{argmax}}\, \ln\left(\theta^2(1-\theta)\right)$$

Find θ that maximizes log-likelihood (will be the same θ)

$$\frac{d}{d\theta} \ln\left(\theta^2(1-\theta)\right) = 0$$

Set derivative to 0

$$\frac{d}{d\theta}\left[\ln(\theta^2) + \ln(1-\theta)\right] = 0$$

Logarithm rule: products become sums

$$\frac{d}{d\theta}\left[2\ln(\theta) + \ln(1-\theta)\right] = 0$$

Logarithm rule: exponentiation becomes multiplication

$$\frac{d}{d\theta} 2\ln(\theta) + \frac{d}{d\theta}\ln(1-\theta) = 0$$

Now we can derive each term of the original product separately

$$\frac{2}{\theta} - \frac{1}{1-\theta} = 0$$

Reminder: Derivative of ln(θ) is 1/θ

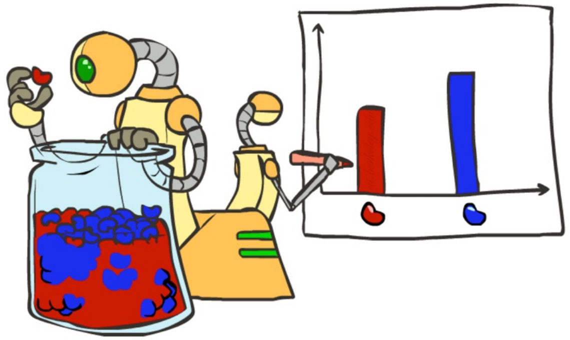
$$\theta = \frac{2}{3}$$

Use algebra to solve for θ. If we used arbitrary red and blue counts r and b instead of r=2 and b=1, we'd get θ = r / (r+b), the count estimate.

Parameter Estimation with Maximum Likelihood (General Case)

- **Model**:

X	red	blue
$P(X\|\theta)$	θ	$1-\theta$



- **Data:** draw N balls, N_r come up red and N_b come up blue
  - Dataset $D = \{x_1, \ldots, x_N\}$ of N ball draws

$$P(D|\theta) = \prod_i P(x_i|\theta) = \theta^{N_r} \cdot (1-\theta)^{N_b}$$

- **Maximum Likelihood Estimation**: find θ that maximizes $\mathrm{P}(\mathrm{D}|\theta)$:

$$\hat{\theta} = \underset{\theta}{\operatorname{argmax}}\, P(D|\theta) = \underset{\theta}{\operatorname{argmax}} \log P(D|\theta) \leftarrow N_r \log(\theta) + N_b \log(1-\theta)$$

Take derivative and set to 0:

$$\frac{\partial \log P(D|\theta)}{\partial \theta} = \frac{N_r}{\theta} - \frac{N_b}{1-\theta} = 0$$

$$\rightarrow \hat{\theta} = \frac{N_r}{N_r + N_b} = \frac{\text{\# of red balls}}{\text{total \# of balls}}$$

Parameter Estimation with Maximum Likelihood (General Case)

- **Maximum Likelihood Estimation**: find θ that maximizes $\mathrm{P}(\mathrm{D}|\theta)$:

$$\hat{\theta} = \underset{\theta}{\operatorname{argmax}}\, P(D|\theta) = \underset{\theta}{\operatorname{argmax}}\, \boxed{\log P(D|\theta)} \leftarrow N_r \log(\theta) + N_b \log(1-\theta)$$

Take derivative and set to 0:

$$\frac{\partial}{\partial\theta} \log P(D|\theta) = \frac{\partial}{\partial\theta}[N_r \log(\theta) + N_b \log(1-\theta)]$$

$$= N_r \frac{\partial}{\partial\theta}[\log(\theta)] + N_b \frac{\partial}{\partial\theta}[\log(1-\theta)]$$

