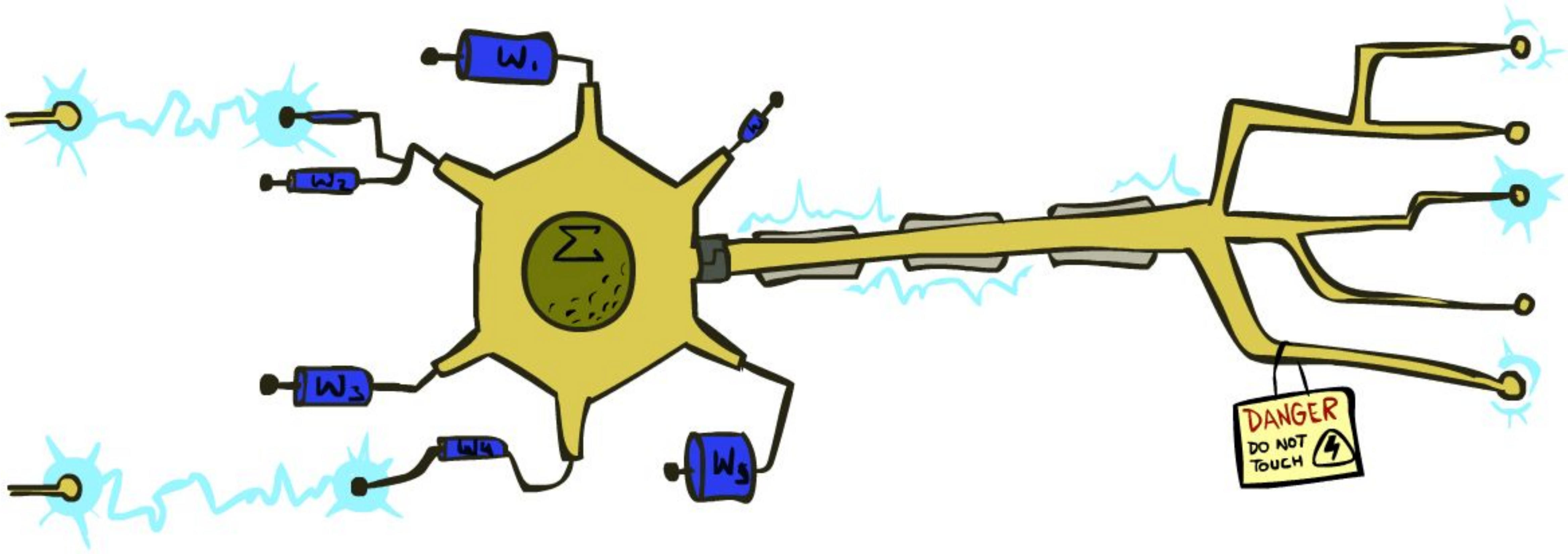
CS 188: Artificial Intelligence

Naïve Bayes and Perceptrons



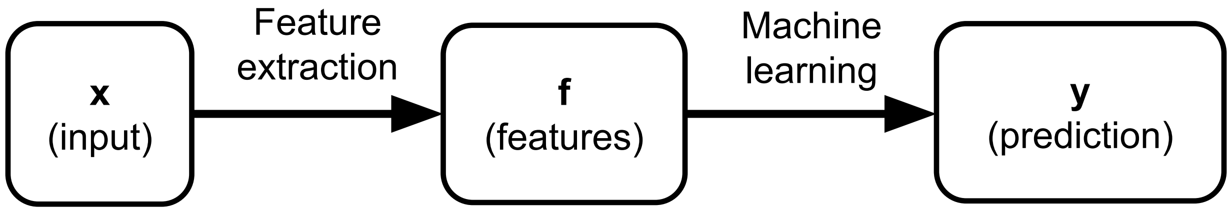
[These slides were created by Dan Klein, Pieter Abbeel, Anca Dragan, Sergey Levine. All CS188 materials are at http://ai.berkeley.edu.]

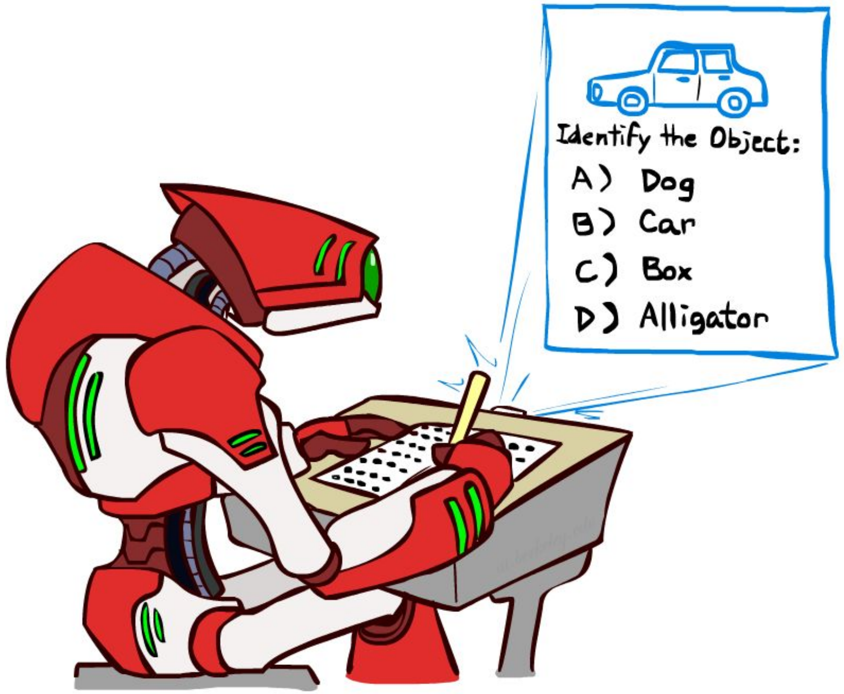
Today's Topics

- Naïve Bayes
  - Review: making predictions / inference
  - Learning parameters
- Machine Learning Workflow
- Perceptrons
  - Making predictions
  - Learning parameters

Last Time

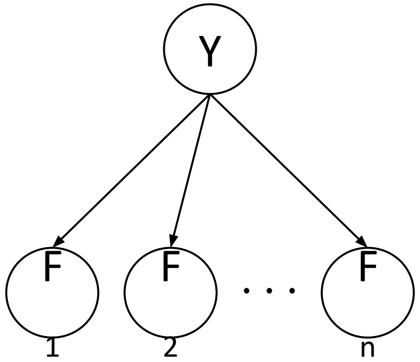
- **Classification:** given inputs x, predict labels (classes) y
  - Convert input x into a collection of *features* $f_1, \ldots, f_n$





Last Time

- *Naïve Bayes* model: $P(Y, F_1, \dots, F_n) = P(Y) \prod_i P(F_i|Y)$
  - Features and label are random variables
  - Input features $F_1, \dots, F_n$ are conditionally independent given label Y
  - Parameters θ: probability tables $P(Y), P(F_1|Y), \dots, P(F_n|Y)$



- Classification is inference in a Bayes Net:
  - Inference by enumeration
  - Given features $f_1, \dots, f_n$ probability over class labels is:

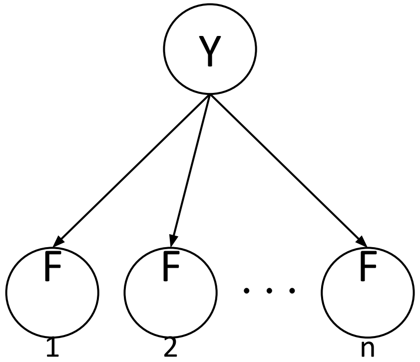
$$P(Y|f_1, \dots, f_n) \propto P(Y, f_1, \dots, f_n) = P(Y) \prod_i P(f_i|Y)$$

Enumerate over every label y:

$$\begin{bmatrix} P(y_1) \prod_i P(f_i|y_1) \\ P(y_2) \prod_i P(f_i|y_2) \\ \vdots \\ P(y_k) \prod_i P(f_i|y_k) \end{bmatrix} \xrightarrow{\text{Normalize}} \begin{bmatrix} P(y_1|\, f_1 \dots f_n) \\ P(y_2|\, f_1 \dots f_n) \\ \vdots \\ P(y_k|\, f_1 \dots f_n) \end{bmatrix}$$

Last Time

- *Naïve Bayes* model: $P(Y, F_1, \ldots, F_n) = P(Y) \prod_i P(F_i|Y)$
  - Features and label are random variables
  - Input features $F_1, \ldots, F_n$ are conditionally independent given label Y
  - Parameters θ: probability tables $P(Y), P(F_1|Y), \ldots, P(F_n|Y)$



- Learn parameters by counting:
  - $P(\text{observing } x) = \dfrac{\#\text{ of times } x \text{ occured}}{\text{total \# of events}}$

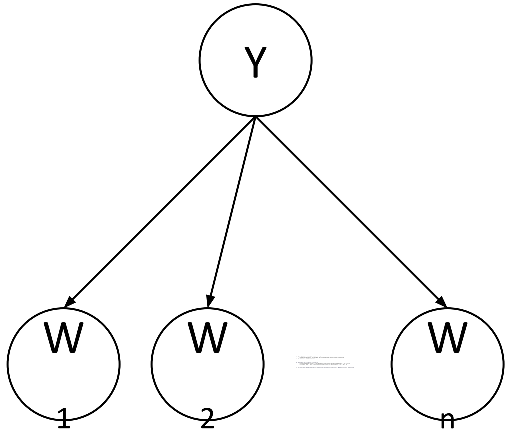
    For example: r r b $P(\text{red}) = \dfrac{2}{3}$

  - Comes from *Maximum Likelihood* estimation: find θ that maximizes $\mathrm{P}(\text{observations}|\theta)$
    - $\theta = \underset{\theta}{\operatorname{argmax}} P(\text{observations}|\theta)$
    - Take derivative and set to 0
    - In practice, maximize log P instead because derivatives are easier

  - In general for Naïve Bayes maximum likelihood estimates of probability tables are:

$$P(y) = \frac{\#\text{ of occurences of class } y}{\text{total \# of observations}} \qquad P(f \mid y) = \frac{\#\text{ of occurences of feature } f \text{ and class } y}{\text{total \# of occurences of class } y}$$

Example: Naïve Bayes for Spam Filtering



- Predict if an email is spam or not
  - Features: W_i is the i'th word in the email (domain: words in the dictionary)
  - Labels: $Y \in \{spam, ham\}$
- Estimated parameters:

$P(Y)$

Y	spam	ham
P(Y)	2/6	4/6

$P(W|Y)$

W	"now"	"the"	"buy"
P(W | Y=spam)	2/6	1/6	3/6
P(W | Y=ham)	1/6	4/6	1/6

- What is $P(Y|\text{"buy"}, \text{"now"})$?
  - $P(spam|\text{"buy"}, \text{"now"}) \propto P(spam)P(\text{"buy"}|spam)P(\text{"now"}|spam) = \frac{2}{6} \cdot \frac{3}{6} \cdot \frac{2}{6} = \frac{12}{6^3}$
  - $P(ham|\text{"buy"}, \text{"now"}) \propto P(ham)P(\text{"buy"}|ham)P(\text{"now"}|ham) = \frac{4}{6} \cdot \frac{1}{6} \cdot \frac{1}{6} = \frac{4}{6^3}$
  - Renormalize:

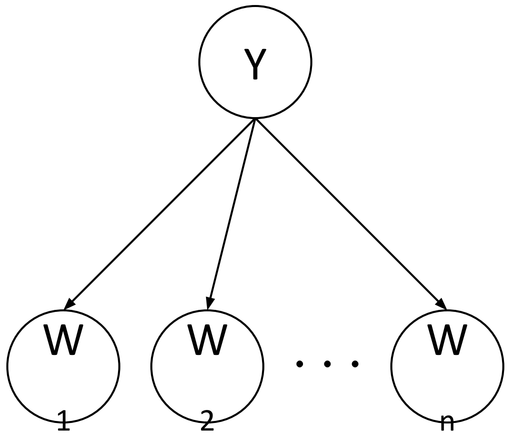
$P(Y|\text{"buy"}, \text{"now"})$

spam	ham
12/16	4/16

- Prediction: pick label with highest probability, so predict ***spam*** for text *"buy now"*

Example: Naïve Bayes for Spam Filtering

- Predict if an email is spam or not
  - Features: W_i is the i'th word in the email (domain: words in the dictionary)
  - Labels: $Y \in \{spam, ham\}$
- Estimated parameters:



$P(Y)$

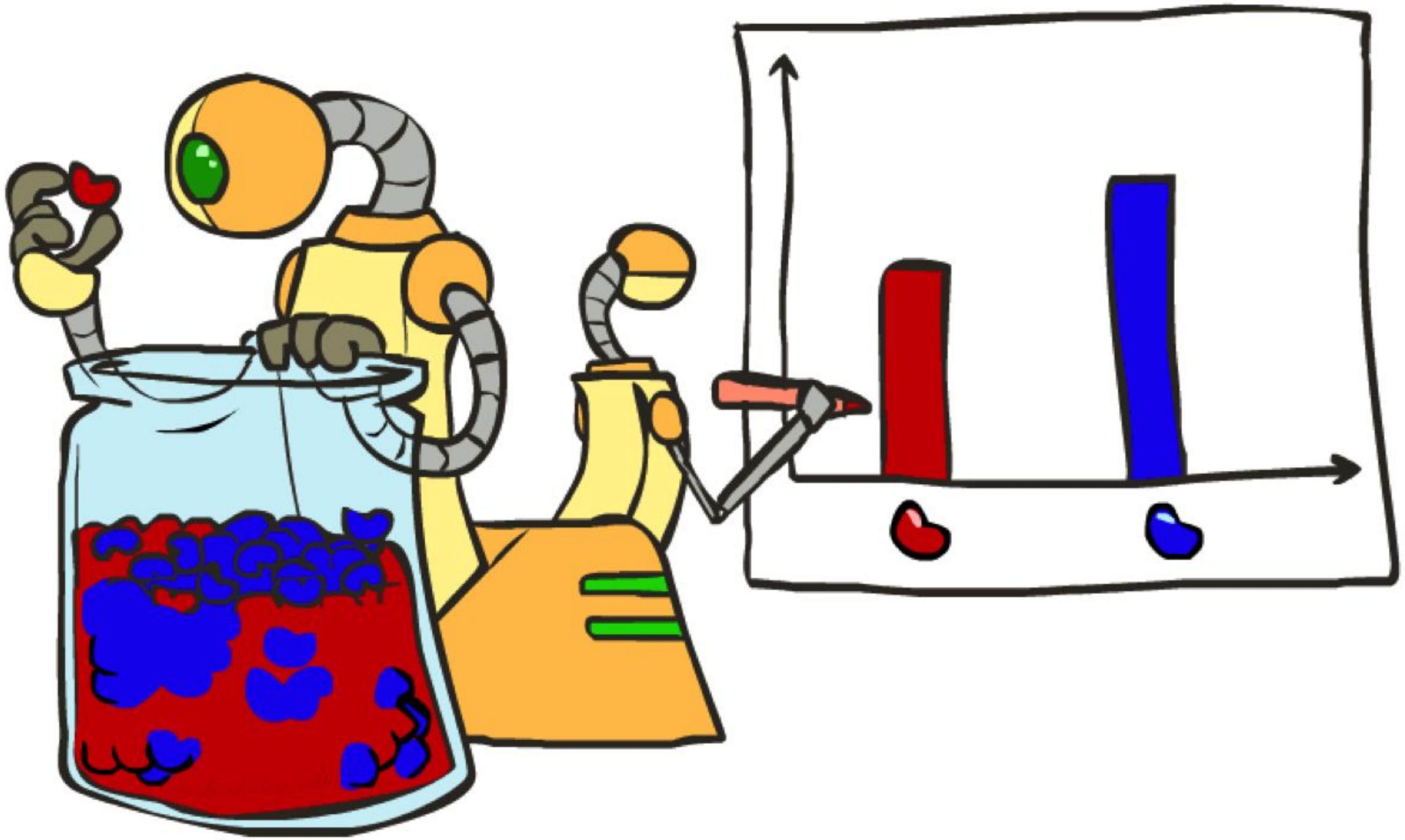
Y	spam	ham
P(Y)	2/6	4/6

$P(W|Y)$

W	"now"	"the"	"buy"
P(W | Y=spam)	2/6	1/6	3/6
P(W | Y=ham)	1/6	4/6	1/6

- How can we estimate these model parameters?

