CS 188: Artificial Intelligence
Spring 2011

Lecture 9: MDPs
2/16/2011

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Many slides over the course adapted from either Dan Klein, Stuart Russell or Andrew Moore

Announcements

- Midterm: Tuesday March 15, 5-8pm
- P2: Due Friday 4:59pm
- W3: Minimax, expectimax and MDPs---out tonight, due Monday February 28.
- Online book: Sutton and Barto

  http://www.cs.ualberta.ca/~sutton/book/ebook/the-book.html

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Outline

- Markov Decision Processes (MDPs)
  - Formalism
  - Value iteration
- Expectimax Search vs. Value Iteration
  - Value Iteration:
    - No exponential blow-up with depth [cf. graph search vs. tree search]
    - Can handle infinite duration games
- Policy Evaluation and Policy Iteration

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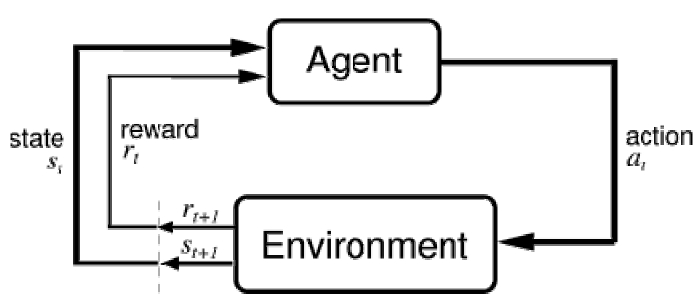
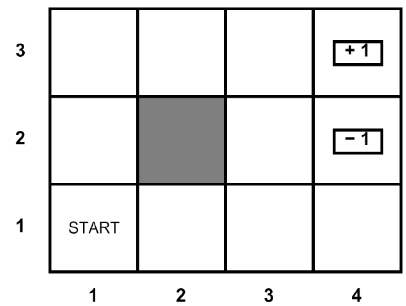
Reinforcement Learning

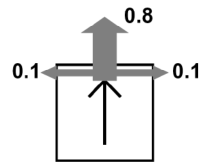
- Basic idea:
  - Receive feedback in the form of rewards
  - Agent's utility is defined by the reward function
  - Must learn to act so as to maximize expected rewards



Grid World

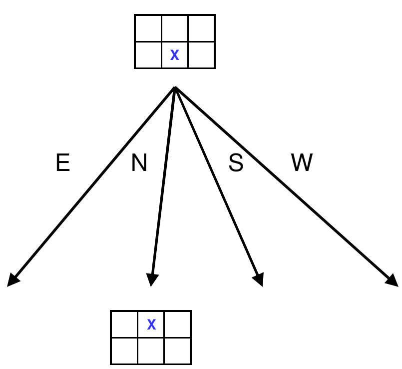
- The agent lives in a grid
- Walls block the agent's path
- The agent's actions do not always go as planned:
  - 80% of the time, the action North takes the agent North (if there is no wall there)
  - 10% of the time, North takes the agent West; 10% East
  - If there is a wall in the direction the agent would have been taken, the agent stays put
- Small "living" reward each step
- Big rewards come at the end
- Goal: maximize sum of rewards



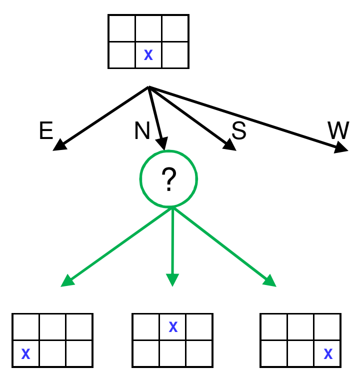


Grid Futures

Deterministic Grid World



Stochastic Grid World



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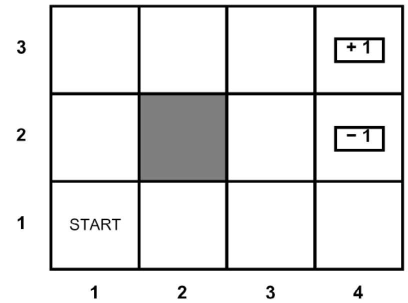
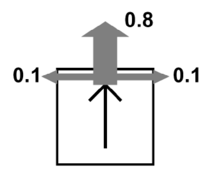
Markov Decision Processes

- An MDP is defined by:
  - A set of states $s \in S$
  - A set of actions $a \in A$
  - A transition function T(s,a,s')
    - Prob that a from s leads to s'
    - i.e., P(s' | s,a)
    - Also called the model
  - A reward function R(s, a, s')
    - Sometimes just R(s) or R(s')
  - A start state (or distribution)
  - Maybe a terminal state



- MDPs are a family of non-deterministic search problems
  - Reinforcement learning: MDPs where we don't know the transition or reward functions



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What is Markov about MDPs?