$$= N_r \frac{1}{\theta} + N_b \frac{1}{1-\theta} \cdot -1$$

$$= N_r(1-\theta) - N_b\theta$$

$$= N_r - \theta(N_r + N_b) = 0$$

$$\rightarrow \hat{\theta} = \frac{N_r}{N_r + N_b}$$

Parameter Estimation with Maximum Likelihood

- Collectively name all model parameters (i.e. probability tables) as θ

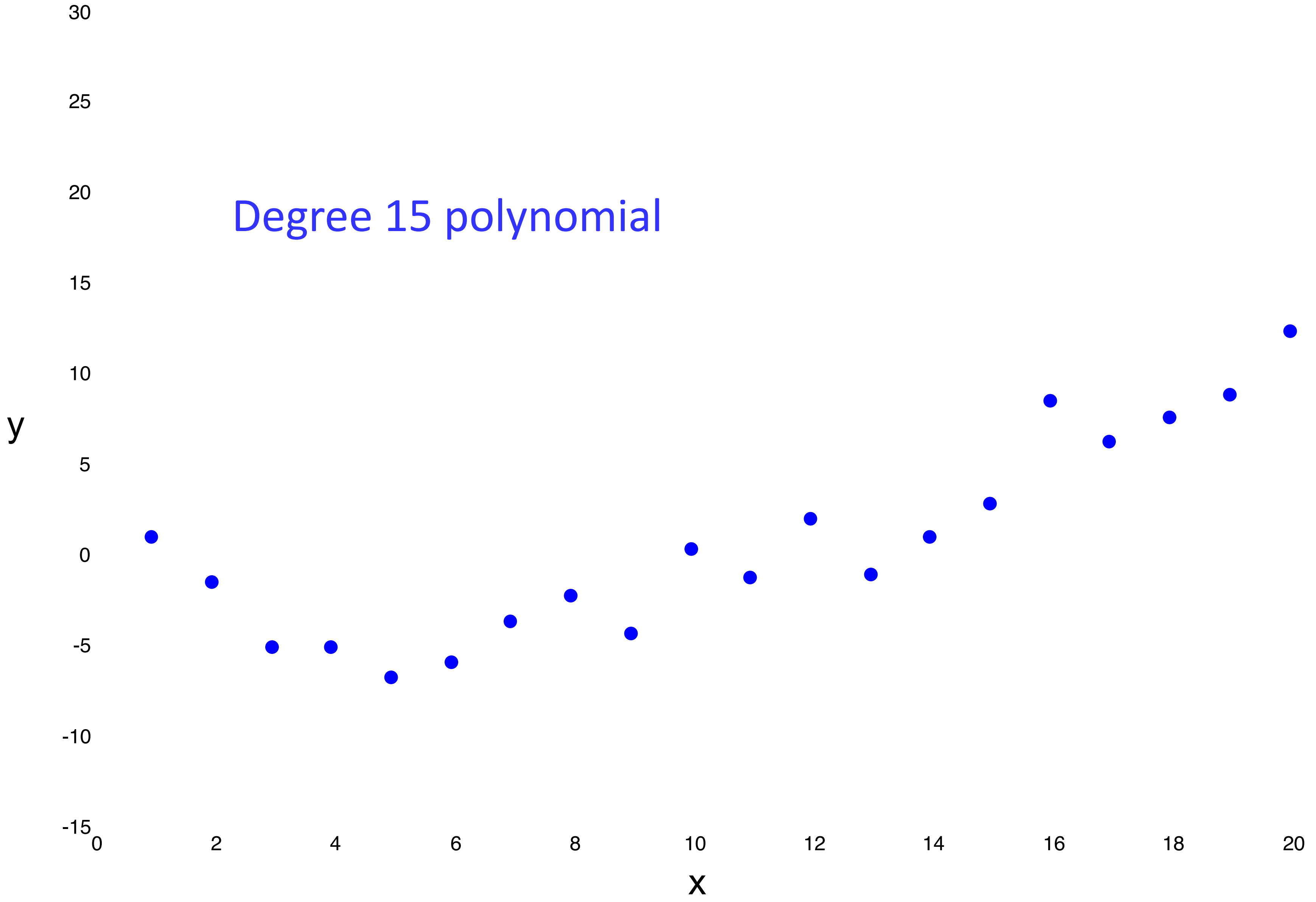
- **Maximum Likelihood Estimation**: find θ that maximizes $\mathrm{P(Data}|\theta)$
  - In practice, maximize log P instead because computation is easier
  - To solve, either take derivative and set to 0, or use numerical optimization (next lectures)

- For Naïve Bayes maximum likelihood estimates of prob. tables are:

$$P(y) = \frac{\text{\# of occurences of class } y}{\text{total \# of observations}} \qquad P(f \mid y) = \frac{\text{\# of occurences of feature } f \text{ and class } y}{\text{total \# of occurences of class } y}$$

- Need to be careful though … let's see what can go wrong..