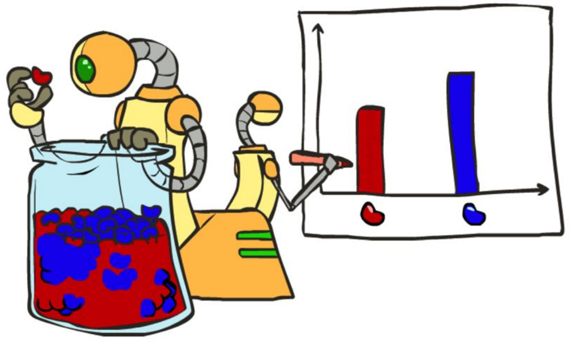
Parameter Estimation



Parameter Estimation with Maximum Likelihood

- Estimating the distribution of a random variable
- Use training data (learning!)
  - For each outcome x, look at the ***empirical rate*** of that value:

$$P_{\text{ML}}(x) = \frac{\text{count}(x)}{\text{total samples}}$$



- Example: probability of x=red given training data r , r , b

$$P_{\text{ML}}(\text{r}) = 2/3$$

X	red	blue

- This estimate maximizes the ***likelihood of the data*** for parametric model:

$$L(\theta) = P(red, red, blue \mid \theta) = P_\theta(red) \cdot P_\theta(red) \cdot P_\theta(blue) = \theta^2 \cdot (1-\theta)$$

  - Why?

$$\hat{\theta} = \underset{\theta}{\operatorname{argmax}}\, L(\theta) = \underset{\theta}{\operatorname{argmax}}\, \boxed{\log L(\theta)}$$

$$2\log(\theta) + \log(1-\theta)$$

Take derivative and set to 0:

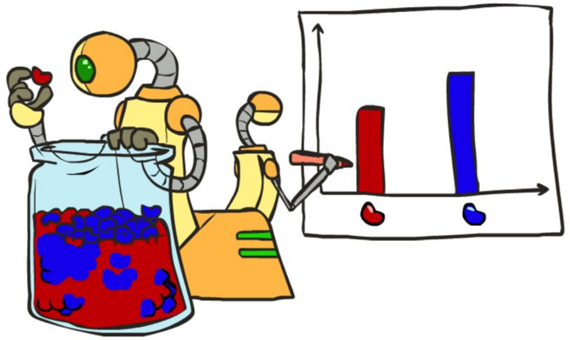
$$\frac{\partial \log L(\theta)}{\partial \theta} = \frac{2}{\theta} + \frac{1}{1-\theta} \cdot -1 = 2(1-\theta) - \theta = 2 - 3\theta = 0$$

$$\boxed{\rightarrow \hat{\theta} = \frac{2}{3}}$$

Parameter Estimation with Maximum Likelihood (General Case)

- **Model:**

X	red	blue



- **Data:** draw N balls, N_r come up red and N_b come up blue
  - Dataset $D = \{x_1, \dots, x_N\}$
  - Ball draws are independent and identically distributed (i.i.d):

$$P(D|\theta) = \prod_i P(x_i|\theta) = \prod_i P_\theta(x_i) = \theta^{N_r} \cdot (1-\theta)^{N_b}$$

- **Maximum Likelihood Estimation**: find θ that maximizes $\mathrm{P}(\mathrm{D}|\theta)$:

$$\hat{\theta} = \underset{\theta}{\operatorname{argmax}}\, P(D|\theta) = \underset{\theta}{\operatorname{argmax}} \boxed{\log P(D|\theta)} \rightarrow N_r \log(\theta) + N_b \log(1-\theta)$$

Take derivative and set to 0:

$$\frac{\partial \log P(D|\theta)}{\partial \theta} = \frac{N_r}{\theta} - \frac{N_b}{1-\theta} = 0$$

$$\boxed{\rightarrow \hat{\theta} = \frac{N_r}{N_r + N_b} = \frac{\text{\# of red balls}}{\text{total \# of balls}}}$$

Parameter Estimation with Maximum Likelihood (General Case)

- **Maximum Likelihood Estimation**: find θ that maximizes $\mathrm{P}(\mathrm{D}|\theta)$:

$$\hat{\theta} = \underset{\theta}{\operatorname{argmax}}\, P(D|\theta) = \underset{\theta}{\operatorname{argmax}}\, \boxed{\log P(D|\theta)} \rightarrow N_r \log(\theta) + N_b \log(1-\theta)$$

Take derivative and set to 0:

$$\frac{\partial}{\partial\theta}\log P(D|\theta) = \frac{\partial}{\partial\theta}[N_r \log(\theta) + N_b \log(1-\theta)]$$

$$= N_r \frac{\partial}{\partial\theta}[\log(\theta)] + N_b \frac{\partial}{\partial\theta}[\log(1-\theta)]$$

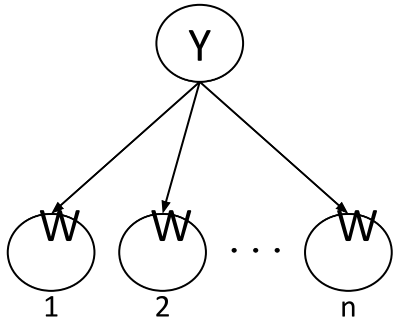
$$= N_r \frac{1}{\theta} + N_b \frac{1}{1-\theta} \cdot -1$$

$$= N_r(1-\theta) - N_b\theta$$

$$= N_r - \theta(N_r + N_b) = 0$$

$$\rightarrow \hat{\theta} = \frac{N_r}{N_r + N_b}$$

Example: Spam Filtering Parameter Estimation



- Predict if an email is spam or not
  - Features: W_i is the i'th word in the email (domain: words in the dictionary)
  - Labels: $Y \in \{spam, ham\}$
- Naïve Bayes **model** parameters:

$P_\theta(Y)$

Y	spam	ham
P(Y)		

$P_\theta(W|Y)$

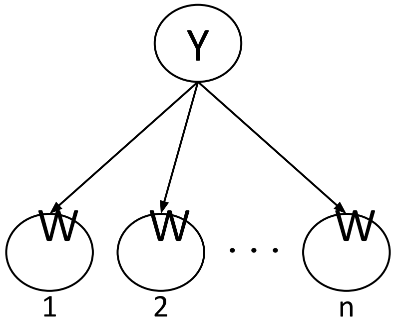
W	"now"	"the"	…
P(W | Y=spam)			…
P(W | Y=ham)			…

- **Dataset:** M emails where k'th email contains words $\{w_1, \ldots, w_{N_k}\}$ and label y_k
  - Emails are independent and identically distributed:

$$P(D|\theta) = \prod_k^M P(y_k, w_1, \ldots, w_{N_k}) = \prod_k^M P(y_k) \prod_i^{N_k} P(w_i|y_k)$$

- **Maximum Likelihood Estimation**: find θ that maximizes $\mathrm{P}(\mathrm{D}|\theta)$

Example: Spam Filtering Parameter Estimation



- Predict if an email is spam or not
  - Features: W_i is the i'th word in the email (domain: words in the dictionary)
  - Labels: $Y \in \{spam, ham\}$
- Naïve Bayes **model** parameters:

$P_\theta(Y)$

Y	spam	ham
P(Y)		

$P_\theta(W|Y)$

W	"now"	"the"	…
P(W | Y=spam)			…
P(W | Y=ham)			…

- **Maximum Likelihood Estimation Result:**
  - $$\theta_{spam} = \frac{\text{\# of spam emails}}{\text{total \# of emails}}$$
  - $$\theta_{w,spam} = \frac{\text{\# of occurences of word w in spam emails}}{\text{total \# of spam emails}}$$
  - $$\theta_{w,ham} = \frac{\text{\# of occurences of word w in ham emails}}{\text{total \# of ham emails}}$$

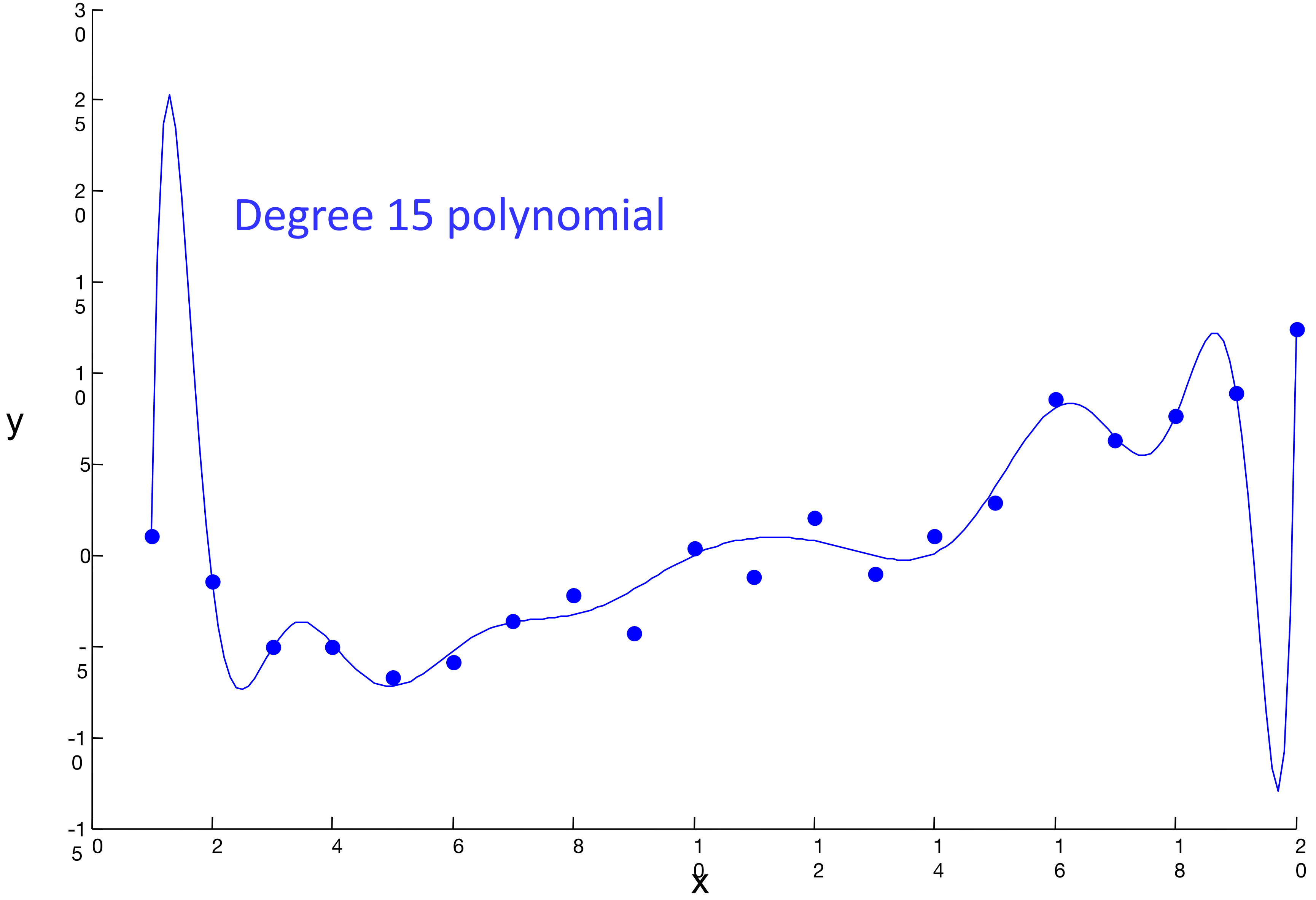
More generally: parameters for model $P_\theta(F|Y)$ are:

$$\theta_{f,y} = \frac{\text{\# of occurences of feature f and class y}}{\text{total \# of occurences of class y}}$$

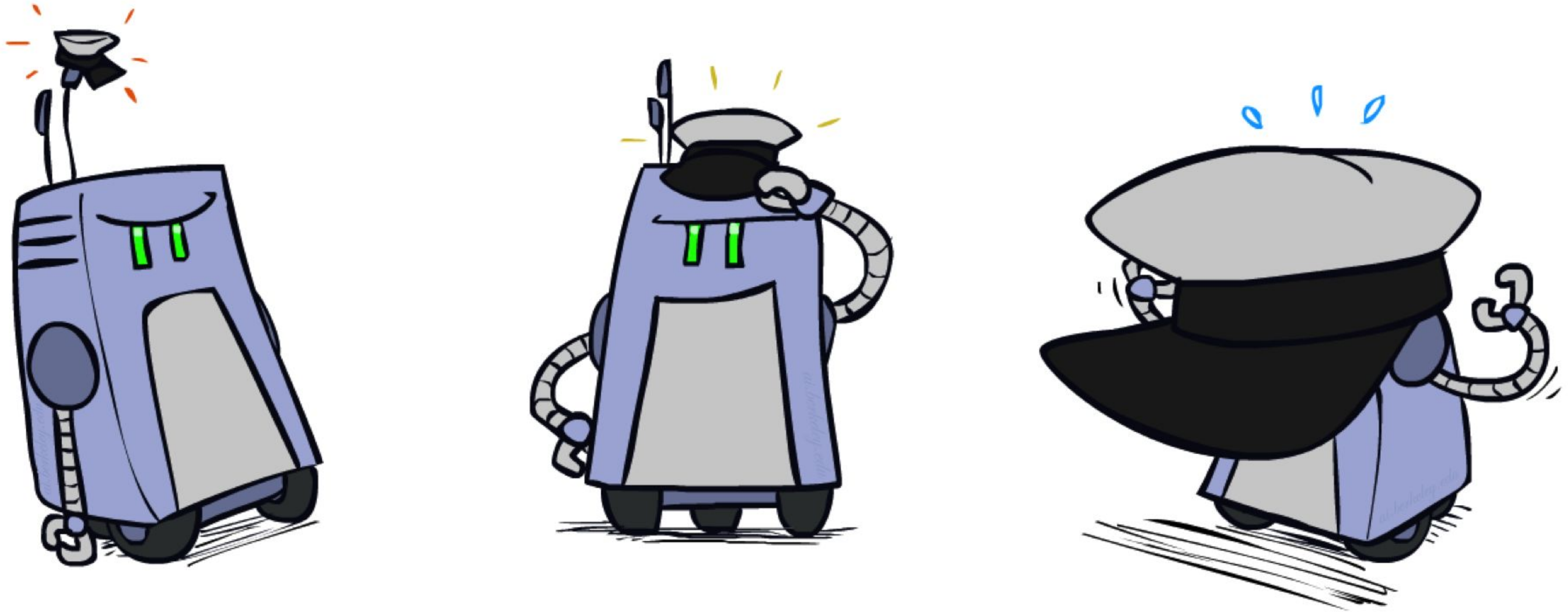
Parameter Estimation with Maximum Likelihood

- How do we estimate the conditional probability tables?
  - Maximum Likelihood, which corresponds to counting
- Need to be careful though … let’s see what can go wrong..

What is the best way to fit this data?



Underfitting and Overfitting



Example: Overfitting

$P(\text{features}, C = 2)$

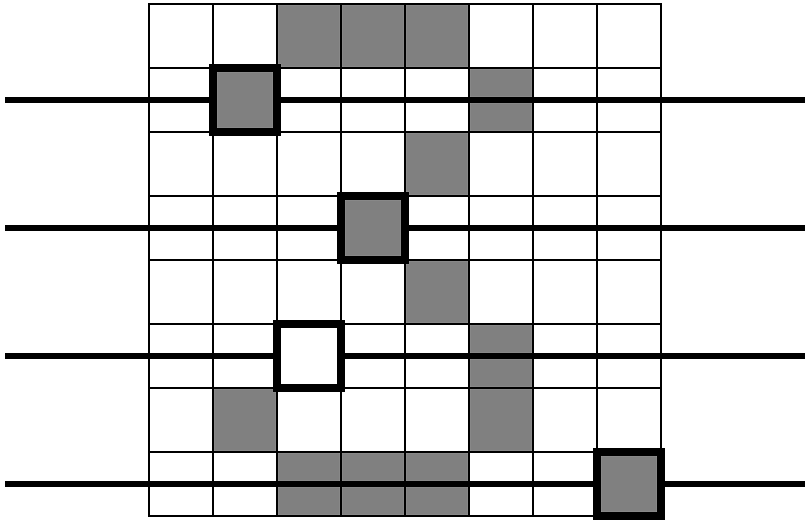
$P(C = 2) = 0.1$

$P(\text{on}|C = 2) = 0.8$

$P(\text{on}|C = 2) = 0.1$

$P(\text{off}|C = 2) = 0.1$

$P(\text{on}|C = 2) = 0.01$



$P(\text{features}, C = 3)$

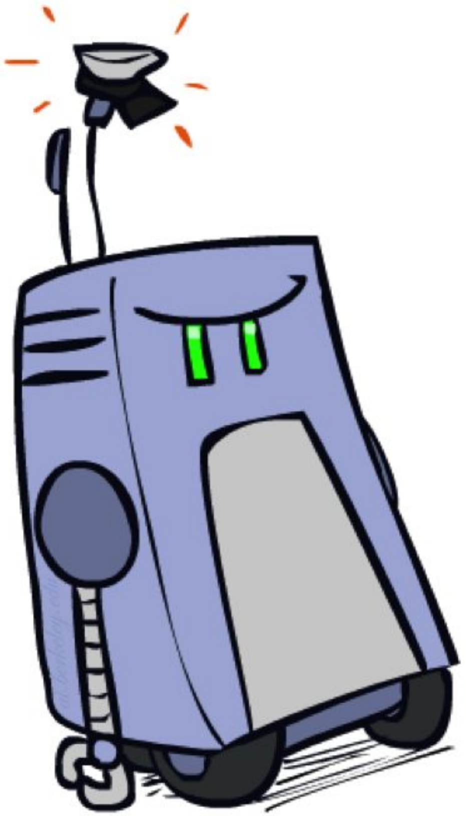
$P(C = 3) = 0.1$

$P(\text{on}|C = 3) = 0.8$

$P(\text{on}|C = 3) = 0.9$

$P(\text{off}|C = 3) = 0.7$

$P(\text{on}|C = 3) = 0.0$



2 wins!!