- Andrey Markov (1856-1922)
- "Markov" generally means that given the present state, the future and the past are independent
- For Markov decision processes, "Markov" means:

$$P(S_{t+1} = s' | S_t = s_t, A_t = a_t, S_{t-1} = s_{t-1}, A_{t-1}, \ldots S_0 = s_0)$$

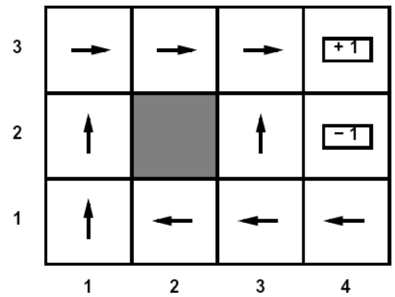
$$=$$

$$P(S_{t+1} = s' | S_t = s_t, A_t = a_t)$$

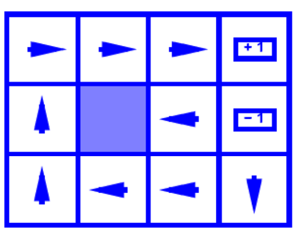
Solving MDPs

- In deterministic single-agent search problems, want an optimal plan, or sequence of actions, from start to a goal
- In an MDP, we want an optimal policy $\pi^*: S \rightarrow A$
  - A policy π gives an action for each state
  - An optimal policy maximizes expected utility if followed
  - Defines a reflex agent

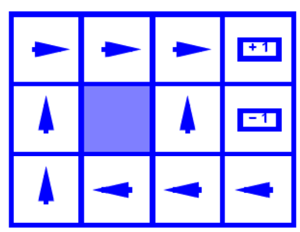
Optimal policy when R(s, a, s') = -0.03 for all non-terminals s



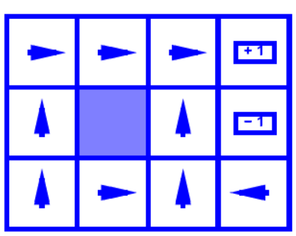
Example Optimal Policies



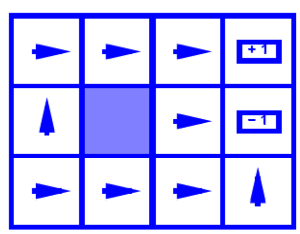
R(s) = -0.01



R(s) = -0.03



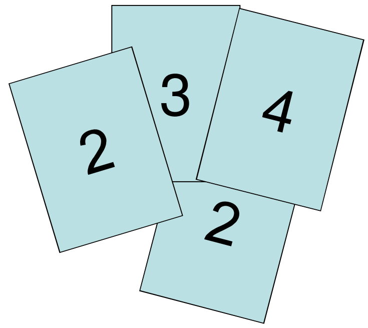
R(s) = -0.4



R(s) = -2.0

Example: High-Low

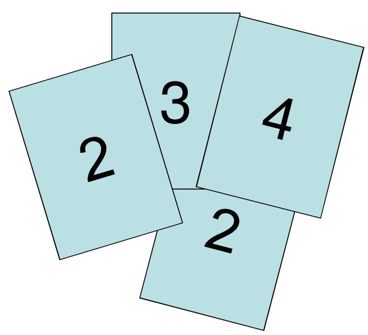
- Three card types: 2, 3, 4
- Infinite deck, twice as many 2's
- Start with 3 showing
- After each card, you say "high" or "low"
- New card is flipped
- If you're right, you win the points shown on the new card
- Ties are no-ops
- If you're wrong, game ends



- Differences from expectimax:
  - #1: get rewards as you go
  - #2: you might play forever!