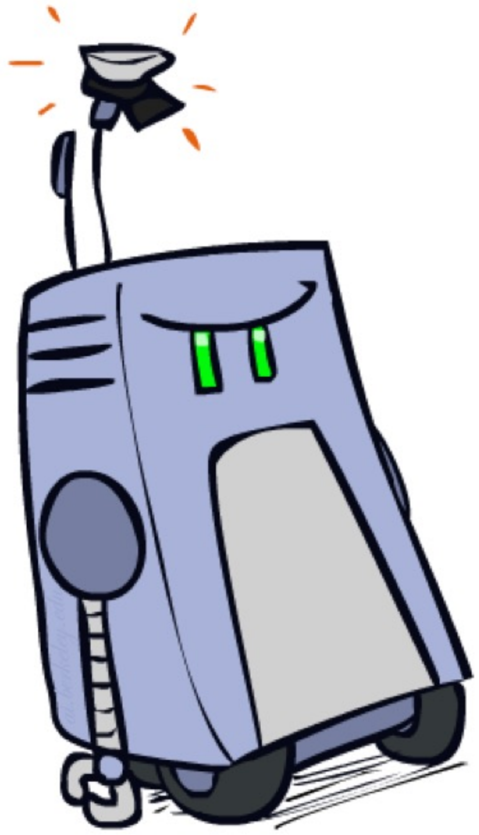
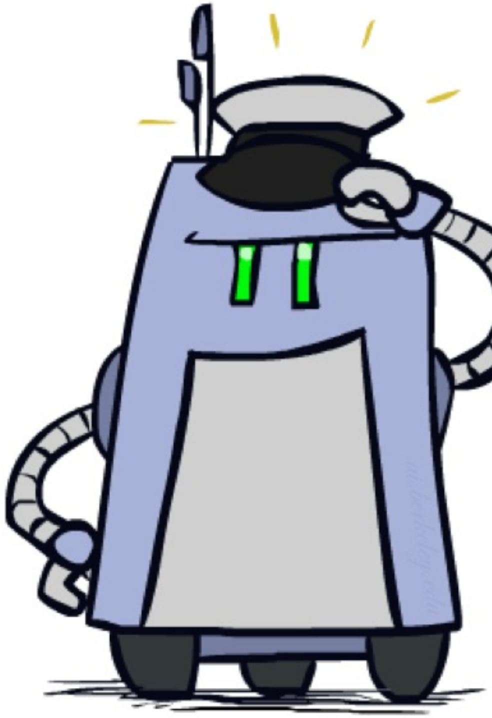
What is the best way to fit this data?



Empirical Risk Minimization

- How should we evaluate the quality of our model?
- Empirical risk minimization
  - Basic principle of machine learning
  - We want the model (classifier, etc) that does best on the true test distribution
  - Don’t know the true distribution so pick the best model on our actual training set
  - Finding “the best” model on the training set is phrased as an optimization problem
- Main worry: overfitting to the training set
  - Better with more training data (less sampling variance, training more like test)
  - Better if we limit the complexity of our hypotheses (regularization and/or small hypothesis spaces)
- Another worry: our training distribution doesn’t match true distribution

Underfitting and Overfitting


Example: Overfitting

$P(\text{features}, C = 2)$

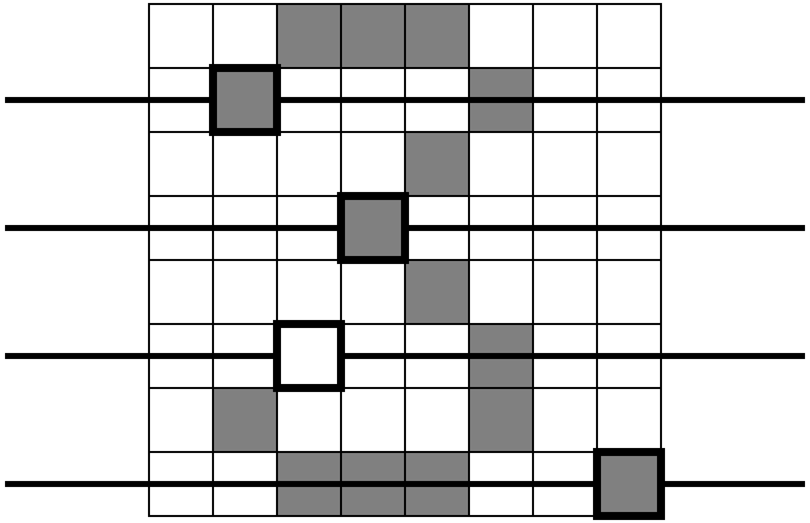
$P(C = 2) = 0.1$

$P(\text{on}|C = 2) = 0.8$

$P(\text{on}|C = 2) = 0.1$

$P(\text{off}|C = 2) = 0.1$

$P(\text{on}|C = 2) = 0.01$



$P(\text{features}, C = 3)$

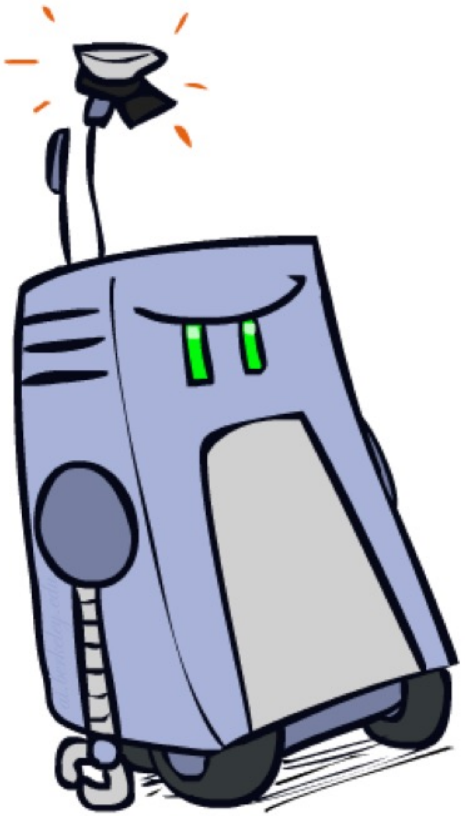
$P(C = 3) = 0.1$

$P(\text{on}|C = 3) = 0.8$

$P(\text{on}|C = 3) = 0.9$

$P(\text{off}|C = 3) = 0.7$

$P(\text{on}|C = 3) = 0.0$



2 wins!!