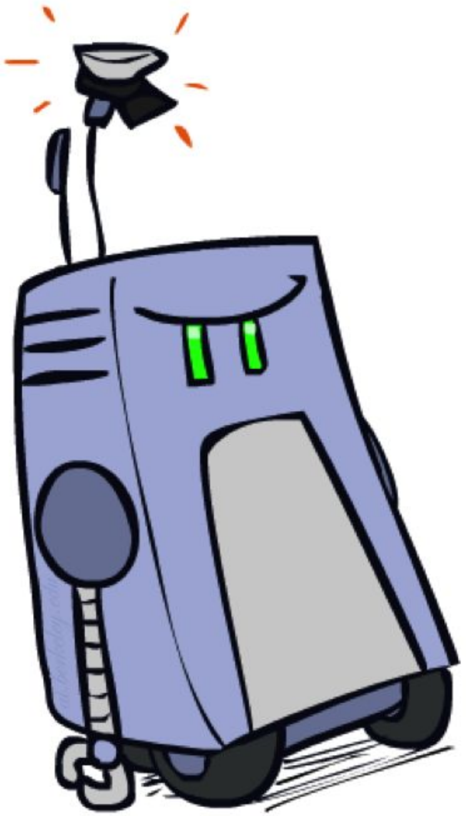
Example: Overfitting

- *relative* probabilities (odds ratios):

$$\frac{P(W|\text{ham})}{P(W|\text{spam})} \qquad \frac{P(W|\text{spam})}{P(W|\text{ham})}$$

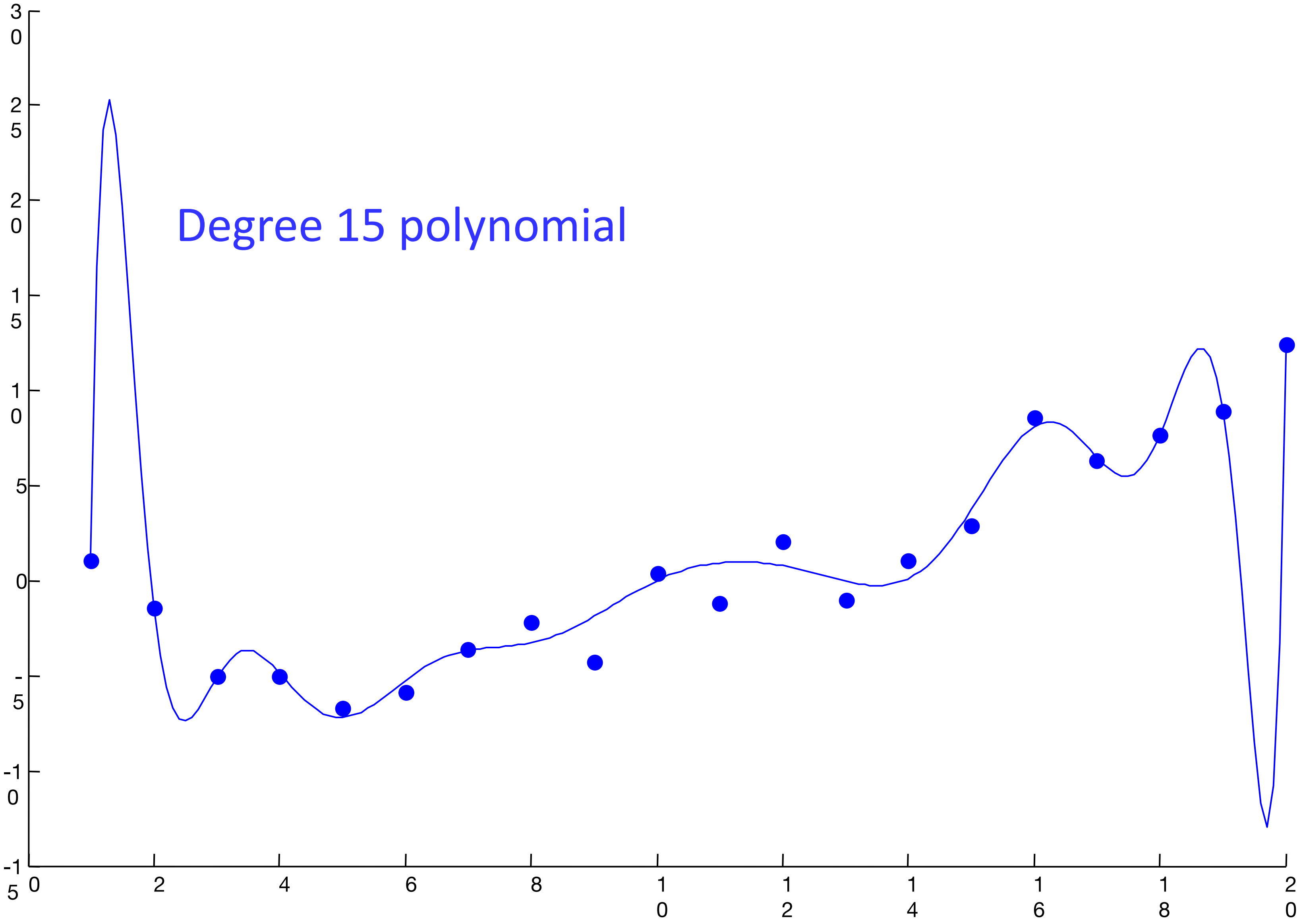
```
south-west  : inf
nation      : inf
morally     : inf
nicely      : inf
extent      : inf
seriously   : inf
...
```

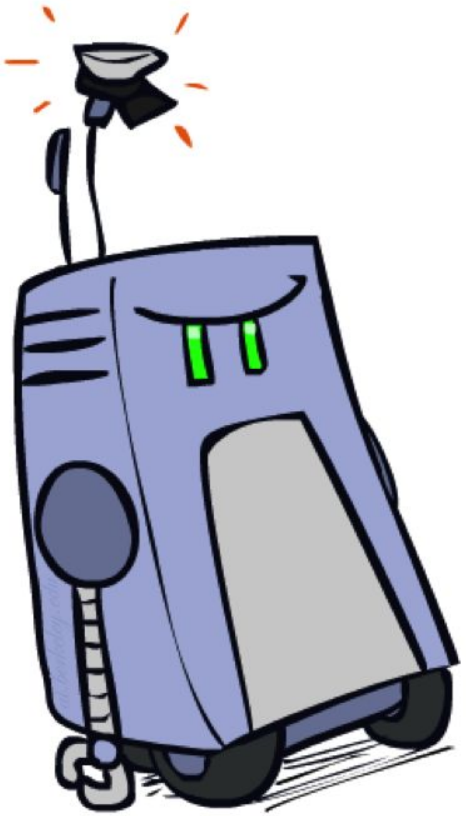
```
screens     : inf
minute      : inf
guaranteed  : inf
$205.00     : inf
delivery    : inf
signature   : inf
...
```



What went wrong here?

Overfitting





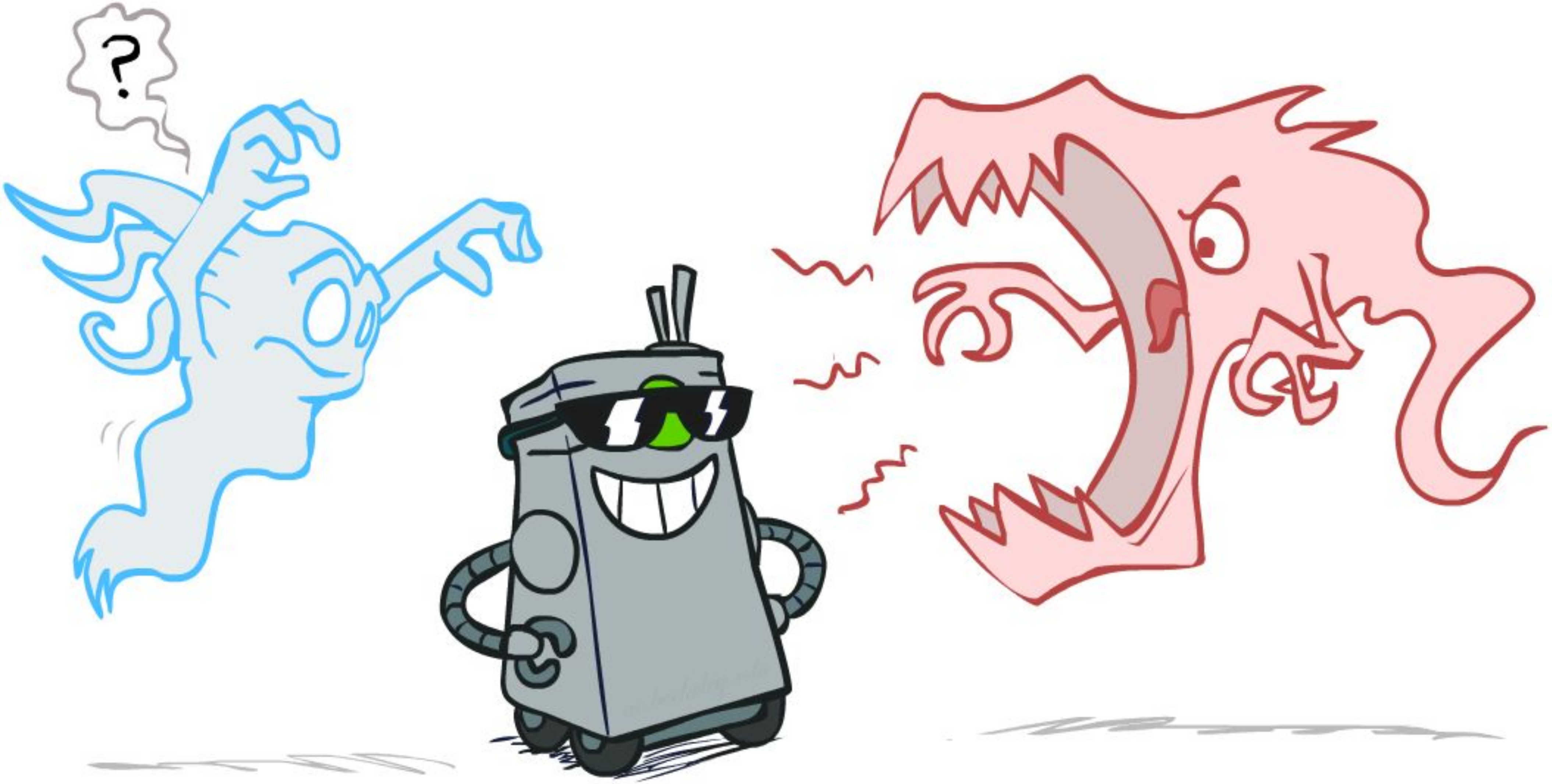
Generalization and Overfitting

- Relative frequency parameters will overfit the training data!
  - Just because we never saw a 3 with pixel (15,15) on during training doesn't mean we won't see it at test time
  - Unlikely that every occurrence of "minute" is 100% spam
  - Unlikely that every occurrence of "seriously" is 100% ham
  - What about all the words that don't occur in the training set at all?
  - In general, we can't go around giving unseen events zero probability

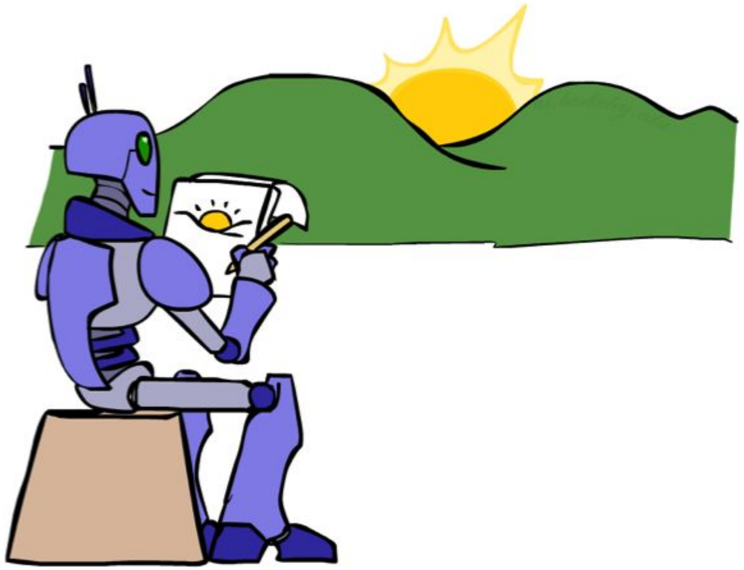
- As an extreme case, imagine using the entire email as the only feature
  - Would get the training data perfect (if deterministic labeling)
  - Wouldn't *generalize* at all
  - Just making the bag-of-words assumption gives us some generalization, but isn't enough

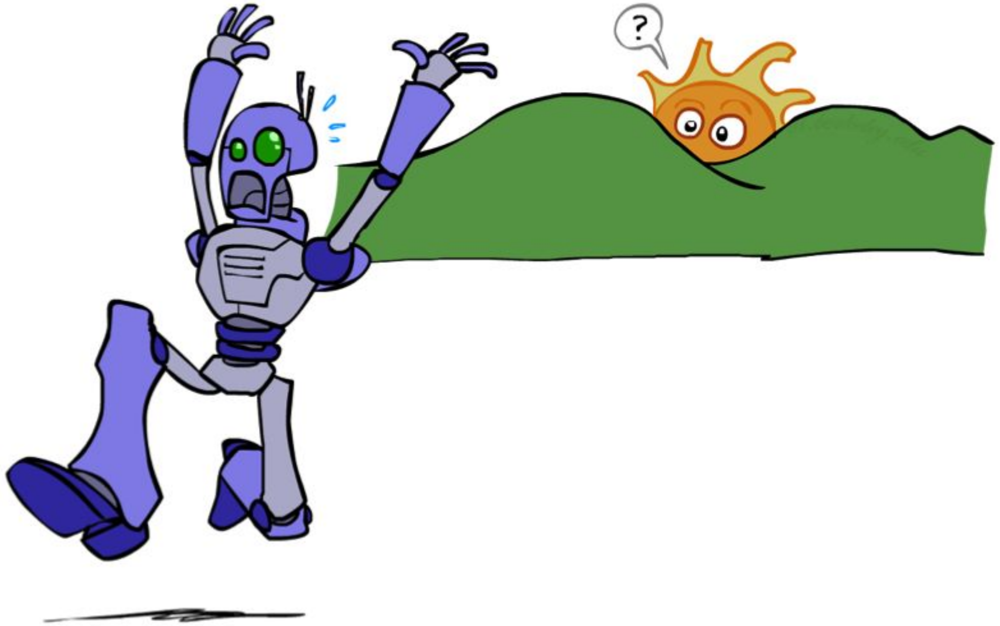
- To generalize better: we need to smooth or regularize the estimates

Smoothing



Unseen Events



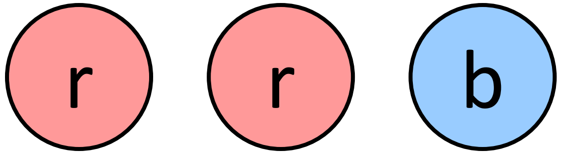


Laplace Smoothing

- Laplace's estimate:
  - Pretend you saw every outcome once more than you actually did

$$P_{LAP}(x) = \frac{c(x) + 1}{\sum_x [c(x) + 1]}$$

$$= \frac{c(x) + 1}{N + |X|}$$

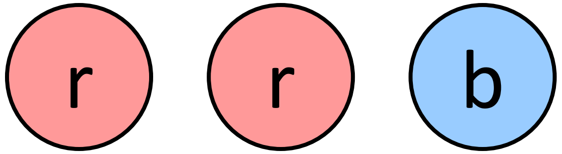


$$P_{ML}(X) =$$

$$P_{LAP}(X) =$$

Laplace Smoothing

- Laplace's estimate (extended):
  - Pretend you saw every outcome k extra times



$$P_{LAP,k}(x) = \frac{c(x) + k}{N + k|X|}$$

  - What's Laplace with k = 0?
  - k is the strength of the prior

$$P_{LAP,0}(X) =$$

$$P_{LAP,1}(X) =$$

$$P_{LAP,100}(X) =$$

Laplace Smoothing Can Be More Formally Derived

- Relative frequencies are the maximum likelihood estimates

$$\theta_{ML} = \arg\max_\theta P(\mathbf{X}|\theta)$$
$$= \arg\max_\theta \prod_i P_\theta(X_i)$$

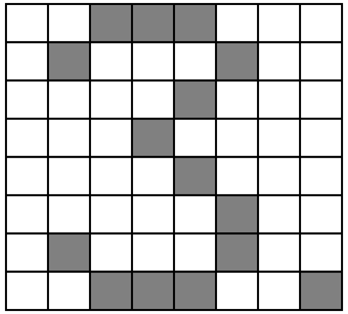
$$P_{\text{ML}}(x) = \frac{\text{count}(x)}{\text{total samples}}$$

- Another option is to consider the most likely parameter value given the data

$$\theta_{MAP} = \arg\max_\theta P(\theta|\mathbf{X})$$
$$= \arg\max_\theta P(\mathbf{X}|\theta)P(\theta)/P(\mathbf{X})$$
$$= \arg\max_\theta P(\mathbf{X}|\theta)P(\theta)$$

"right" choice of P(theta)
-> Laplace estimates

Estimation: Linear Interpolation*



- In practice, Laplace can perform poorly for P(X|Y):
  - When |X| is very large
  - When |Y| is very large
- Another option: linear interpolation
  - Also get the empirical P(X) from the data
  - Make sure the estimate of P(X|Y) isn't too different from the empirical P(X)

$$P_{LIN}(x|y) = \alpha \hat{P}(x|y) + (1.0 - \alpha)\hat{P}(x)$$

  - What if α is 0? 1?
- For even better ways to estimate parameters, as well as details of the math, see cs281a, cs288

Real NB: Smoothing

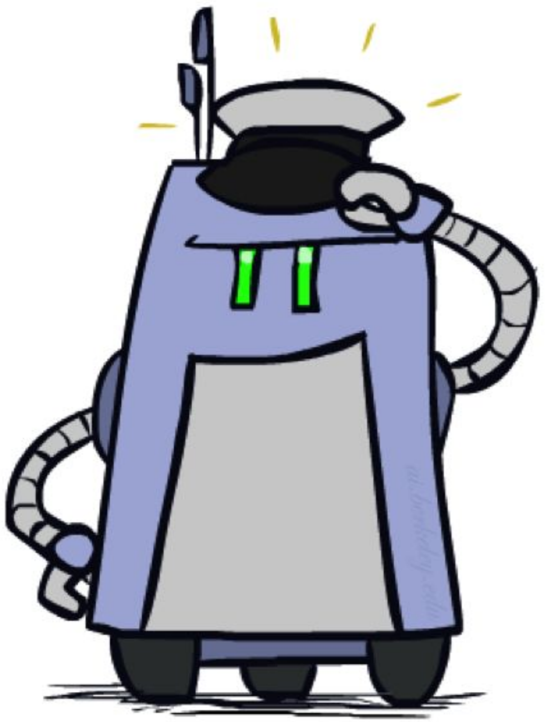
- For real classification problems, smoothing is critical
- New odds ratios:

$$\frac{P(W|\text{ham})}{P(W|\text{spam})}$$

```
helvetica  : 11.4
seems      : 10.8
group      : 10.2
ago        :  8.4
areas      :  8.3
...
```