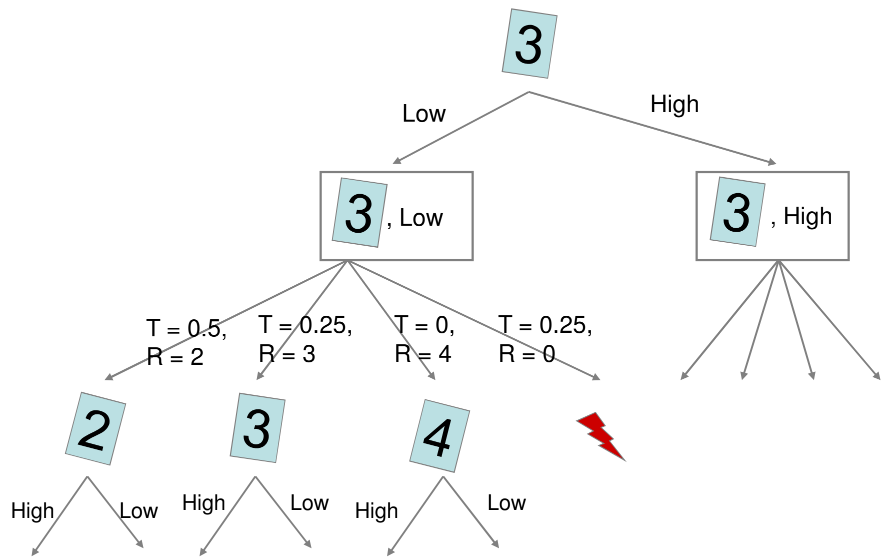
High-Low as an MDP

- States: 2, 3, 4, done
- Actions: High, Low
- Model: T(s, a, s'):
  - P(s'=4 | 4, Low) = 1/4
  - P(s'=3 | 4, Low) = 1/4
  - P(s'=2 | 4, Low) = 1/2
  - P(s'=done | 4, Low) = 0
  - P(s'=4 | 4, High) = 1/4
  - P(s'=3 | 4, High) = 0
  - P(s'=2 | 4, High) = 0
  - P(s'=done | 4, High) = 3/4
  - …



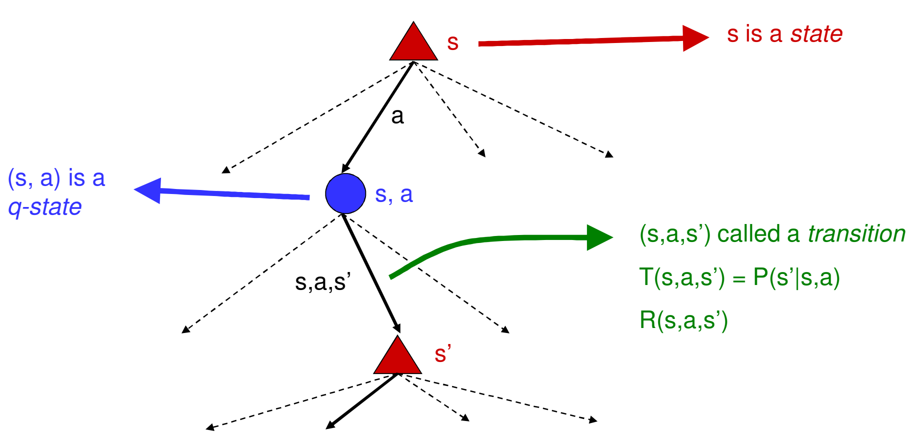
- Rewards: R(s, a, s'):
  - Number shown on s' if $s \neq s'$ and a is "correct"
  - 0 otherwise
- Start: 3

Example: High-Low



MDP Search Trees

- Each MDP state gives an expectimax-like search tree



Utilities of Sequences

- In order to formalize optimality of a policy, need to understand utilities of sequences of rewards
- Typically consider stationary preferences:

$$[r, r_0, r_1, r_2, \ldots] \succ [r, r'_0, r'_1, r'_2, \ldots]$$
$$\Leftrightarrow$$
$$[r_0, r_1, r_2, \ldots] \succ [r'_0, r'_1, r'_2, \ldots]$$

- Theorem: only two ways to define stationary utilities
  - Additive utility:

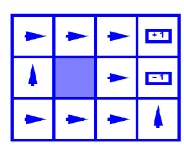
$$U([r_0, r_1, r_2, \ldots]) = r_0 + r_1 + r_2 + \cdots$$

  - Discounted utility:

$$U([r_0, r_1, r_2, \ldots]) = r_0 + \gamma r_1 + \gamma^2 r_2 \cdots$$

Infinite Utilities?!