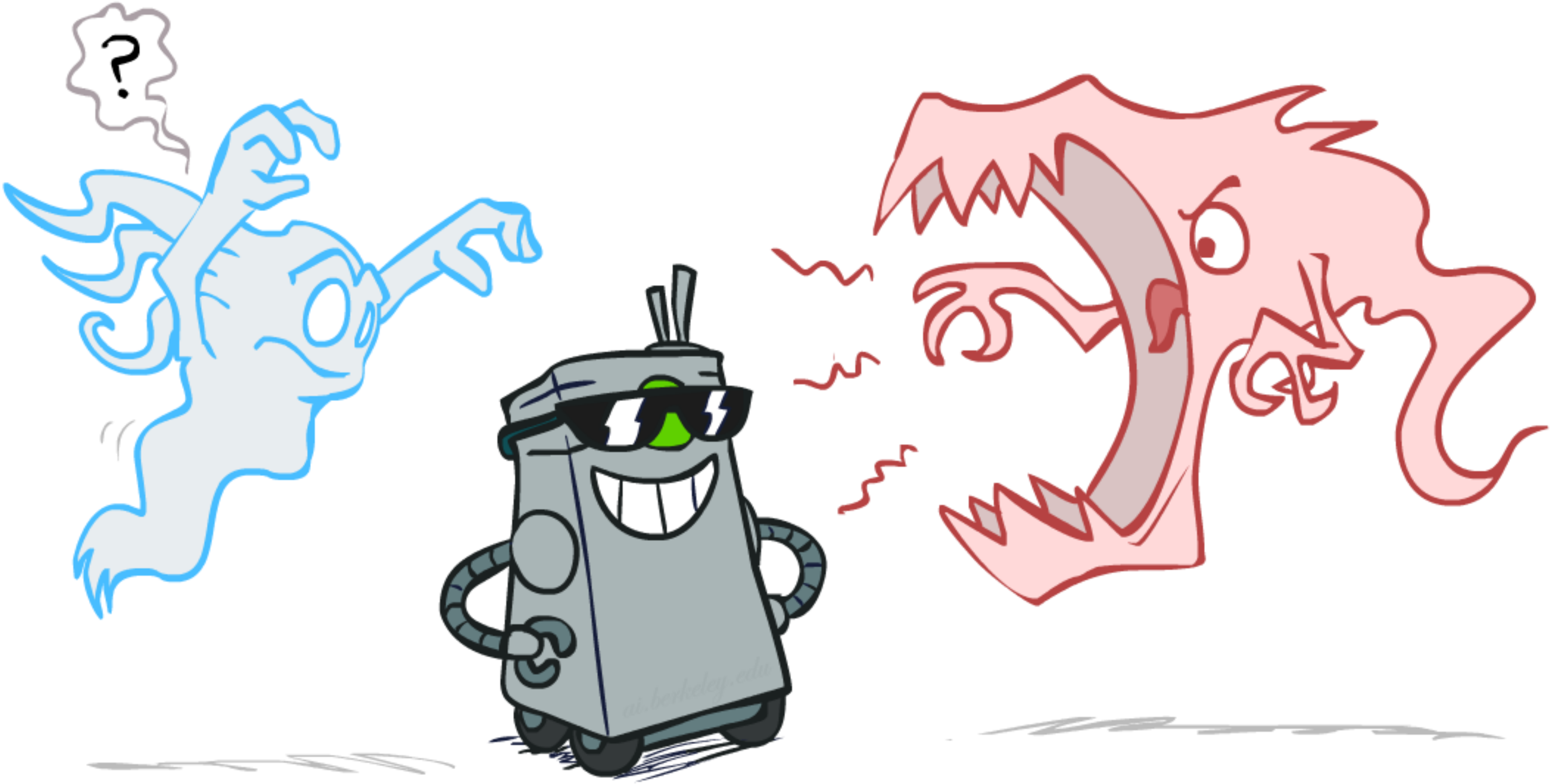
Generalization and Overfitting

- Relative frequency parameters will overfit the training data!
  - Just because we never saw a 3 with pixel (15,15) on during training doesn't mean we won't see it at test time
  - Unlikely that every occurrence of "minute" is 100% spam
  - Unlikely that every occurrence of "seriously" is 100% ham
  - What about all the words that don't occur in the training set at all?
  - In general, we can't go around giving unseen events zero probability