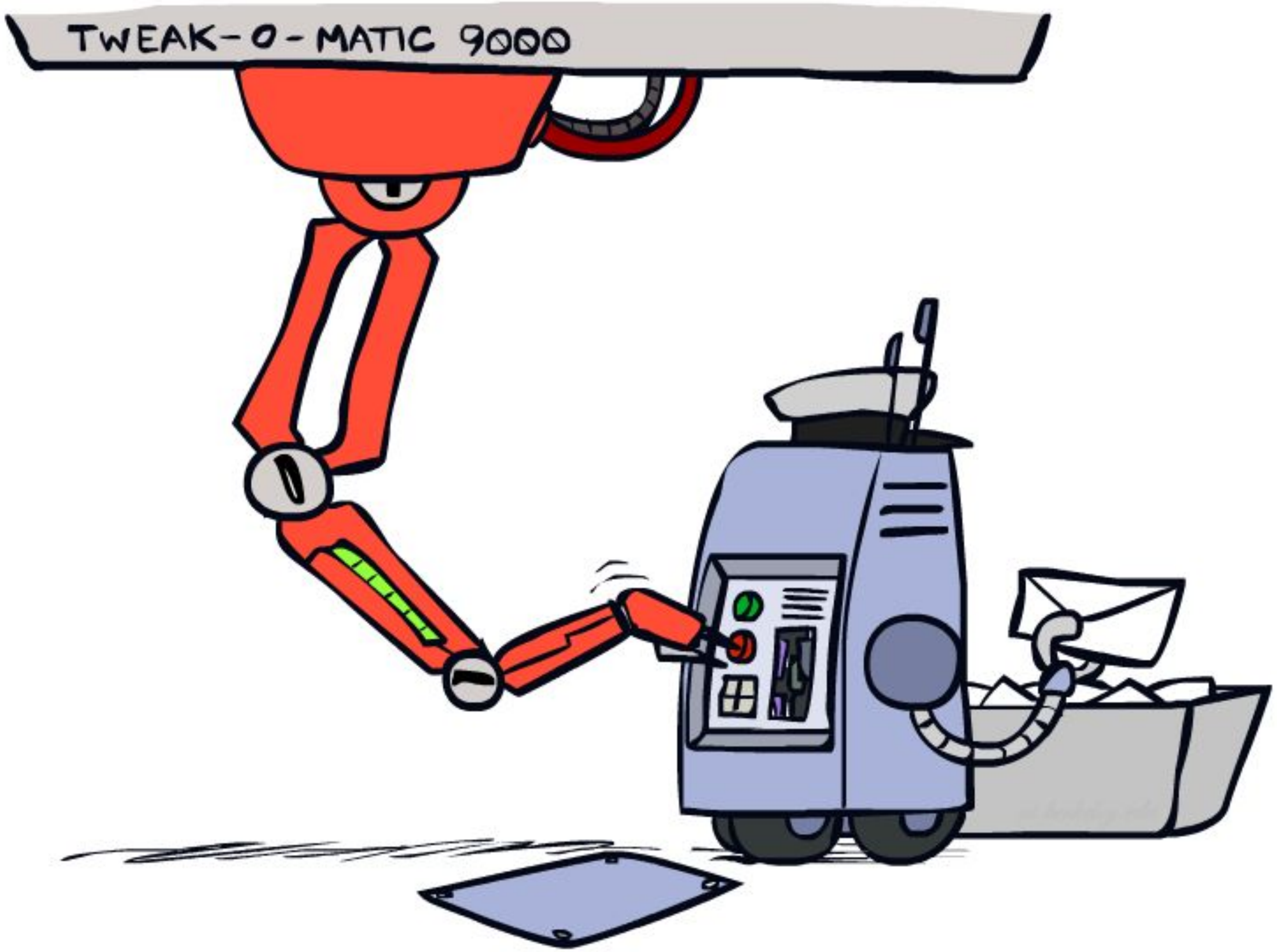
$$\frac{P(W|\text{spam})}{P(W|\text{ham})}$$

```
verdana : 28.8
Credit  : 28.4
ORDER   : 27.2
<FONT>  : 26.9
money   : 26.5
...
```



Do these make more sense?

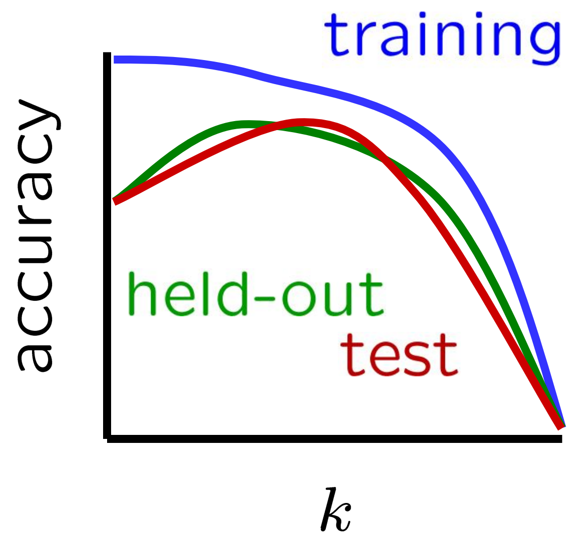
Tuning

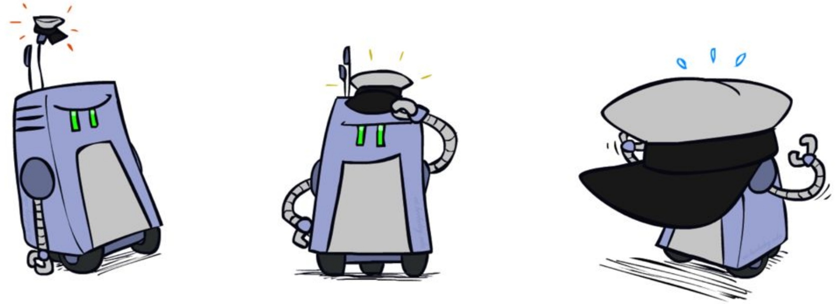


Tuning on Held-Out Data

- Now we've got two kinds of unknowns
  - Parameters: the probabilities P(X|Y), P(Y)
  - Hyperparameters: e.g. the amount of smoothing *k*

- What should we learn where?
  - Learn parameters from training data
  - Tune hyperparameters on different data
    - Why?
  - For each value of the hyperparameters, train and test on the held-out data
  - Choose the best value and do a final test on the test data

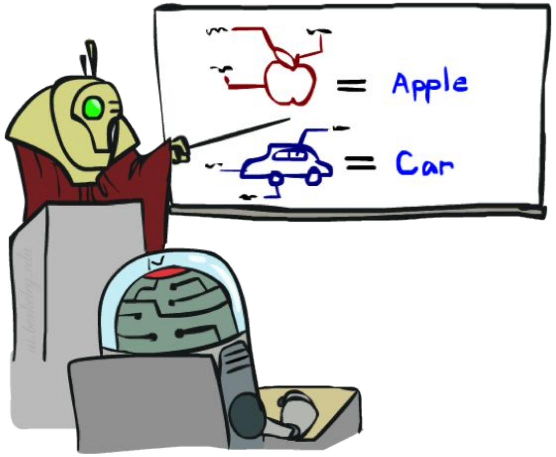




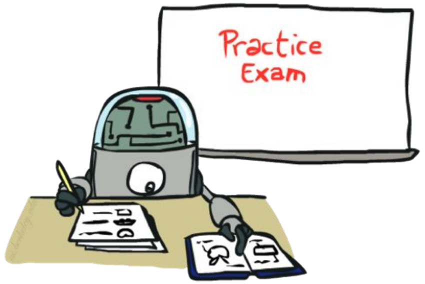
Important Concepts

- Data: labeled instances, e.g. emails marked spam/ham
  - Training set
  - Held out set
  - Test set
- Features: attribute-value pairs which characterize each input
- Experimentation cycle
  - Learn parameters (e.g. model probabilities) on training set
  - (Tune hyperparameters on held-out set)
  - Compute accuracy on test set
  - Very important: never "peek" at the test set!
- Evaluation
  - Accuracy: fraction of instances predicted correctly
- Overfitting and generalization
  - Want a classifier which does well on *test* data
  - <u>Overfitting</u>: fitting the training data very closely, but not generalizing well
  - <u>Underfitting</u>: fits the training set poorly

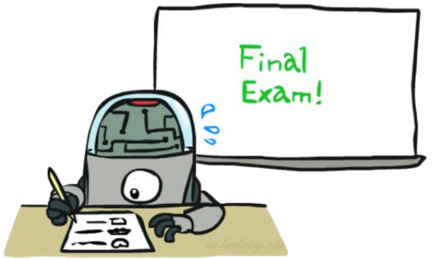
Training Data



Held-Out Data



Test Data



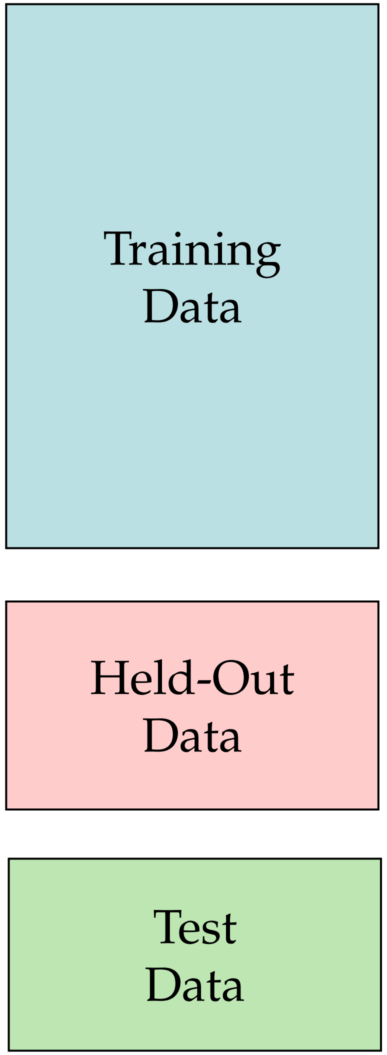
Workflow

- **Phase 1: Train model on Training Data. Choice points for "tuning"**
  - Attributes / Features
  - Model types: Naïve Bayes vs. Perceptron vs. Logistic Regression vs. Neural Net etc..
  - Model hyperparameters
    - E.g. Naïve Bayes – Laplace k
    - E.g. Logistic Regression – weight regularization
    - E.g. Neural Net – architecture, learning rate, …
  - Make sure good performance on training data (why?)

- **Phase 2: Evaluate on Hold-Out Data**
  - If Hold-Out performance is close to Train performance
    - We achieved good generalization, onto Phase 3! ☺
  - If Hold-Out performance is much worse than Train performance
    - We overfitted to the training data! ☹
    - Take inspiration from the errors and:
      - Either: go back to Phase 1 for tuning (typically: make the model less expressive)
      - Or: if we are out of options for tuning while maintaining high train accuracy, collect more data (i.e., let the data drive generalization, rather than the tuning/regularization) and go to Phase 1

- **Phase 3: Report performance on Test Data**

Possible outer-loop: Collect more data



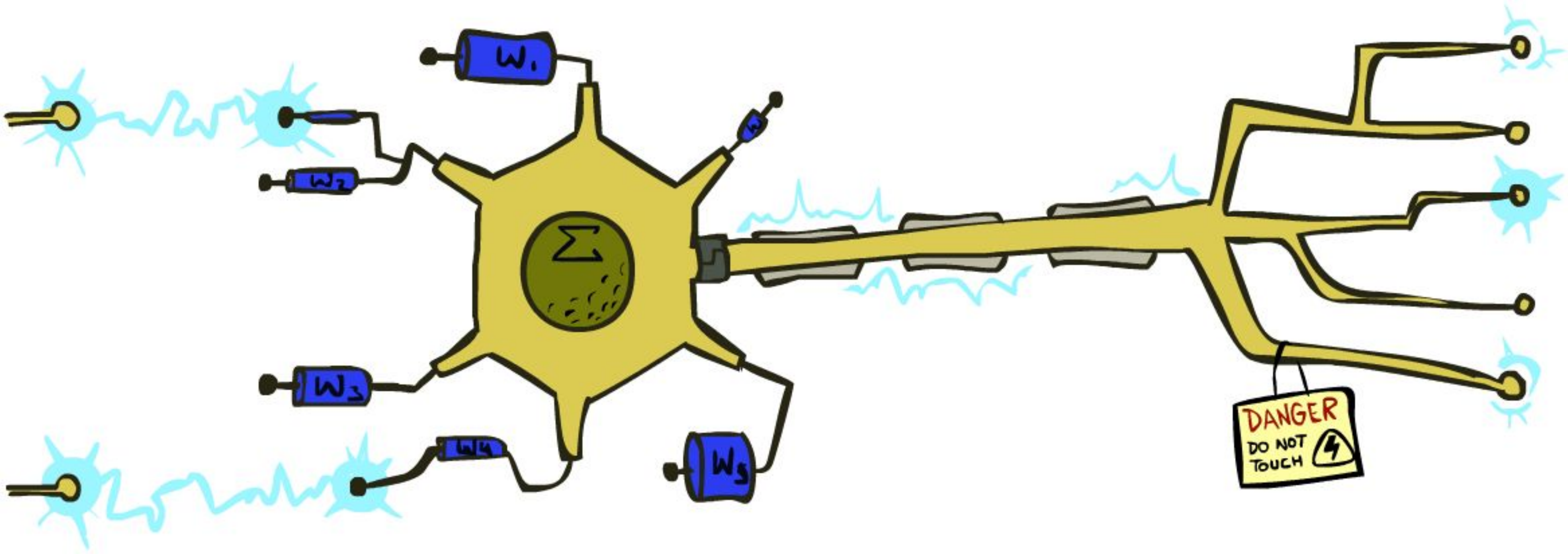
Practical Tip: Baselines

- First step: get a baseline
  - Baselines are very simple “straw man” procedures
  - Help determine how hard the task is
  - Help know what a “good” accuracy is

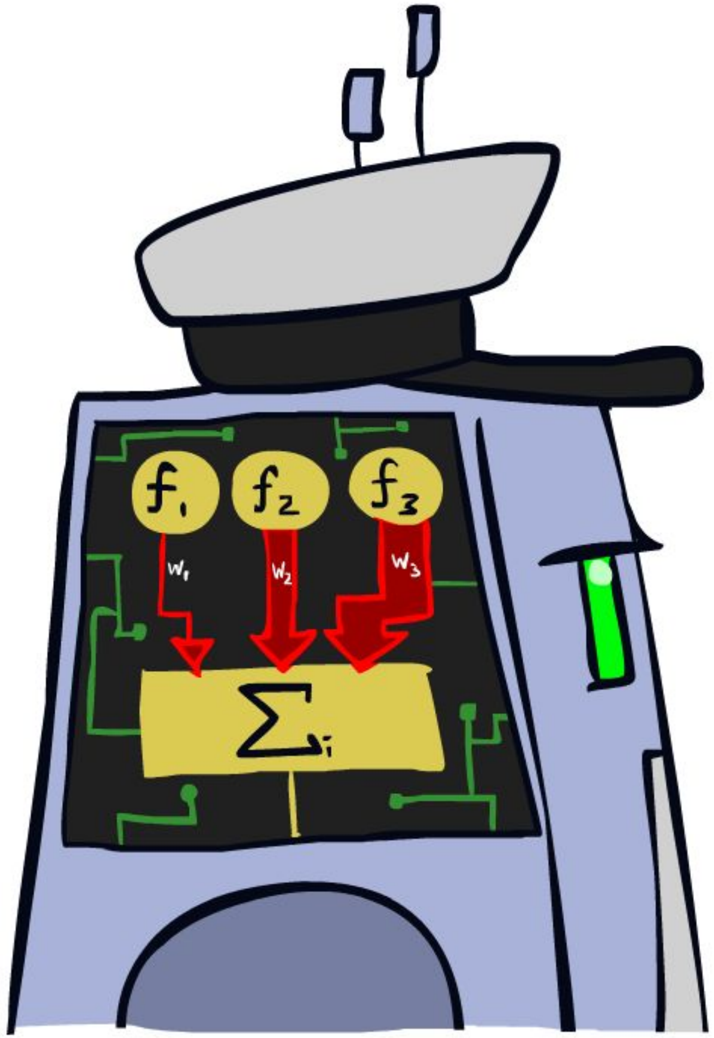
- Weak baseline: most frequent label classifier
  - Gives all test instances whatever label was most common in the training set
  - E.g. for spam filtering, might label everything as ham
  - Accuracy might be very high if the problem is skewed
  - E.g. calling everything “ham” gets 66%, so a classifier that gets 70% isn’t very good…

- For real research, usually use previous work as a (strong) baseline

Perceptrons



Linear Classifiers



Feature Vectors

x $\quad\quad$ $f(x)$ $\quad\quad$ y

```
Hello,

Do you want free printr
cartriges?  Why pay more
when you can get them
ABSOLUTELY FREE!  Just
```

```
# free       : 2
YOUR_NAME    : 0
MISSPELLED   : 2
FROM_FRIEND  : 0
...
```