- Problem: infinite state sequences have infinite rewards
- Solutions:



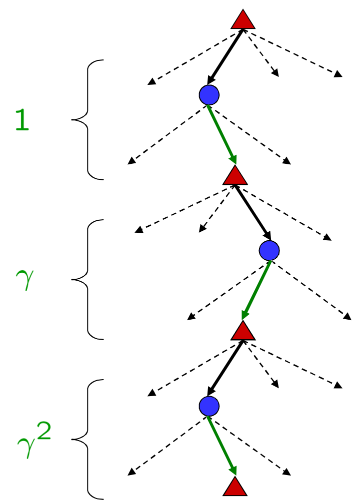
  - Finite horizon:
    - Terminate episodes after a fixed T steps (e.g. life)
    - Gives nonstationary policies (π depends on time left)
  - Absorbing state: guarantee that for every policy, a terminal state will eventually be reached (like "done" for High-Low)
  - Discounting: for $0 < \gamma < 1$

$$U([r_0, \ldots r_\infty]) = \sum_{t=0}^{\infty} \gamma^t r_t \leq R_{\max}/(1-\gamma)$$

    - Smaller γ means smaller "horizon" – shorter term focus

Discounting

- Typically discount rewards by $\gamma < 1$ each time step
  - Sooner rewards have higher utility than later rewards
  - Also helps the algorithms converge



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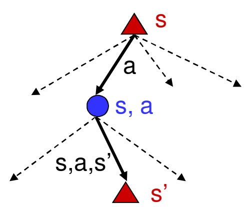
Recap: Defining MDPs

- Markov decision processes:
  - States S
  - Start state s_0
  - Actions A
  - Transitions P(s'|s,a) (or T(s,a,s'))
  - Rewards R(s,a,s') (and discount γ)



- MDP quantities so far:
  - Policy = Choice of action for each state
  - Utility (or return) = sum of discounted rewards

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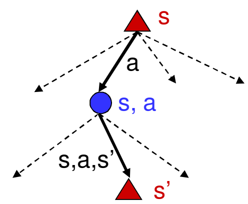
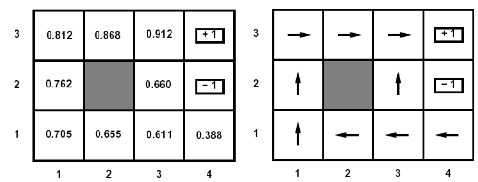
Optimal Utilities

- Fundamental operation: compute the values (optimal expectimax utilities) of states s
- Why? Optimal values define optimal policies!
- Define the value of a state s:
  $V^*(s)$ = expected utility starting in s and acting optimally
- Define the value of a q-state (s,a):
  $Q^*(s,a)$ = expected utility starting in s, taking action a and thereafter acting optimally
- Define the optimal policy:
  $\pi^*(s)$ = optimal action from state s





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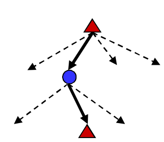
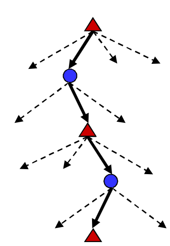
Value Estimates

- Calculate estimates $V_k^*(s)$
  - Not the optimal value of s!
  - The optimal value considering only next k time steps (k rewards)
  - As k $\to \infty$, it approaches the optimal value
- Almost solution: recursion (i.e. expectimax)
- Correct solution: dynamic programming

$V_1^*(s)$



$V_2^*(s)$



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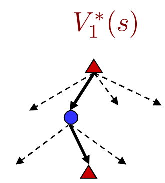
Value Iteration: V*$_1$



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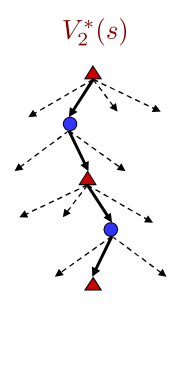
Value Iteration: V*$_2$



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Value Iteration V^*_{i+1}

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Value Iteration

- Idea:
  - $V_i^*(s)$: the expected discounted sum of rewards accumulated when starting from state s and acting optimally for a horizon of i time steps.
  - Start with $V_0^*(s) = 0$, which we know is right (why?)
  - Given V_i^*, calculate the values for all states for horizon i+1:

$$V_{i+1}^*(s) \leftarrow \max_a \sum_{s'} T(s,a,s') \left[R(s,a,s') + \gamma V_i^*(s') \right]$$