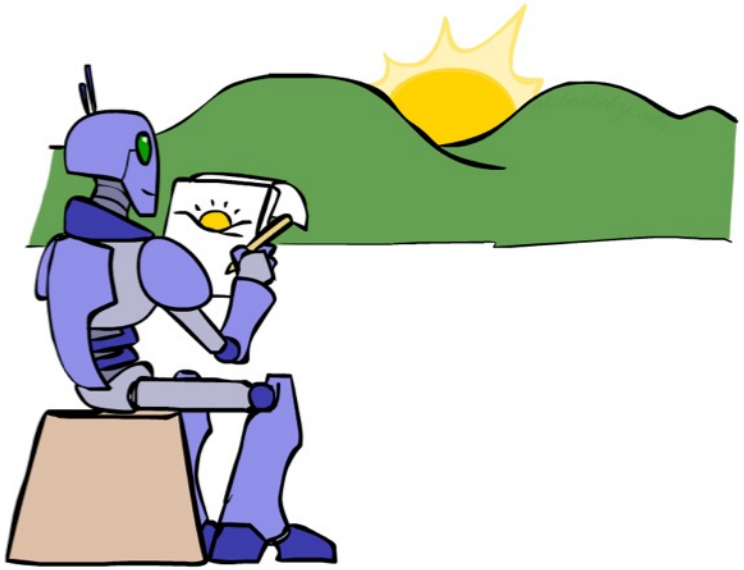
- As an extreme case, imagine using the entire email as the only feature
  - Would get the training data perfect (if deterministic labeling)
  - Wouldn't *generalize* at all
  - Just making the bag-of-words assumption gives us some generalization, but isn't enough

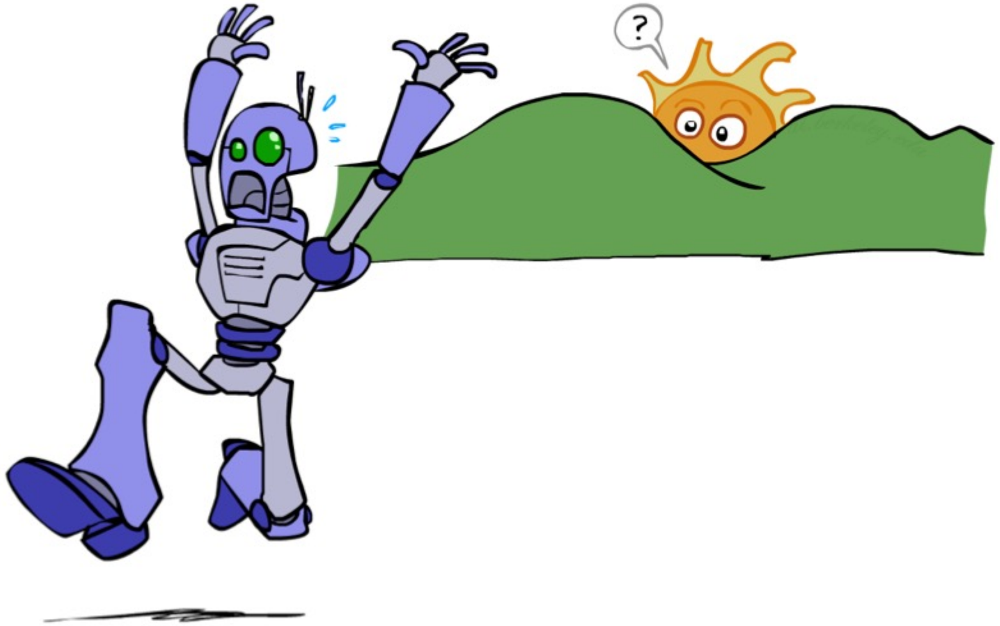
- To generalize better: we need to smooth or regularize the estimates

Smoothing



Unseen Events





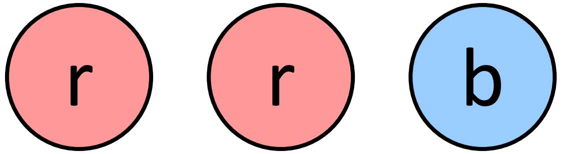
Laplace Smoothing

- Laplace's estimate:
  - Pretend you saw every outcome once more than you actually did

$$P_{LAP}(x) = \frac{c(x) + 1}{\sum_x [c(x) + 1]}$$

$$= \frac{c(x) + 1}{N + |X|}$$

  - Can derive this estimate with *Dirichlet priors* (see cs281a)



$$P_{ML}(X) =$$

$$P_{LAP}(X) =$$

Laplace Smoothing

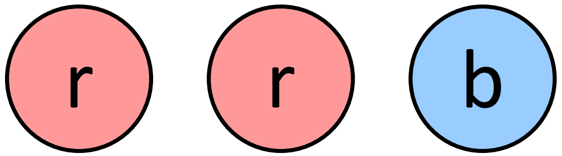
- Laplace's estimate (extended):
  - Pretend you saw every outcome k extra times

$$P_{LAP,k}(x) = \frac{c(x) + k}{N + k|X|}$$

  - What's Laplace with k = 0?
  - k is the strength of the prior

- Laplace for conditionals:
  - Smooth each condition independently:

$$P_{LAP,k}(x|y) = \frac{c(x, y) + k}{c(y) + k|X|}$$



$$P_{LAP,0}(X) =$$

$$P_{LAP,1}(X) =$$

$$P_{LAP,100}(X) =$$

Naïve Bayes: No Smoothing (Overfitting)