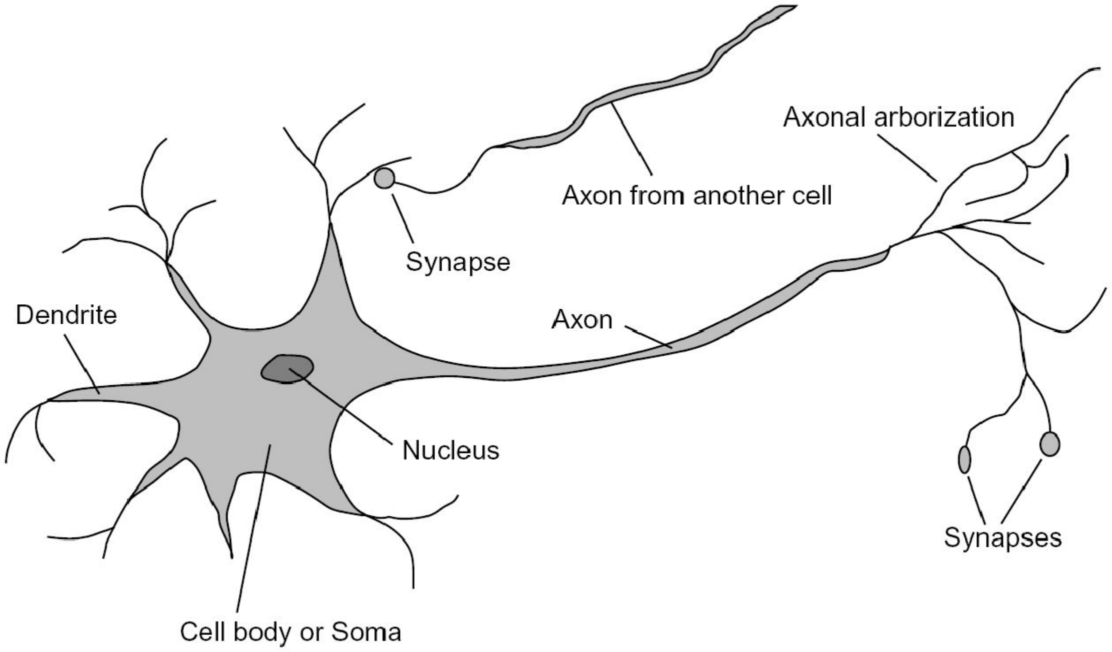
SPAM
or
HAM

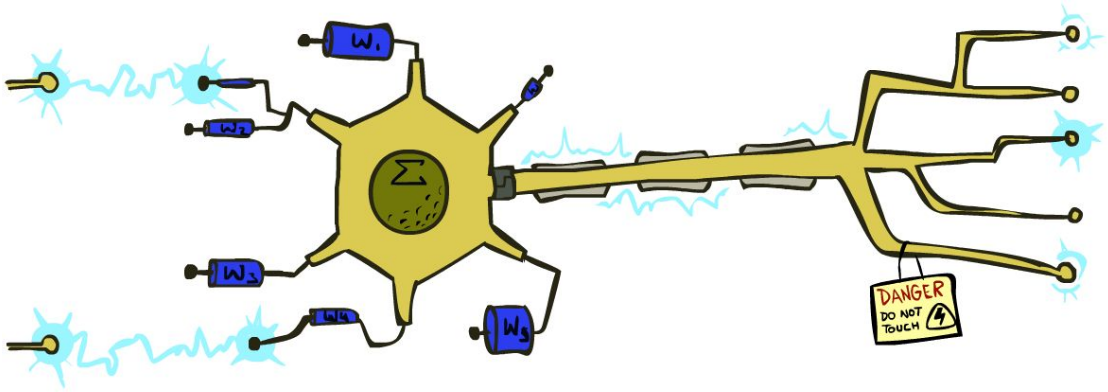
```
PIXEL-7,12  : 1
PIXEL-7,13  : 0
...
NUM_LOOPS   : 1
...
```

“2”

Some (Simplified) Biology

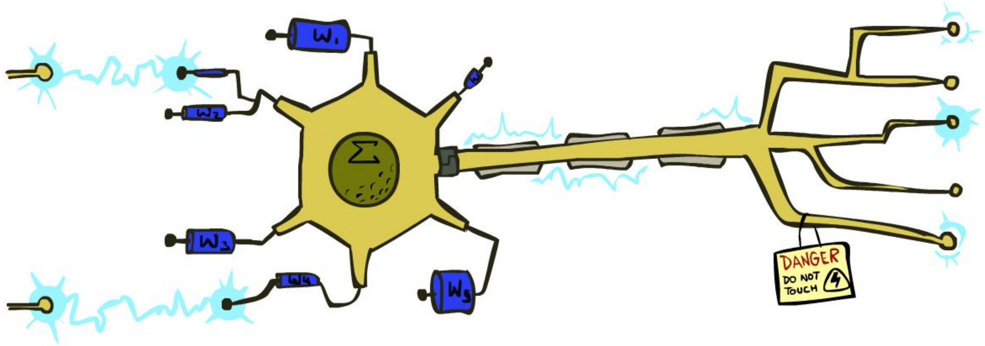
- Very loose inspiration: human neurons





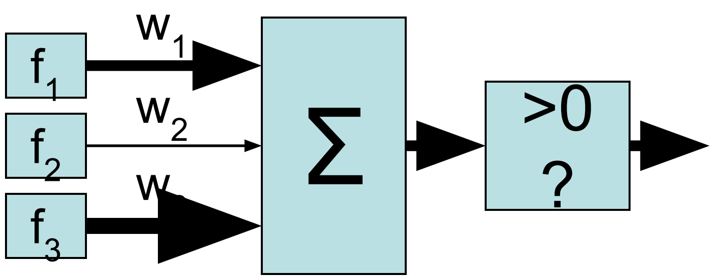
Linear Classifiers

- Inputs are feature values
- Each feature has a weight
- Sum is the activation



$$\text{activation}_w(x) = \sum_i w_i \cdot f_i(x) = w \cdot f(x)$$

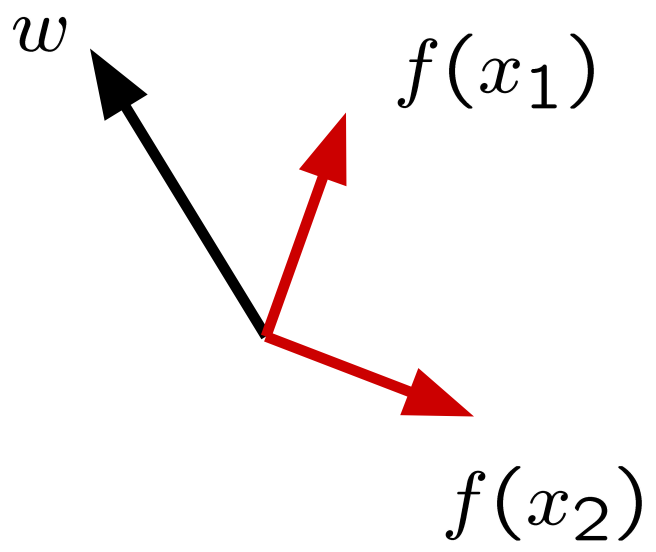
- If the activation is:
  - Positive, output +1
  - Negative, output -1



Weights

- Binary case: compare features to a weight vector
- Learning: figure out the weight vector from examples

```
# free      : 4
YOUR_NAME   :-1
MISSPELLED  : 1
FROM_FRIEND :-3
...
```

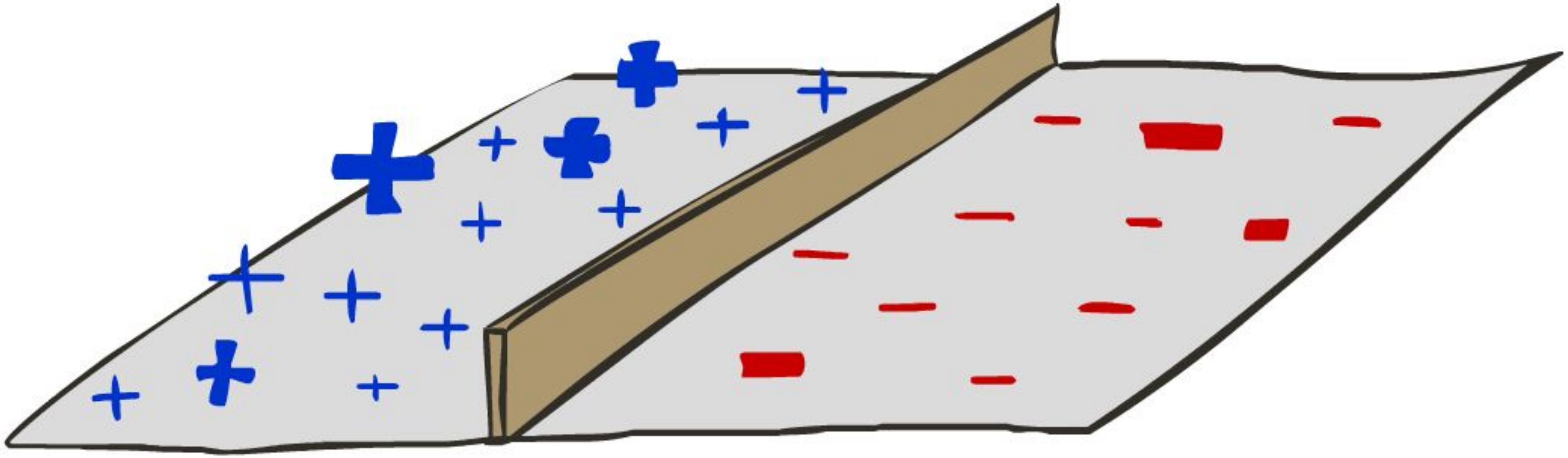



```
# free      : 2
YOUR_NAME   : 0
MISSPELLED  : 2
FROM_FRIEND : 0
...
```

```
# free      : 0
YOUR_NAME   : 1
MISSPELLED  : 1
FROM_FRIEND : 1
...
```

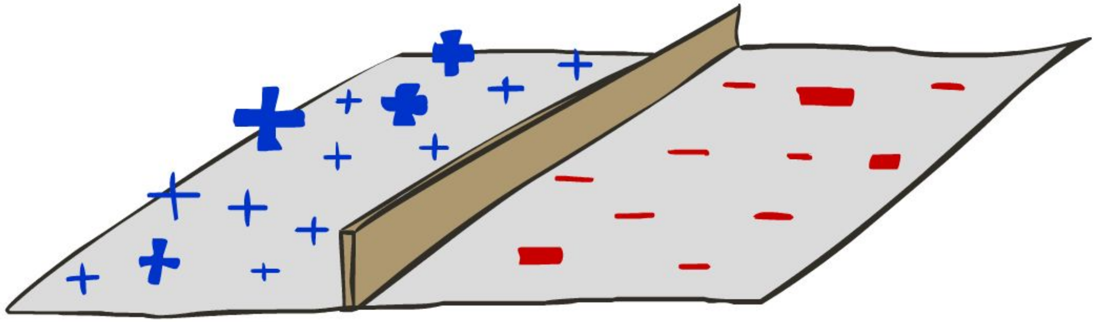
Dot product $w \cdot f$ positive means the positive class

Decision Rules



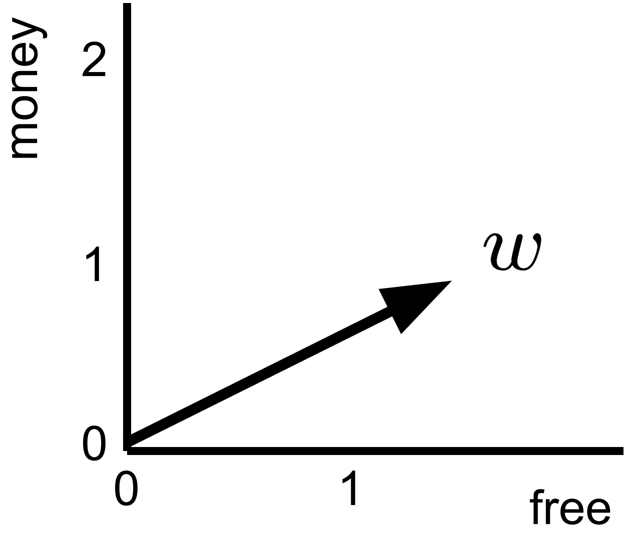
Binary Decision Rule

- In the space of feature vectors
  - Examples are points
  - Any weight vector is a hyperplane
  - One side corresponds to Y=+1
  - Other corresponds to Y=-1



w

free	:	4
money	:	2

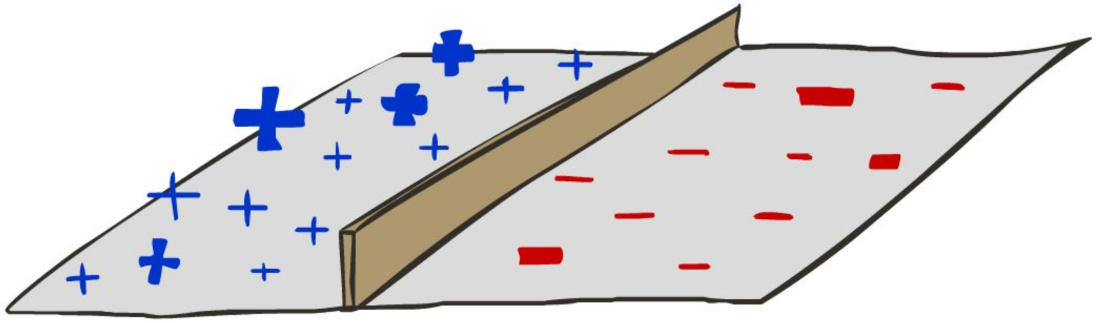


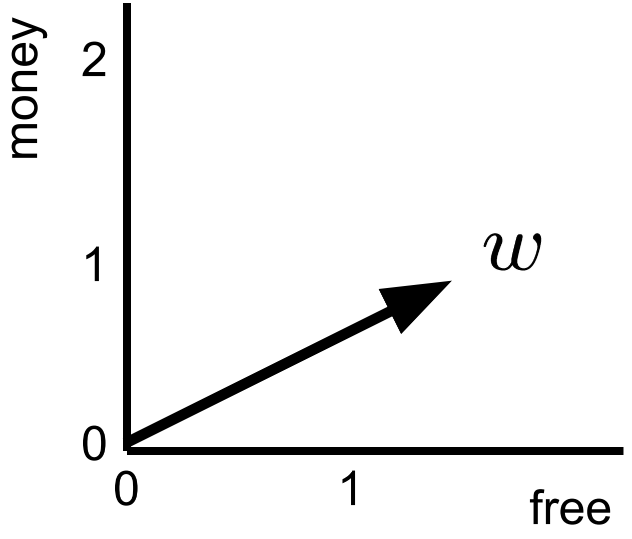
Binary Decision Rule

- In the space of feature vectors
  - Examples are points
  - Any weight vector is a hyperplane
  - One side corresponds to Y=+1
  - Other corresponds to Y=-1

w

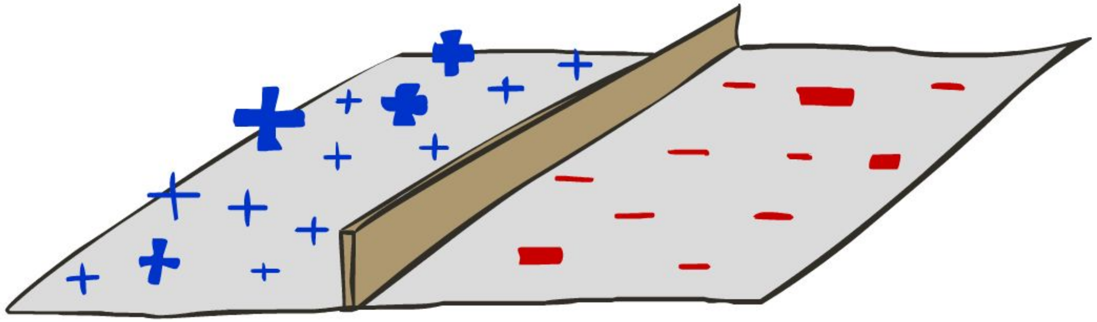
```
BIAS  : -3
free  :  4
money :  2
...
```





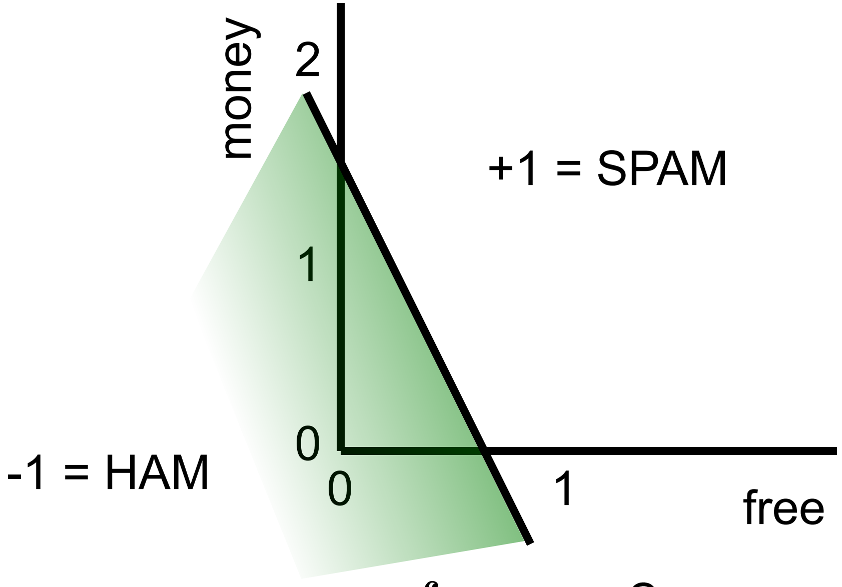
Binary Decision Rule

- In the space of feature vectors
  - Examples are points
  - Any weight vector is a hyperplane
  - One side corresponds to Y=+1
  - Other corresponds to Y=-1



w

```
BIAS  : -3
free  :  4
money :  2
...
```

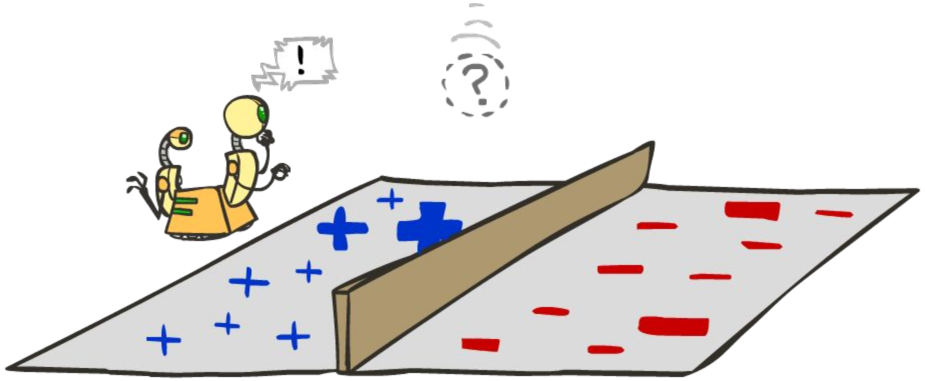


$f \cdot w = 0$

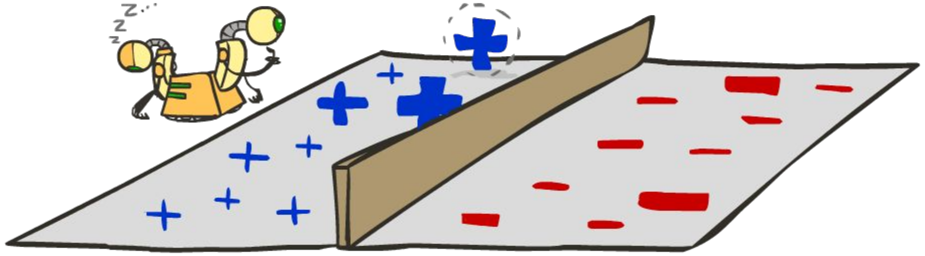
Weight Updates

Learning: Binary Perceptron

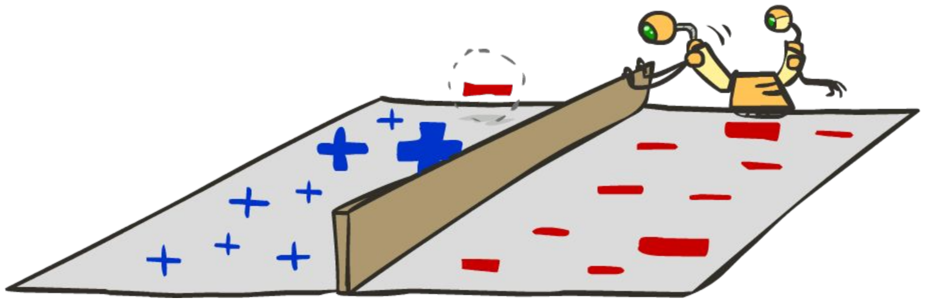
- Start with weights w = 0
- For each training instance f(x), y*:
  - Classify with current weights



  - If correct (i.e., y=y*), no change!