  - This is called a value update or Bellman update
  - Repeat until convergence
- Theorem: will converge to unique optimal values
  - Basic idea: approximations get refined towards optimal values
  - Policy may converge long before values do

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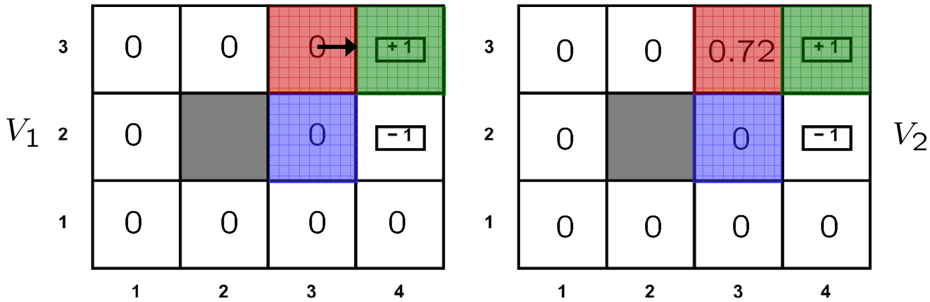
Example: γ=0.9, living reward=0, noise=0.2

Example: Bellman Updates



$$V_{i+1}(s) = \max_a \sum_{s'} T(s,a,s') \left[R(s,a,s') + \gamma V_i(s')\right]$$

$$V_2(\langle 3,3\rangle) = \sum_{s'} T(\langle 3,3\rangle, \text{right}, s') \left[R(\langle 3,3\rangle) + 0.9\, V_1(s')\right]$$

max happens for a=right, other actions not shown

$$= 0.9\,[0.8 \cdot 1 + 0.1 \cdot 0 + 0.1 \cdot 0]$$

Convergence*

- Define the max-norm: $||U|| = \max_s |U(s)|$
- Theorem: For any two approximations U and V

$$||U_{i+1} - V_{i+1}|| \leq \gamma\, ||U_i - V_i||$$

  - I.e. any distinct approximations must get closer to each other, so, in particular, any approximation must get closer to the true U and value iteration converges to a unique, stable, optimal solution
- Theorem:

$$||U_{i+1} - U_i|| < \epsilon, \;\Rightarrow\; ||U_{i+1} - U|| < 2\epsilon\gamma/(1-\gamma)$$

  - I.e. once the change in our approximation is small, it must also be close to correct

At Convergence

- At convergence, we have found the optimal value function V* for the discounted infinite horizon problem, which satisfies the Bellman equations:

$$\forall s \in S: \quad V^*(s) = \max_a \sum_{s'} T(s,a,s') \left[R(s,a,s') + \gamma V^*(s') \right]$$

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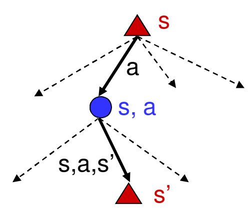
The Bellman Equations

- Definition of "optimal utility" leads to a simple one-step lookahead relationship amongst optimal utility values:

  Optimal rewards = maximize over first action and then follow optimal policy



- Formally:

$$V^*(s) = \max_a Q^*(s,a)$$

$$Q^*(s,a) = \sum_{s'} T(s,a,s') \left[R(s,a,s') + \gamma V^*(s') \right]$$

$$V^*(s) = \max_a \sum_{s'} T(s,a,s') \left[R(s,a,s') + \gamma V^*(s') \right]$$

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Practice: Computing Actions

- Which action should we chose from state s:
  - Given optimal values V?

$$\arg\max_a \sum_{s'} T(s,a,s')[R(s,a,s') + \gamma V^*(s')]$$

  - Given optimal q-values Q?

$$\arg\max_a Q^*(s,a)$$

  - Lesson: actions are easier to select from Q's!

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Complete Procedure

- 1. Run value iteration (off-line)

→Returns V, which (assuming sufficiently many iterations is a good approximation of V*)

- 2. Agent acts. At time t the agent is in state s_t and takes the action a_t:

$$\arg\max_a \sum_{s'} T(s_t,a,s')[R(s_t,a,s') + \gamma V^*(s')]$$

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Complete Procedure

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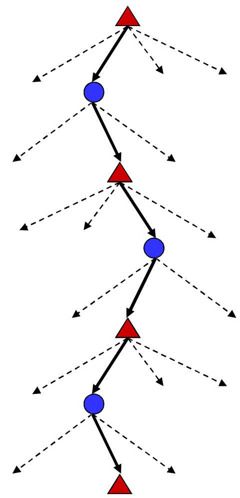
Why Not Search Trees?