- *Relative* probabilities (odds ratios):

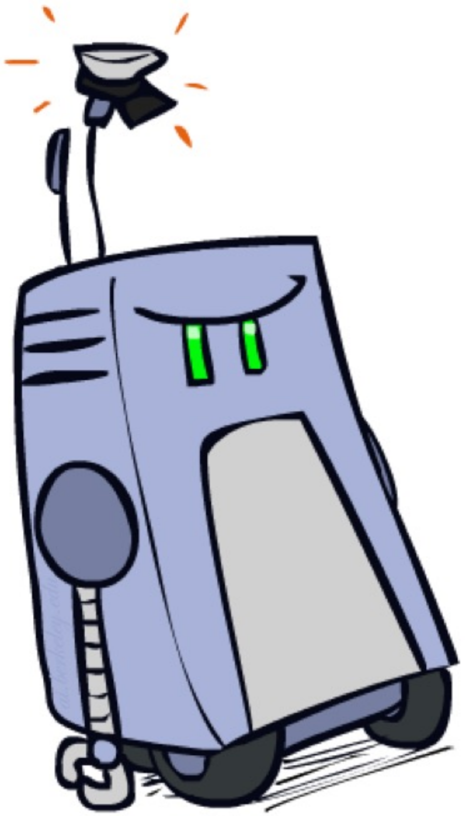
$$\frac{P(W|\text{ham})}{P(W|\text{spam})}$$

```
south-west  : inf
nation      : inf
morally     : inf
nicely      : inf
extent      : inf
seriously   : inf
...
```

$$\frac{P(W|\text{spam})}{P(W|\text{ham})}$$

```
screens     : inf
minute      : inf
guaranteed  : inf
$205.00     : inf
delivery    : inf
signature   : inf
...
```

What went wrong here?



Naïve Bayes: With Smoothing

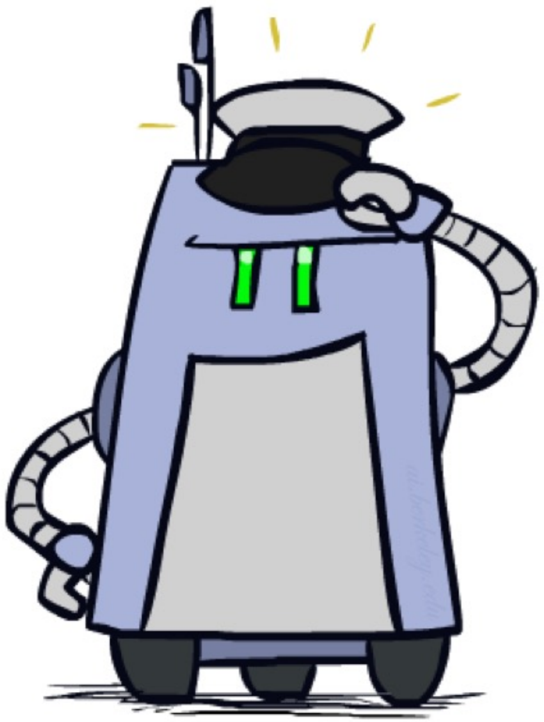
- For real classification problems, smoothing is critical
- New odds ratios:

$$\frac{P(W|\text{ham})}{P(W|\text{spam})}$$

```
helvetica  : 11.4
seems      : 10.8
group      : 10.2
ago        :  8.4
areas      :  8.3
...
```

$$\frac{P(W|\text{spam})}{P(W|\text{ham})}$$

```
verdana  : 28.8
Credit   : 28.4
ORDER    : 27.2
<FONT>   : 26.9
money    : 26.5
...
```