  - If wrong: adjust the weight vector



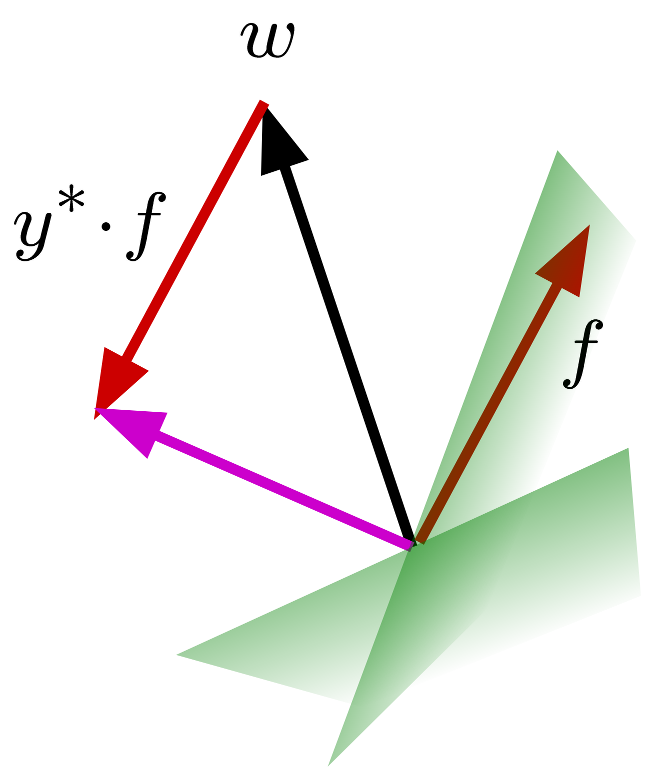
Learning: Binary Perceptron

- Start with weights w = 0
- For each training instance f(x), y*:
  - Classify with current weights

$$y = \begin{cases} +1 & \text{if } \ w \cdot f(x) \geq 0 \\ -1 & \text{if } \ w \cdot f(x) < 0 \end{cases}$$

  - If correct (i.e., y=y*), no change!
  - If wrong: adjust the weight vector by adding or subtracting the feature vector. Subtract if y* is -1.

$$w = w + y^* \cdot f$$



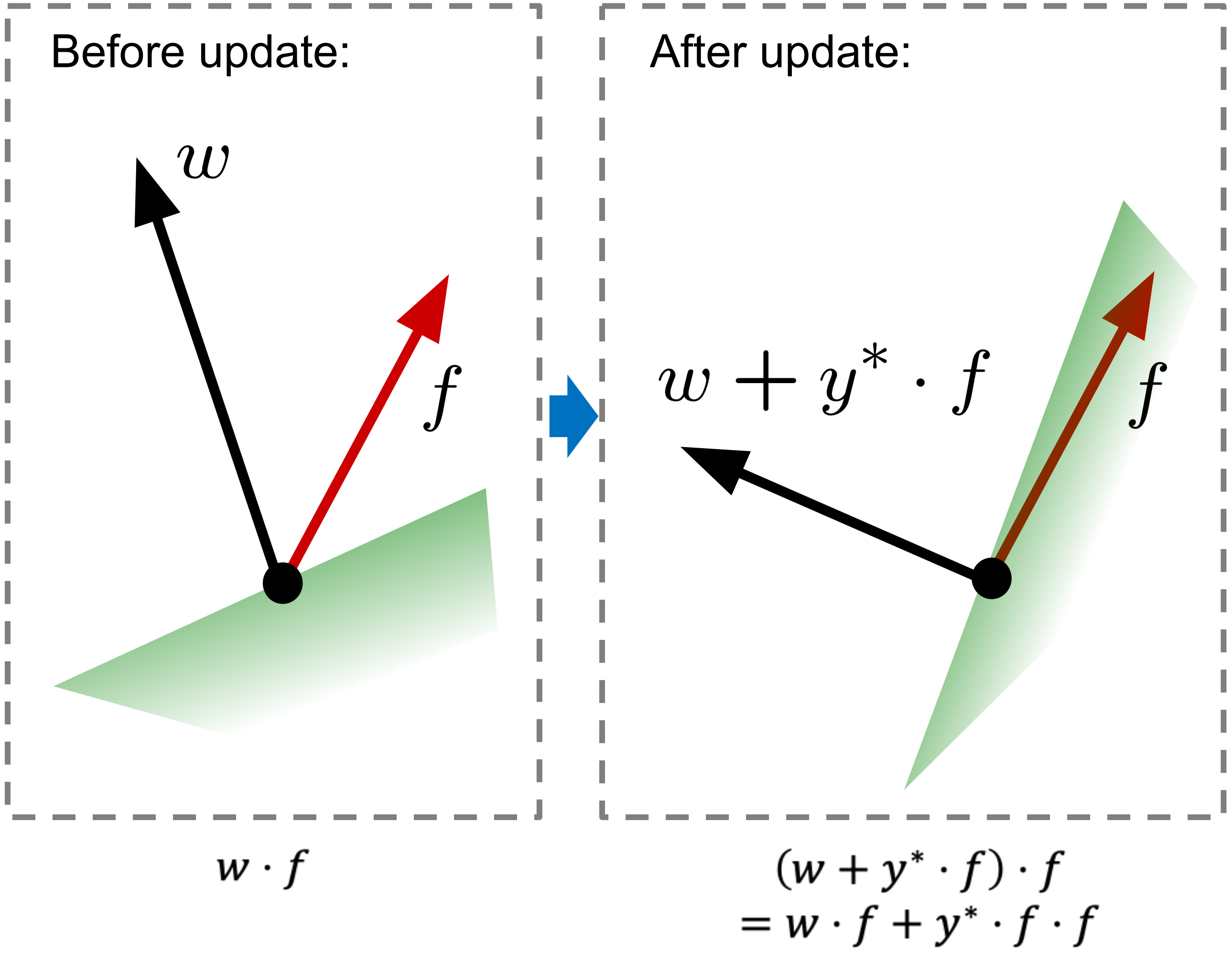
Learning: Binary Perceptron

- Start with weights w = 0
- For each training instance f(x), y*:
  - Classify with current weights

$$y = \begin{cases} +1 & \text{if } w \cdot f(x) \geq 0 \\ -1 & \text{if } w \cdot f(x) < 0 \end{cases}$$

  - If correct (i.e., y=y*), no change!
  - If wrong: adjust the weight vector by adding or subtracting the feature vector. Subtract if y* is -1.

$$w = w + y^* \cdot f$$



Example: Perceptron

Iteration 0: x: "*win the vote*" f(x): [1 1 0 1 1] y*: -1

Iteration 1: x: "*win the election*" f(x): [1 1 0 0 1] y*: -1

Iteration 2: x: "*win the game*" f(x): [1 1 1 0 1] y*: +1

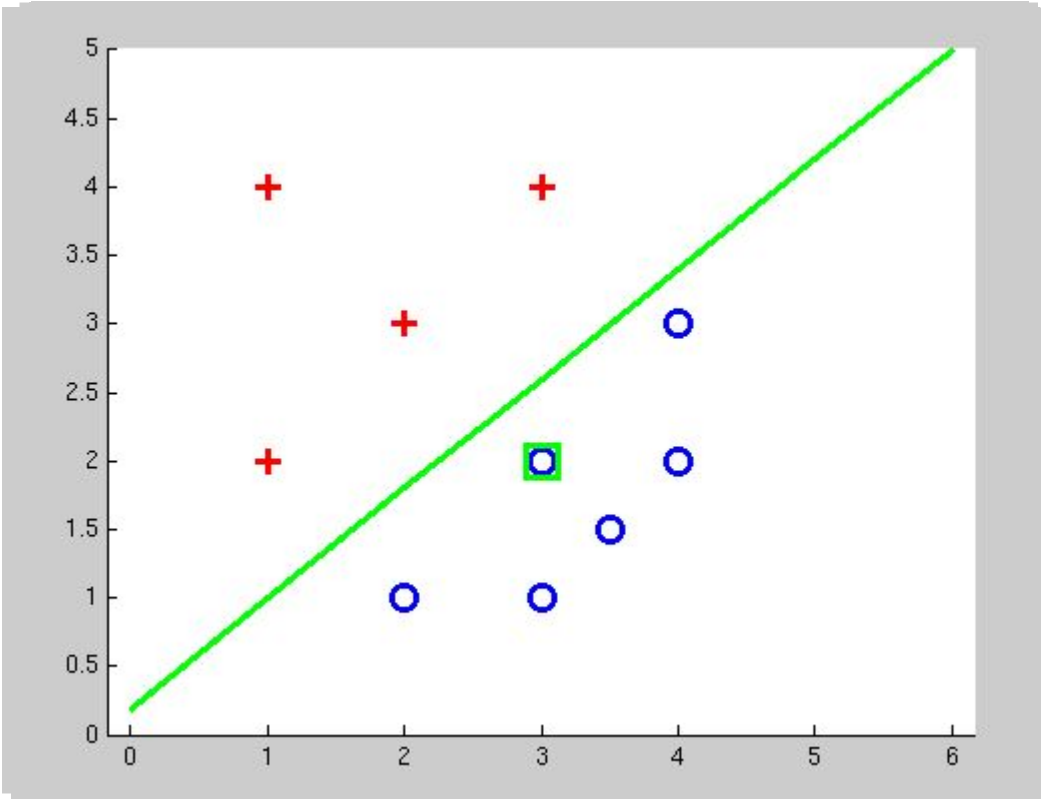
Iteration 3: x: "*win the game*" f(x): [1 1 1 0 1] y*: +1

w

BIAS	1	0	0	1
win	0	-1	-1	0
game	0	0	0	1
vote	0	-1	-1	-1
the	0	-1	-1	0
$w \cdot f(x)$:	1	-2	-2	2

Example: Perceptron

- Separable Case



Multiclass Decision Rule

- If we have multiple classes:
  - A weight vector for each class:

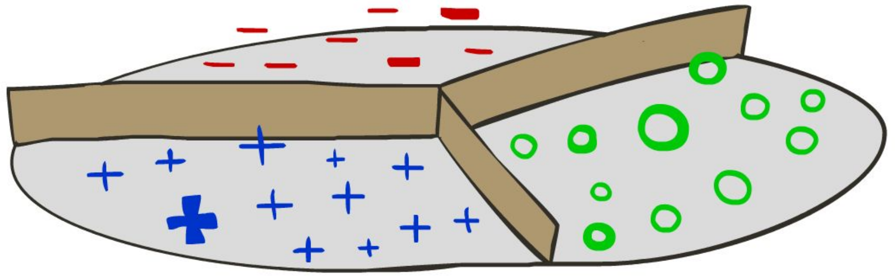
    $$w_y$$

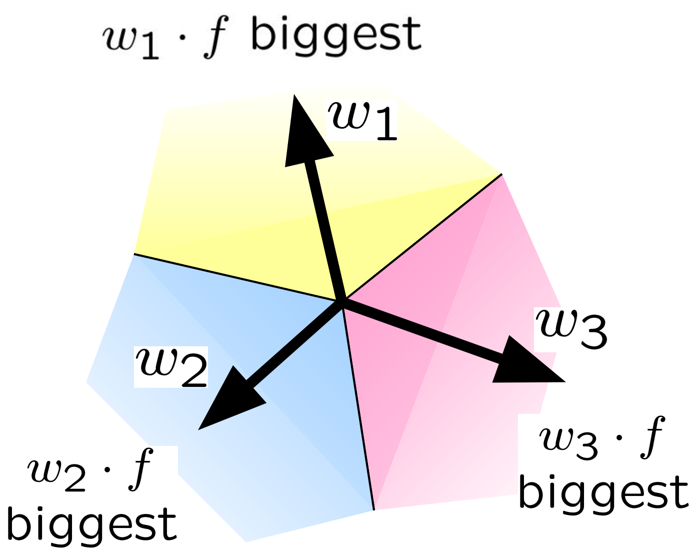
  - Score (activation) of a class y:

    $$w_y \cdot f(x)$$

  - Prediction highest score wins

    $$y = \arg\max_y \; w_y \cdot f(x)$$





Binary = multiclass where the negative class has weight zero

Learning: Multiclass Perceptron

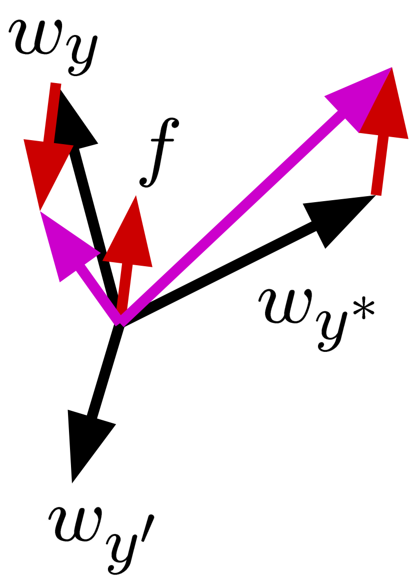
- Start with all weights = 0
- Pick up training examples f(x), y* one by one
- Predict with current weights

$$y = \arg\max_y \; w_y \cdot f(x)$$

- If correct, no change!
- If wrong: lower score of wrong answer, raise score of right answer

$$w_y = w_y - f(x)$$

$$w_{y^*} = w_{y^*} + f(x)$$



Example: Multiclass Perceptron

Iteration 0: x: "*win the vote*" f(x): [1 1 0 1 1] y*: politics

Iteration 1: x: "*win the election*" f(x): [1 1 0 0 1] y*: politics

Iteration 2: x: "*win the game*" f(x): [1 1 1 0 1] y*: sports

w_{SPORTS}

BIAS	1	0	0	1
win	0	-1	-1	0
game	0	0	0	1
vote	0	-1	-1	-1
the	0	-1	-1	0
$w \cdot f(x)$:	1	-2	-2	

$w_{POLITICS}$

BIAS	0	1	1	0
win	0	1	1	0
game	0	0	0	-1
vote	0	1	1	1
the	0	1	1	0
$w \cdot f(x)$:	0	3	3	

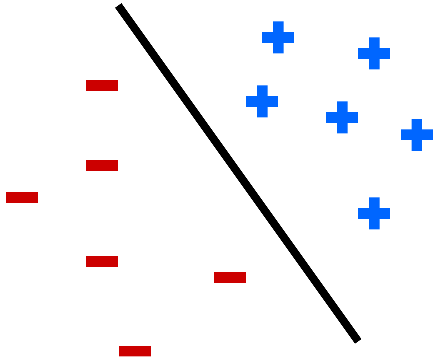
w_{TECH}

BIAS	0	0	0	0
win	0	0	0	0
game	0	0	0	0
vote	0	0	0	0
the	0	0	0	0
$w \cdot f(x)$:	0	0	0	

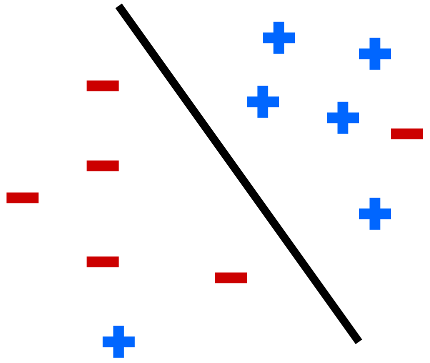
Properties of Perceptrons

- Separability: true if some parameters get the training set perfectly correct
- Convergence: if the training is separable, perceptron will eventually converge (binary case)
- Mistake Bound: the maximum number of mistakes (binary case) related to the *margin* or degree of separability

Separable

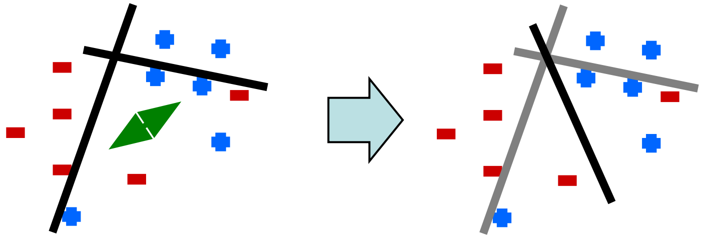


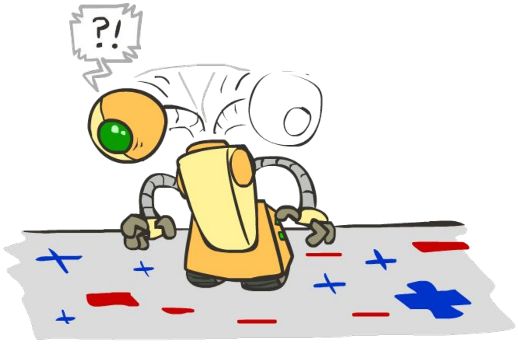
Non-Separable



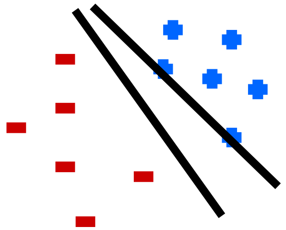
Problems with the Perceptron

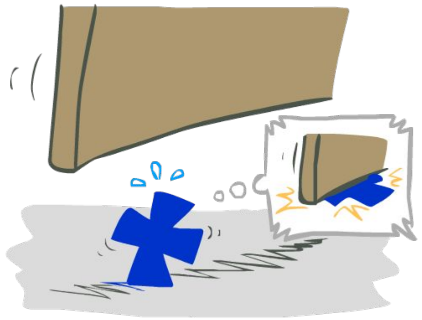
- Noise: if the data isn't separable, weights might thrash
  - Averaging weight vectors over time can help (averaged perceptron)