- Why not solve with expectimax?
- Problems:
  - This tree is usually infinite (why?)
  - Same states appear over and over (why?)
  - We would search once per state (why?)
- Idea: Value iteration
  - Compute optimal values for all states all at once using successive approximations
  - Will be a bottom-up dynamic program similar in cost to memoization
  - Do all planning offline, no replanning needed!



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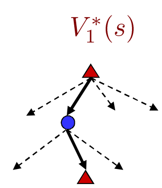
Expectimax vs. Value Iteration: V_1*



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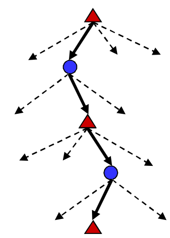
Expectimax vs. Value Iteration: V_2^*



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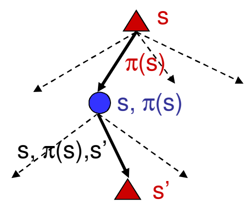
Utilities for Fixed Policies

- Another basic operation: compute the utility of a state s under a fix (general non-optimal) policy



- Define the utility of a state s, under a fixed policy π:
  - $V^{\pi}(s)$ = expected total discounted rewards (return) starting in s and following π
- Recursive relation (one-step look-ahead / Bellman equation):

$$V^{\pi}(s) = \sum_{s'} T(s, \pi(s), s')[R(s, \pi(s), s') + \gamma V^{\pi}(s')]$$

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Policy Evaluation

- How do we calculate the V's for a fixed policy?
- Idea one: modify Bellman updates

$$V_0^{\pi}(s) = 0$$

$$V_{i+1}^{\pi}(s) \leftarrow \sum_{s'} T(s, \pi(s), s')[R(s, \pi(s), s') + \gamma V_i^{\pi}(s')]$$

- Idea two: it's just a linear system, solve with Matlab (or whatever)

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Policy Iteration

- Alternative approach:
  - Step 1: Policy evaluation: calculate utilities for some fixed policy (not optimal utilities!) until convergence
  - Step 2: Policy improvement: update policy using one-step look-ahead with resulting converged (but not optimal!) utilities as future values
  - Repeat steps until policy converges
- This is policy iteration
  - It's still optimal!
  - Can converge faster under some conditions

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Policy Iteration

- Policy evaluation: with fixed current policy π, find values with simplified Bellman updates:
  - Iterate until values converge

$$V_{i+1}^{\pi_k}(s) \leftarrow \sum_{s'} T(s, \pi_k(s), s') \left[R(s, \pi_k(s), s') + \gamma V_i^{\pi_k}(s')\right]$$

- Policy improvement: with fixed utilities, find the best action according to one-step look-ahead

$$\pi_{k+1}(s) = \arg\max_a \sum_{s'} T(s, a, s') \left[R(s, a, s') + \gamma V^{\pi_k}(s')\right]$$

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Comparison

- In value iteration:
  - Every pass (or "backup") updates both utilities (explicitly, based on current utilities) and policy (possibly implicitly, based on current policy)
- In policy iteration:
  - Several passes to update utilities with frozen policy
  - Occasional passes to update policies
- Hybrid approaches (asynchronous policy iteration):
  - Any sequences of partial updates to either policy entries or utilities will converge if every state is visited infinitely often

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Asynchronous Value Iteration*

- In value iteration, we update every state in each iteration
- Actually, *any* sequences of Bellman updates will converge if every state is visited infinitely often
- In fact, we can update the policy as seldom or often as we like, and we will still converge
- Idea: Update states whose value we expect to change: If $|V_{i+1}(s) - V_i(s)|$ is large then update predecessors of s

MDPs recap

- Markov decision processes:
  - States S
  - Actions A
  - Transitions P(s'|s,a) (or T(s,a,s'))
  - Rewards R(s,a,s') (and discount γ)
  - Start state s_0
- Solution methods:
  - Value iteration (VI)
  - Policy iteration (PI)
  - Asynchronous value iteration
- Current limitations:
  - Relatively small state spaces
  - Assumes T and R are known