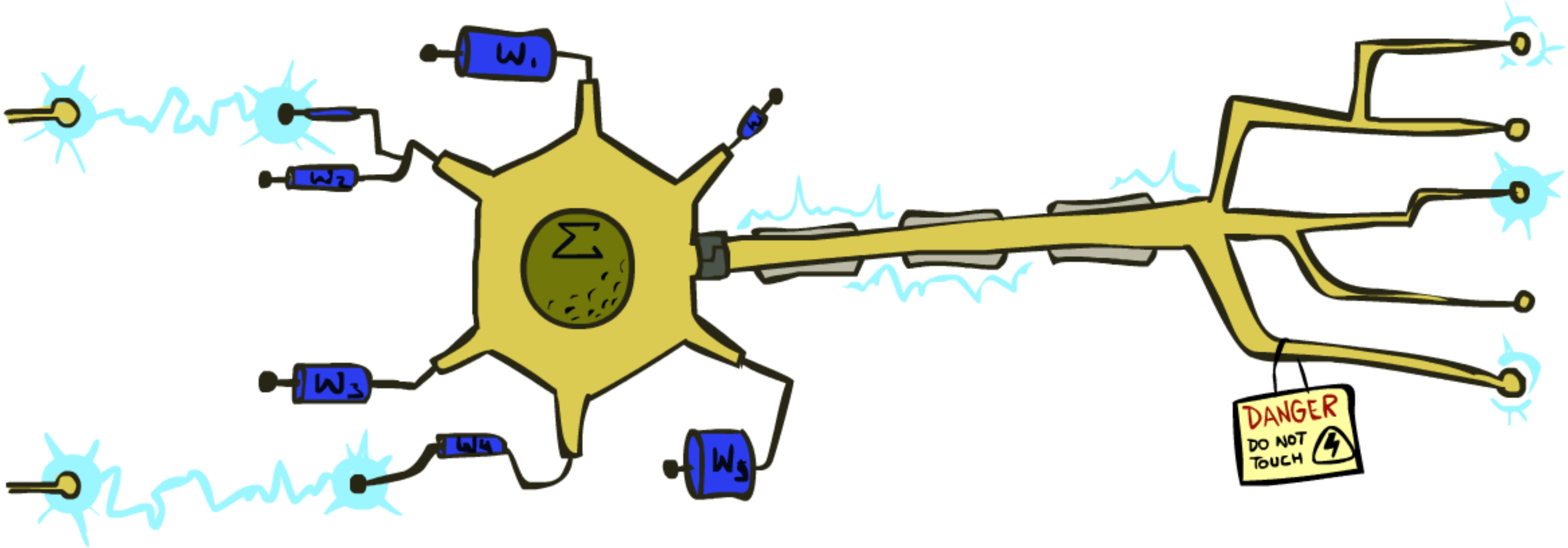


Do these make more sense?

What we did today

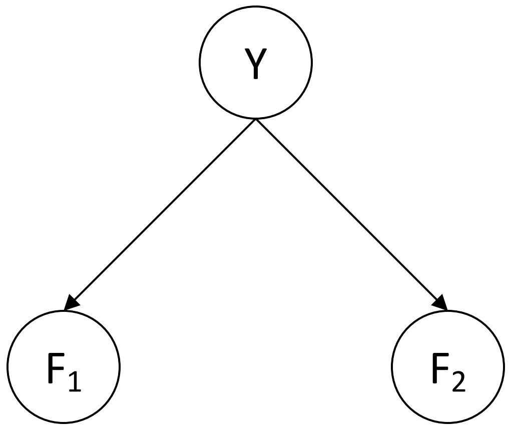
- Discussed various learning problems
  - Supervised (classification or regression), reinforcement, unsupervised
- Saw our first machine learning algorithm: Naïve Bayes
  - Model is a Bayes Net where features are independent given class label
  - Classification is just inference in Bayes Nets
  - Learning is just counting feature occurrences in training data
- Saw *Maximum Likelihood* as a principled way to estimate parameters
  - Maximize probability of the data given model parameters
  - For Naïve Bayes, we solved maximization problem analytically
- Saw that fitting training data too well can cause issues (Overfitting)

Next: Perceptrons



Example: Naïve Bayes for Spam Filter

- Step 1: Select a ML algorithm. We choose to model the problem with Naïve Bayes.
- Step 2: Choose features to use.



Y: The label (spam or ham)	
Y	P(Y)
ham	?
spam	?

F_1: A feature (do I know the sender?)		
F_1	Y	$P(F_1 \mid Y)$
yes	ham	?
no	ham	?
yes	spam	?
no	spam	?

F_2: Another feature (# of occurrences of FREE)		
F_2	Y	$P(F_2 \mid Y)$
0	ham	?
1	ham	?
2	ham	?
0	spam	?
1	spam	?
2	spam	?

Example: Naïve Bayes for Spam Filter

- Step 3: Training: Use training data to fill in the probability tables.

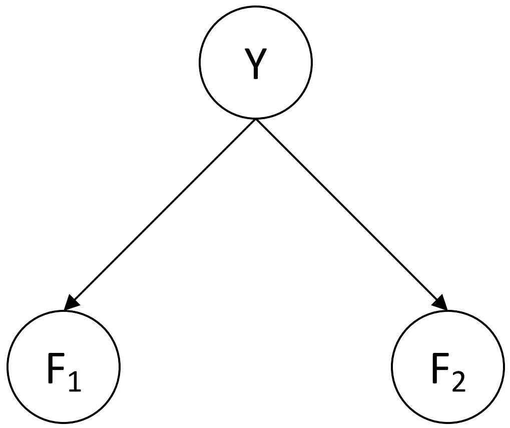
F_2: # of occurrences of FREE		
F_2	Y	$P(F_2 \mid Y)$
0	ham	0.5
1	ham	0.5
2	ham	0.0
0	spam	0.25
1	spam	0.50
2	spam	0.25

Training Data		
#	Email Text	Label
1	Attached is my portfolio.	ham
2	Are you **free** for a meeting tomorrow?	ham
3	**Free** unlimited credit cards!!!!	spam
4	Mail $10,000 check to this address	spam
5	Sign up now for 1 **free** Bitcoin	spam
6	**Free** money **free** money	spam

Row 4: $P(F_2=0 \mid Y=spam) = 0.25$ because 1 out of 4 spam emails contains “free” 0 times.
Row 5: $P(F_2=1 \mid Y=spam) = 0.50$ because 2 out of 4 spam emails contains “free” 1 time.
Row 6: $P(F_2=2 \mid Y=spam) = 0.25$ because 1 out of 4 spam emails contains “free” 2 times.