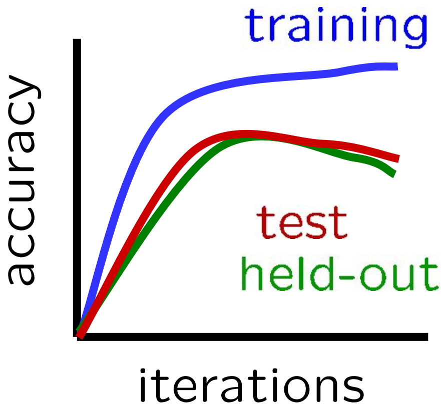


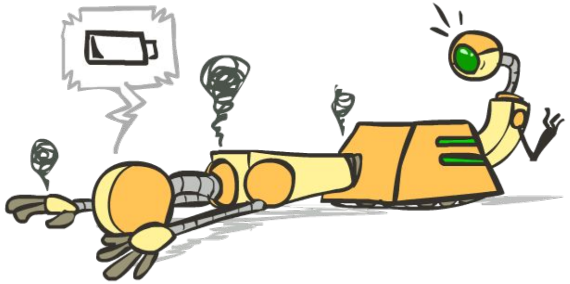
- Mediocre generalization: finds a "barely" separating solution





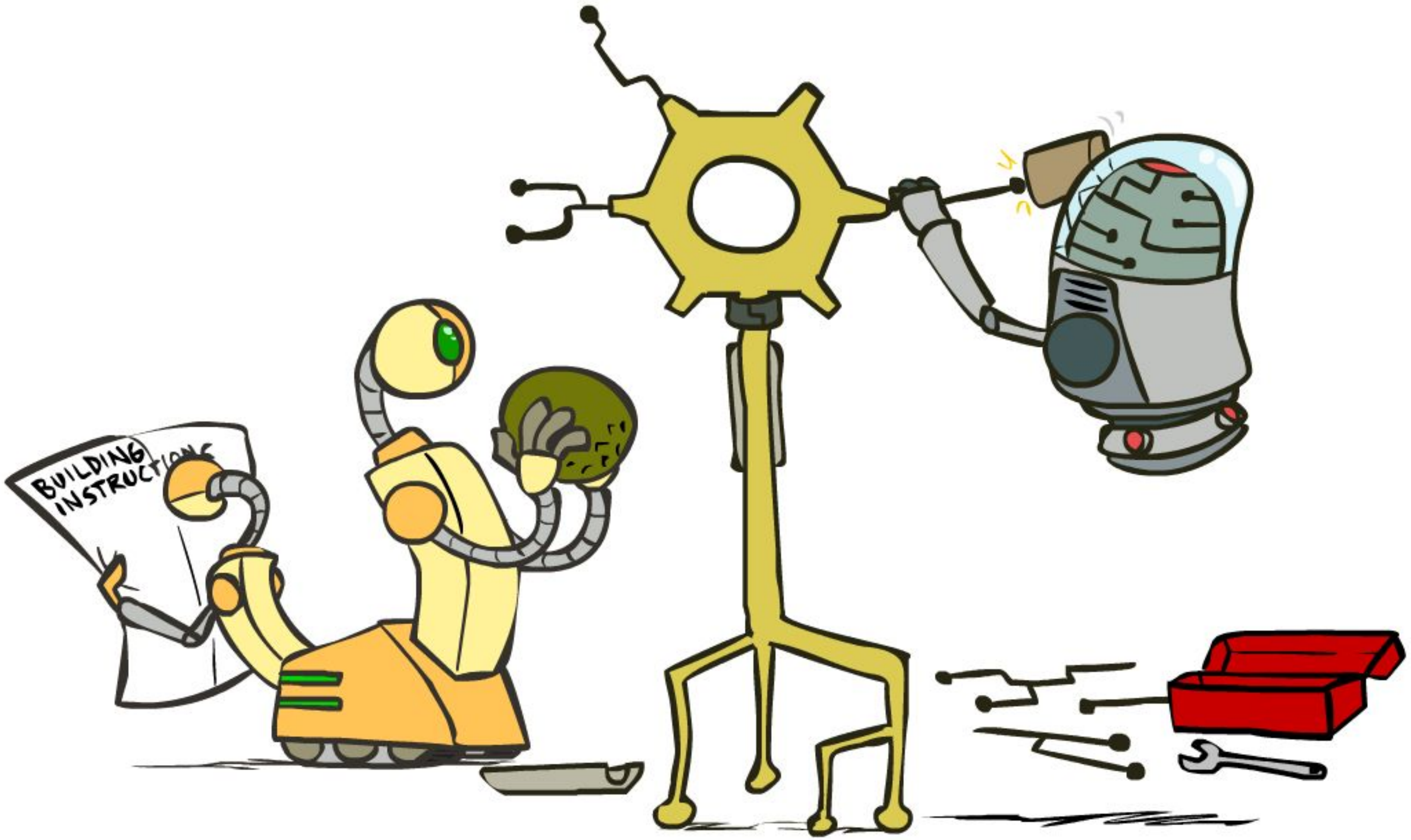
- Overtraining: test / held-out accuracy usually rises, then falls
  - Overtraining is a kind of overfitting





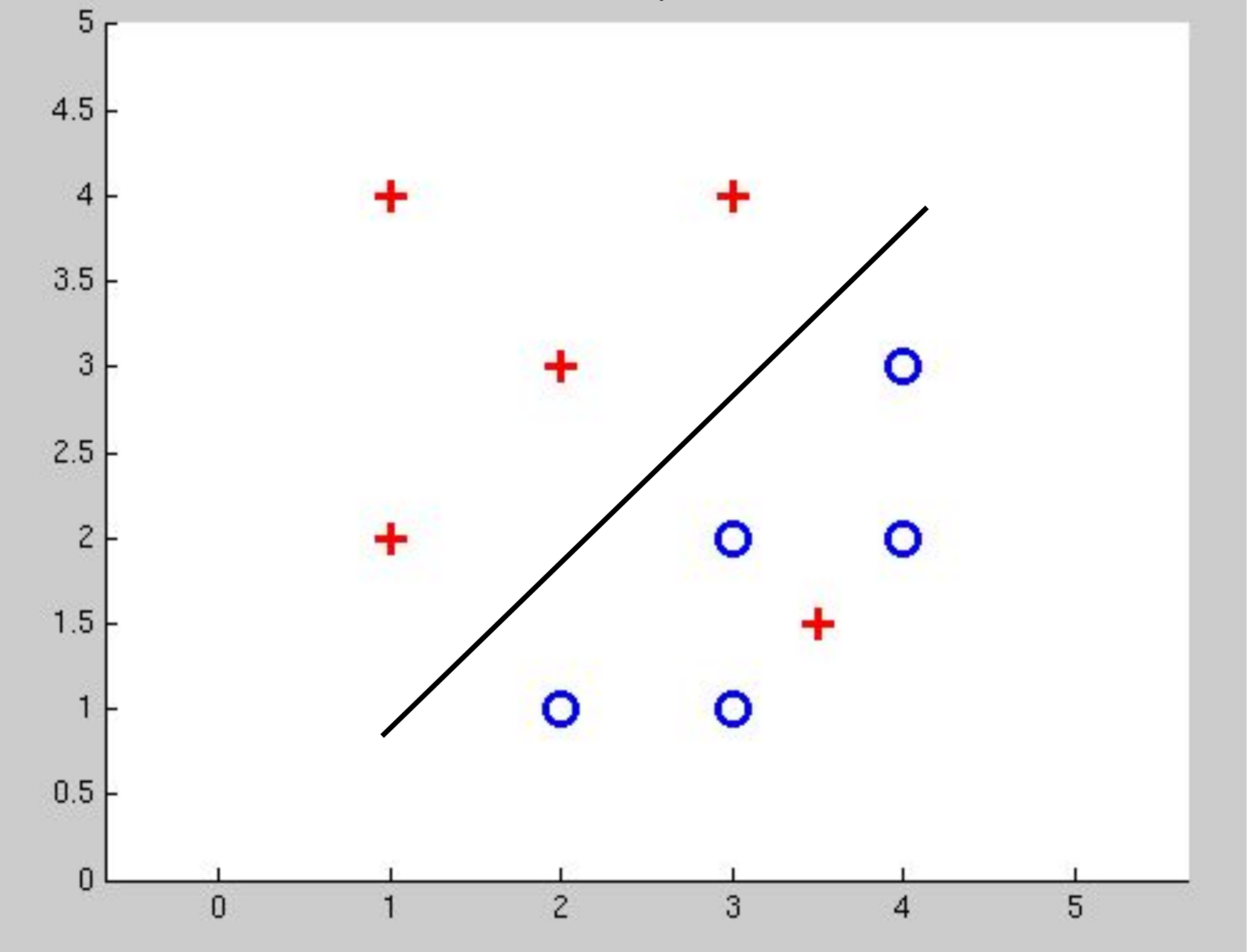
Next Lecture: Improving Perceptron & Optimization

Improving the Perceptron

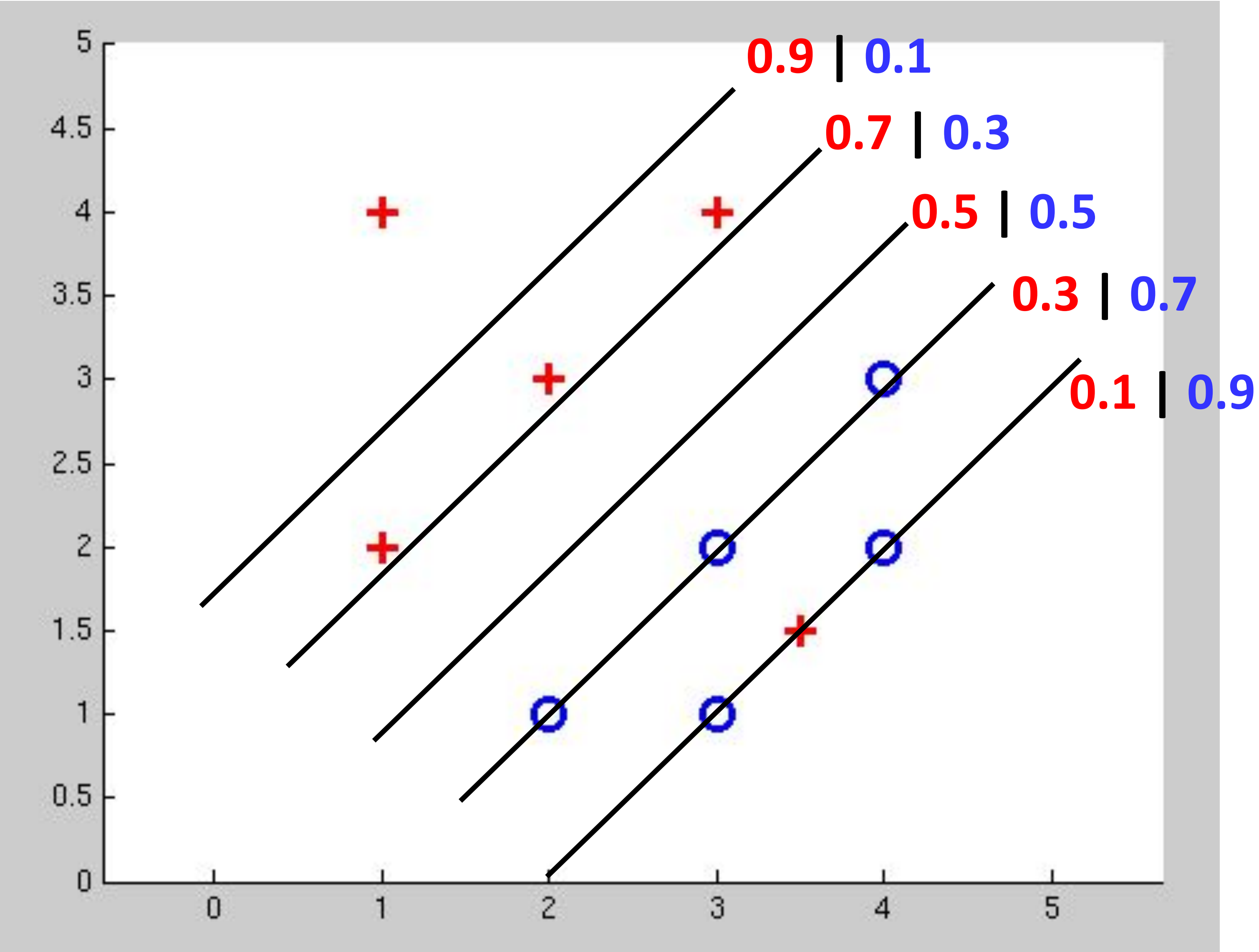


Non-Separable Case: Deterministic Decision

Even the best linear boundary makes at least one mistake



Non-Separable Case: Probabilistic Decision

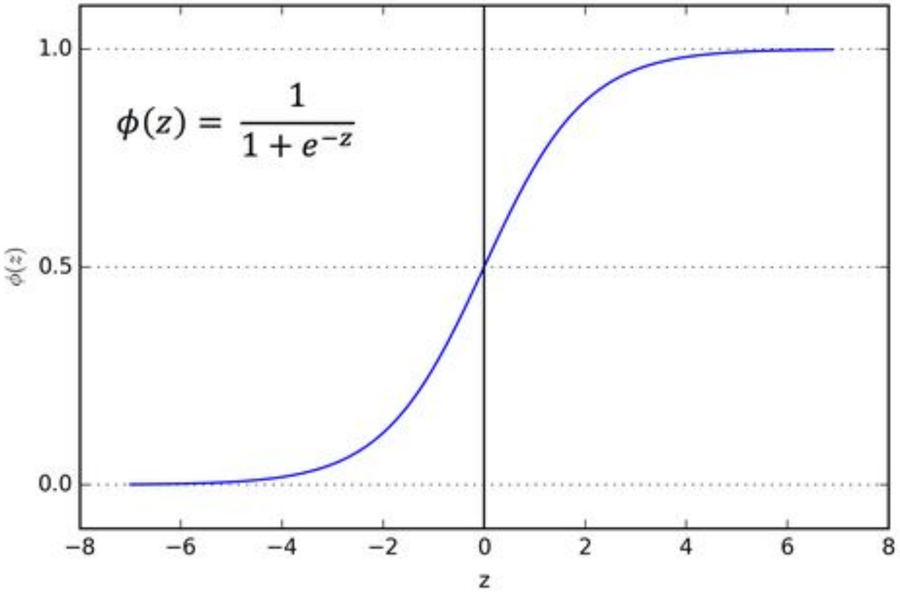


How to get probabilistic decisions?