Example: Naïve Bayes for Spam Filter

- Model trained on a larger dataset:



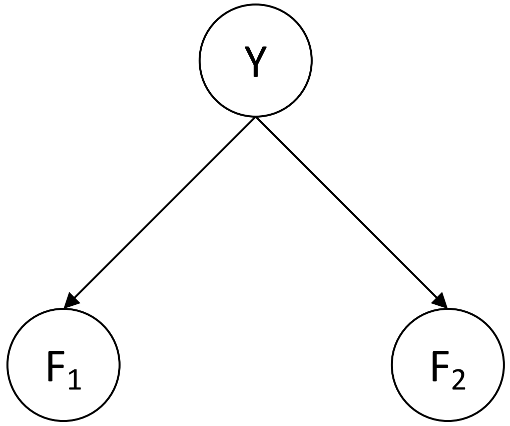
Y: The label (spam or ham)	
Y	P(Y)
ham	0.6
spam	0.4

F_1: A feature (do I know the sender?)		
F_1	Y	$P(F_1 \mid Y)$
yes	ham	0.7
no	ham	0.3
yes	spam	0.1
no	spam	0.9

F_2: Another feature (# of occurrences of FREE)		
F_2	Y	$P(F_2 \mid Y)$
0	ham	0.85
1	ham	0.07
2	ham	0.08
0	spam	0.75
1	spam	0.12
2	spam	0.13

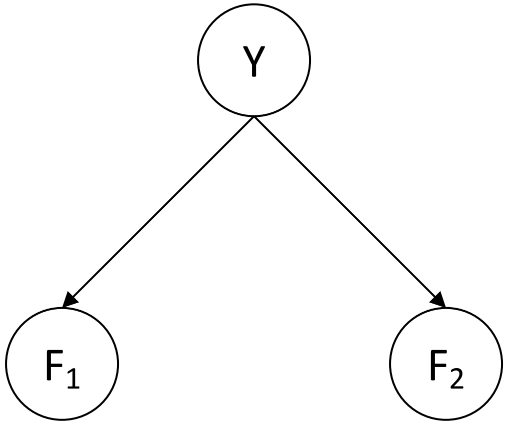
Example: Naïve Bayes for Spam Filter

- Step 4: Classification
- Suppose you want to label this email from a known sender:
  "**Free** food in Soda 430 today"
- Step 4.1: Feature extraction:
  - F_1 = yes, known sender
  - F_2 = 1 occurrence of "free"



Example: Naïve Bayes for Spam Filter

- Step 4.2: Inference
- Instantiate features (evidence):
  - F_1 = yes
  - F_2 = 1
- Compute joint probabilities:
  - $P(Y = \text{spam}, F_1 = \text{yes}, F_2 = 1) = P(Y = \text{spam})\ P(F_1 = \text{yes} \mid \text{spam})\ P(F_2 = 1 \mid \text{spam})$
    $= 0.4 * 0.1 * 0.12 = 0.0048$
  - $P(Y = \text{ham}, F_1 = \text{yes}, F_2 = 1) = P(Y = \text{ham})\ P(F_1 = \text{yes} \mid \text{ham})\ P(F_2 = 1 \mid \text{ham})$
    $= 0.6 * 0.7 * 0.07 = 0.0294$
- Normalize:
  - $P(Y = \text{spam} \mid F_1 = \text{yes}, F_2 = 1) = 0.0048 / (0.0048+0.0294) = 0.14$
  - $P(Y = \text{ham} \mid F_1 = \text{yes}, F_2 = 1) = 0.0294 / (0.0048+0.0294) = 0.86$
- Classification result:
  - 14% chance the email is spam. 86% chance it's ham.
  - Or, if you don't need probabilities, note that 0.0294 > 0.0048 and guess ham.



Y: The label (spam or ham)	
Y	P(Y)
ham	0.6
spam	0.4

F_1: do I know the sender?		
F_1	Y	$P(F_1 \mid Y)$
yes	ham	0.7
no	ham	0.3
yes	spam	0.1
no	spam	0.9

F_2: # of occurrences of FREE		
F_2	Y	$P(F_2 \mid Y)$
0	ham	0.85
1	ham	0.07
2	ham	0.08
0	spam	0.75
1	spam	0.12
2	spam	0.13