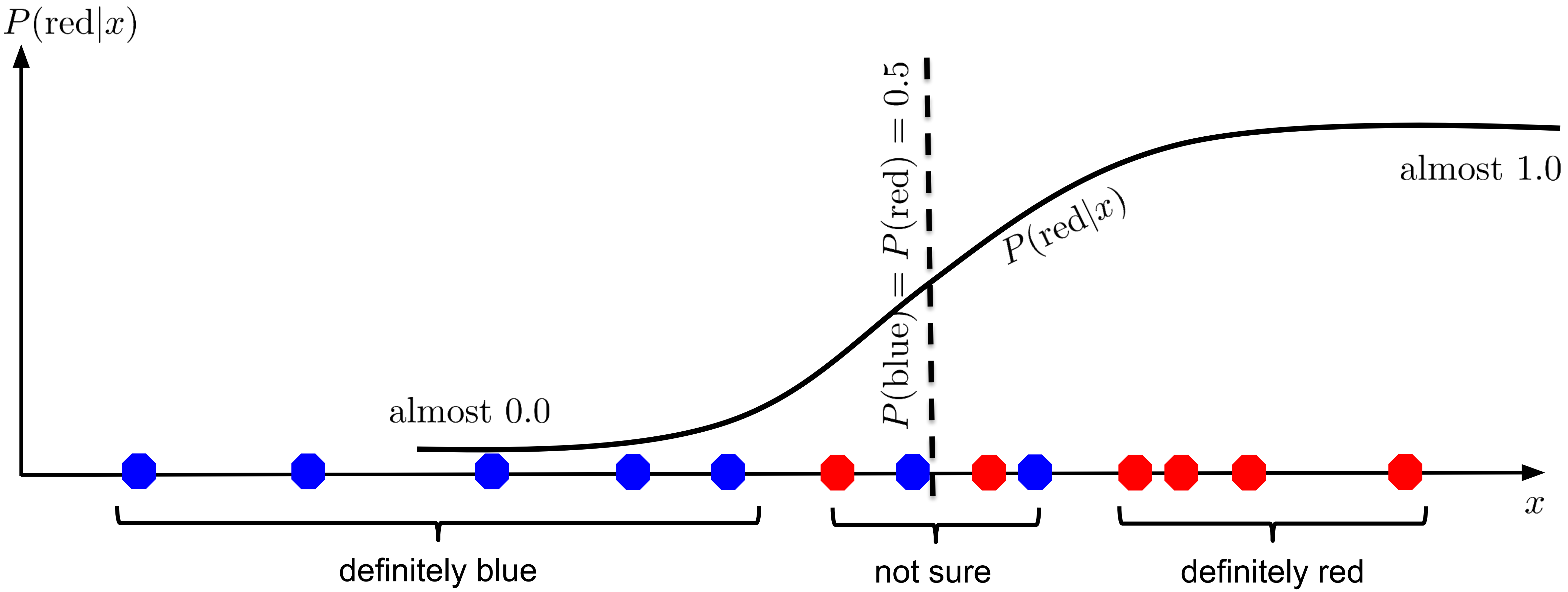
- Perceptron scoring: $z = w \cdot f(x)$
- If $z = w \cdot f(x)$ very positive □ want probability going to 1
- If $z = w \cdot f(x)$ very negative □ want probability going to 0

- Sigmoid function

$$\phi(z) = \frac{1}{1 + e^{-z}}$$



A 1D Example

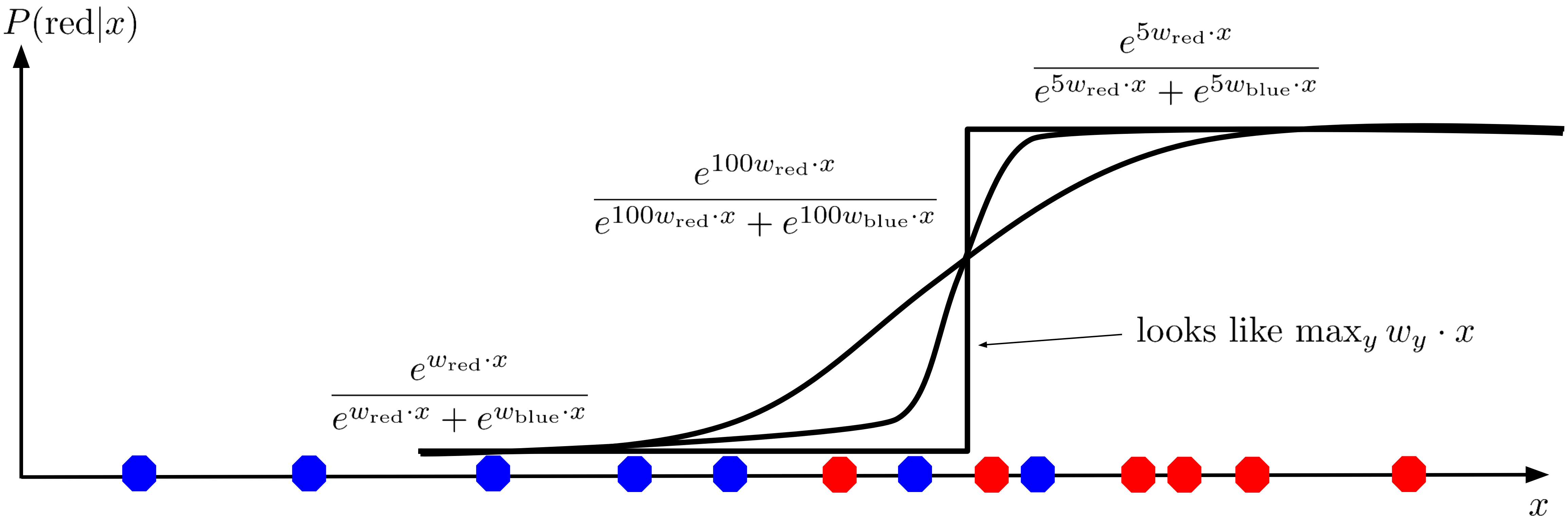


$$P(\text{red}|x) = \frac{e^{w_{\text{red}} \cdot x}}{e^{w_{\text{red}} \cdot x} + e^{w_{\text{blue}} \cdot x}}$$

probability increases exponentially as we move away from boundary

normalizer

The *Soft* Max



$$P(\text{red}|x) = \frac{e^{w_{\text{red}}\cdot x}}{e^{w_{\text{red}}\cdot x}+e^{w_{\text{blue}}\cdot x}}$$

Best w?

- Maximum likelihood estimation:

$$\max_w \quad ll(w) = \max_w \quad \sum_i \log P(y^{(i)}|x^{(i)}; w)$$

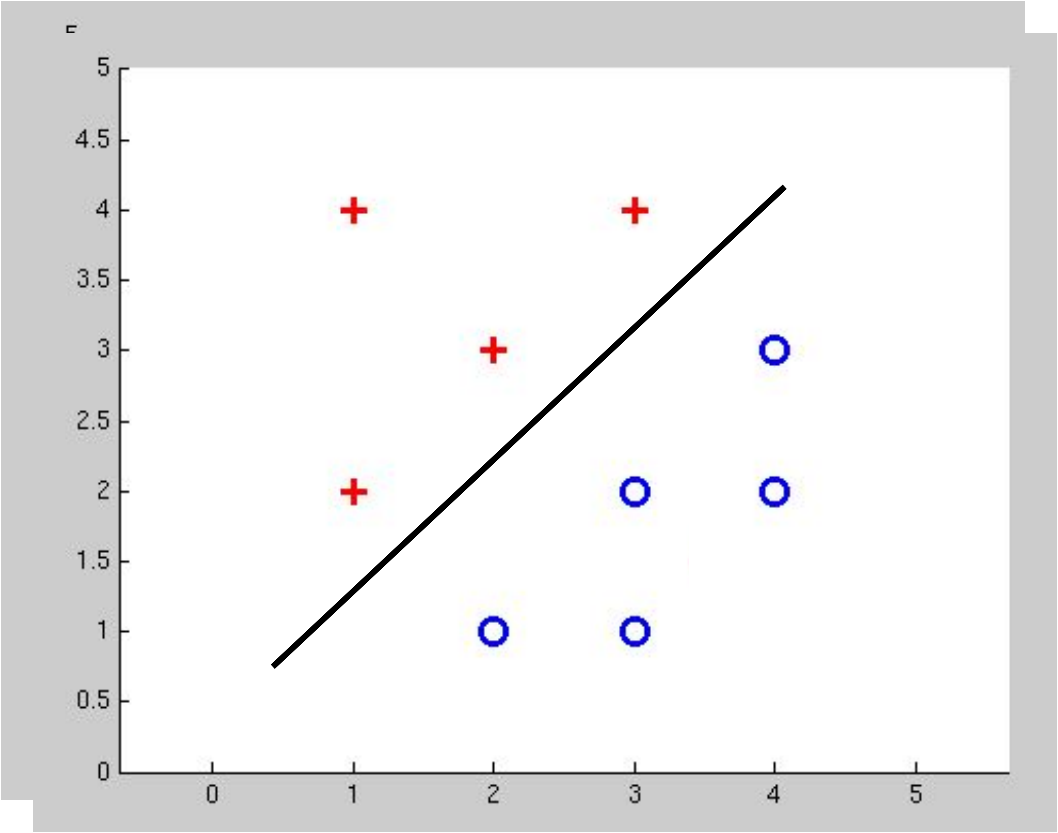
with:

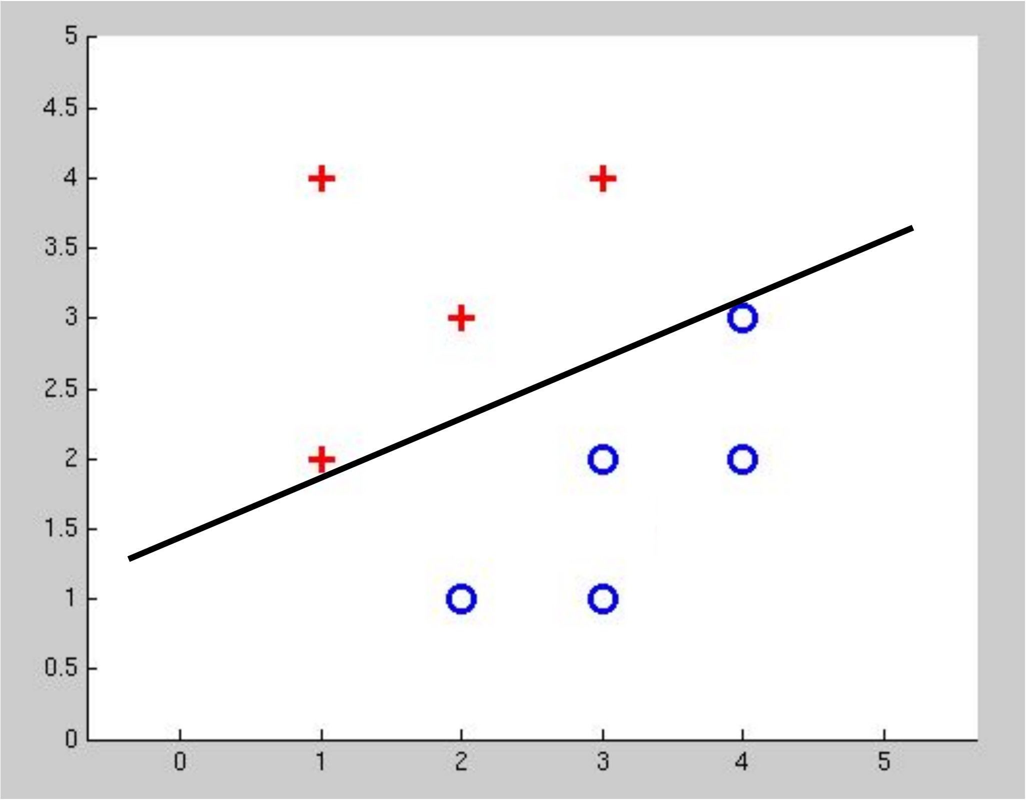
$$P(y^{(i)} = +1|x^{(i)}; w) = \frac{1}{1 + e^{-w \cdot f(x^{(i)})}}$$

$$P(y^{(i)} = -1|x^{(i)}; w) = 1 - \frac{1}{1 + e^{-w \cdot f(x^{(i)})}}$$

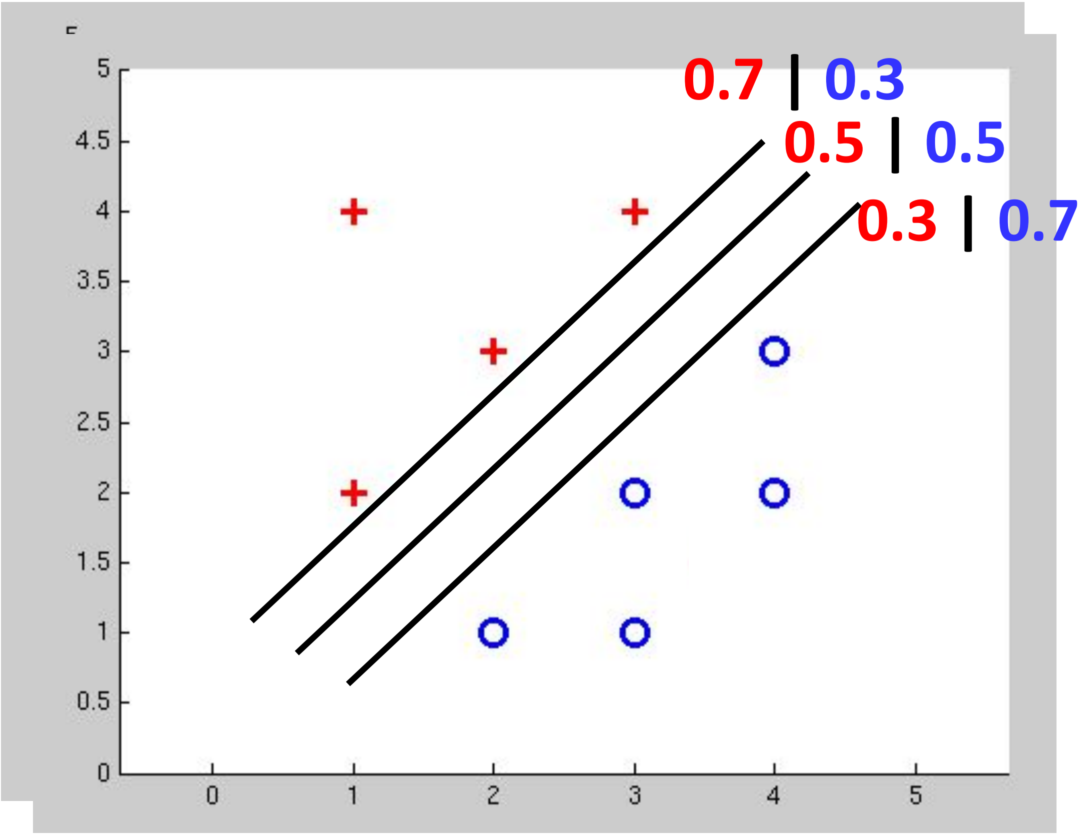
= Logistic Regression

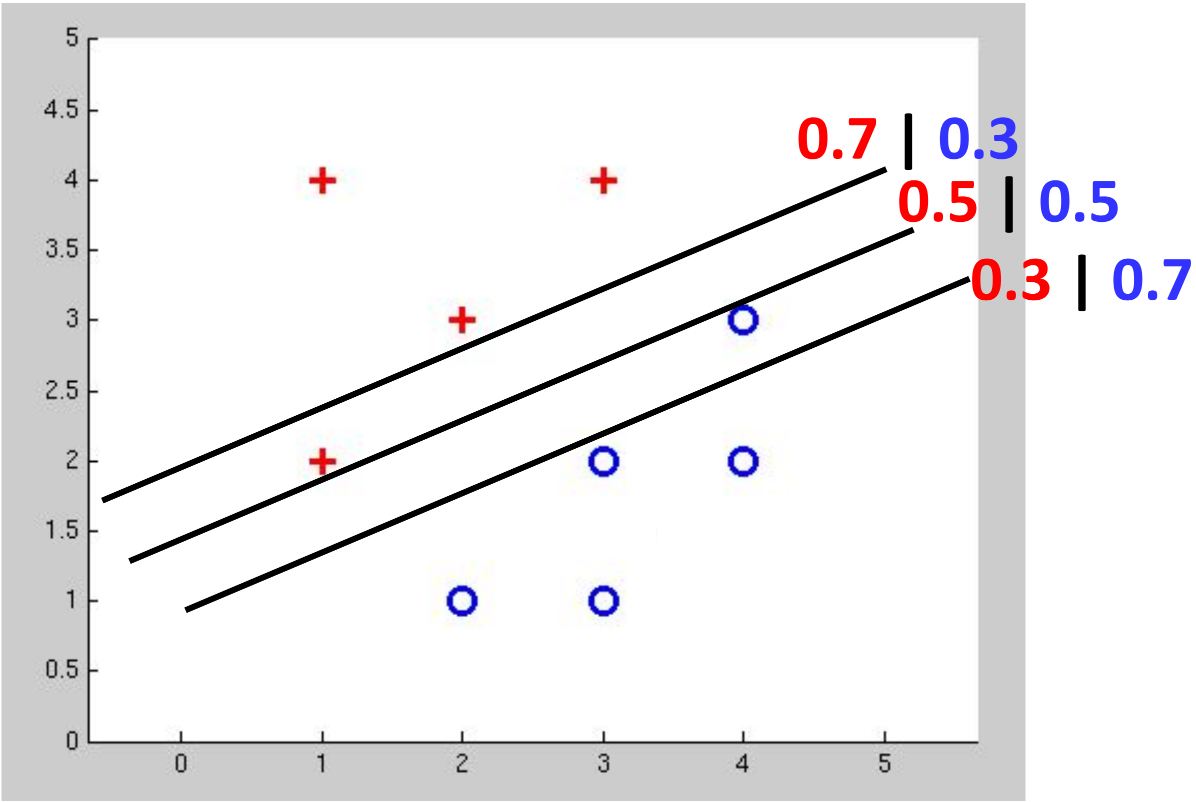
Separable Case: Deterministic Decision – Many Options





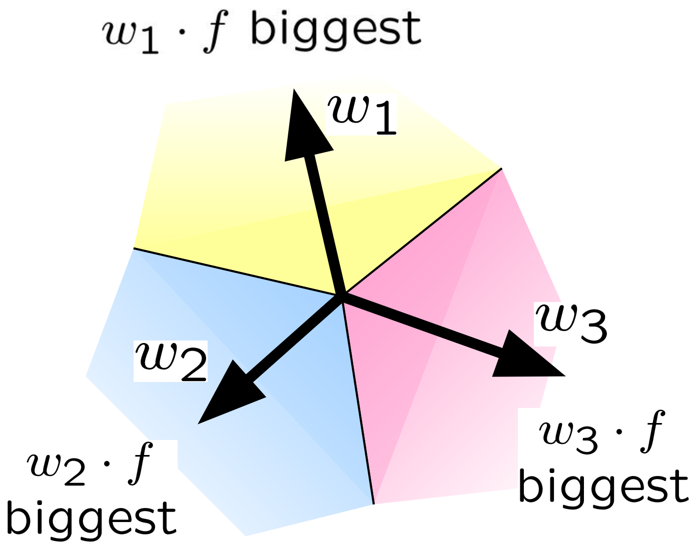
Separable Case: Probabilistic Decision – Clear Preference





Multiclass Logistic Regression

- Recall Perceptron:
  - A weight vector for each class: w_y
  - Score (activation) of a class y: $w_y \cdot f(x)$
  - Prediction highest score wins $y = \arg\max_y \; w_y \cdot f(x)$



- How to make the scores into probabilities?

$$\underbrace{z_1, z_2, z_3}_{\text{original activations}} \rightarrow \underbrace{\frac{e^{z_1}}{e^{z_1}+e^{z_2}+e^{z_3}}, \frac{e^{z_2}}{e^{z_1}+e^{z_2}+e^{z_3}}, \frac{e^{z_3}}{e^{z_1}+e^{z_2}+e^{z_3}}}_{\text{softmax activations}}$$

Best w?

- Maximum likelihood estimation:

$$\max_{w} \quad ll(w) = \max_{w} \quad \sum_{i} \log P(y^{(i)}|x^{(i)};w)$$

with:

$$P(y^{(i)}|x^{(i)};w) = \frac{e^{w_{y^{(i)}} \cdot f(x^{(i)})}}{\sum_{y} e^{w_y \cdot f(x^{(i)})}}$$

= Multi-Class Logistic Regression

Next Lecture

- Optimization
  - i.e., how do we solve:

$$\max_w \quad ll(w) = \max_w \quad \sum_i \log P(y^{(i)}|x^{(i)}; w)$$