CS 188
Spring 2019

# Section Handout 10 Solutions

# CS188 Spring 2019 Section 10: Decision Networks and Naive Bayes

## Decision Networks

- **Chance nodes** - Chance nodes in a decision network behave identically to Bayes' nets. Each outcome in a chance node has an associated probability, which can be determined by running inference on the underlying Bayes' net it belongs to. We'll represent these with ovals.
- **Action nodes** - Action nodes are nodes that we have complete control over; they're nodes representing a choice between any of a number of actions which we have the power to choose from. We'll represent action nodes with rectangles.
- **Utility nodes** - Utility nodes are children of some combination of action and chance nodes. They output a utility based on the values taken on by their parents, and are represented as diamonds in our decision networks.

The **expected utility** of taking an action $A = a$ given evidence $E = e$ and $n$ chance nodes is computed with the following formula:

$$EU(A = a|E = e) = \sum_{x_1,...,x_n} P(X_1 = x_1, ..., X_n = x_n|E = e)U(A = a, X_1 = x_1, ..., X_n = x_n)$$

where each $x_i$ represents a value that the $i^{th}$ chance node can take on. The **maximum expected utility** is the expected utility of the action that has the highest expected utility:

$$MEU(E = e) = \max_a EU(A = a|E = e).$$

## Value of Perfect Information

**Value of perfect information** (VPI) quantifies the amount an agent's maximum expected utility is expected to increase if it were to observe some new evidence. Usually observing new evidence comes at a cost. If we observed some new evidence $E' = e'$ before acting, the maximum expected utility of our action at that point would become

$$MEU(E = e, E' = e') = \max_a \sum_x P(X = x|E = e, E' = e')U(X = x, A = a).$$

However, note that *we don't know what new evidence we'll get*. Because we don't know what what new evidence $e'$ we'll get, we must represent it as a random variable $E'$. We will compute the expected value of the maximum expected utility:

$$MEU(E = e, E') = \sum_{e'} P(E' = e'|E = e)MEU(E = e, E' = e').$$

Observing a new evidence variable yields a different MEU with probabilities corresponding to the probabilities of observing each value for the evidence variable, and so by computing $MEU(E = e, E')$ as above, we compute what we expect our new MEU will be if we choose to observe new evidence. The VPI is the expected maximum expected utility if we were to observe the new evidence, minus the maximum expected utility if we were not to observe the new evidence:

$$VPI(E'|E = e) = MEU(E = e, E') - MEU(E = e).$$

## Properties of VPI

The value of perfect information has several very important properties, namely:

- **Nonnegativity.** $\forall E', e \ VPI(E'|E=e) \geq 0$
  Observing new information always allows you to make a *more informed* decision, and so your maximum expected utility can only increase (or stay the same if the information is irrelevant for the decision you must make).

- **Nonadditivity.** $VPI(E_j, E_k|E=e) \neq VPI(E_j|E=e) + VPI(E_k|E=e)$ in general.
  This is probably the trickiest of the three properties to understand intuitively. It's true because generally observing some new evidence $E_j$ might change how much we care about $E_k$; therefore we can't simply add the VPI of observing $E_j$ to the VPI of observing $E_k$ to get the VPI of observing both of them. Rather, the VPI of observing two new evidence variables is equivalent to observing one, incorporating it into our current evidence, then observing the other. This is encapsulated by the order-independence property of VPI, described more below.

- **Order-independence.** $VPI(E_j, E_k|E=e) = VPI(E_j|E=e) + VPI(E_k|E=e, E_j) = VPI(E_k|E=e) + VPI(E_j|E=e, E_k)$
  Observing multiple new evidences yields the same gain in maximum expected utility regardless of the order of observation. This should be a fairly straightforward assumption - because we don't actually take any action until after observing any new evidence variables, it doesn't actually matter whether we observe the new evidence variables together or in some arbitrary sequential order.

# Naive Bayes

## Prediction

$$\mathit{prediction}(x_1, ..., x_n) = \arg\max_y P(Y=y) \prod_{i=1}^{n} P(X_i = x_i|Y=y)$$

## Parameter Estimation

Given sample your **maximum likelihood** estimate for an outcome $x$ that can take on $|X|$ different values from a sample of size $N$ is

$$P_{MLE}(x) = \frac{\mathit{count}(x)}{N}.$$

With **Laplace smoothing,** the Laplace estimate with strength $k$ is

$$P_{LAP,k}(x) = \frac{\mathit{count}(x) + k}{N + k|X|}.$$

A similar result holds for computing Laplace estimates for conditionals (which is useful for computing Laplace estimates for outcomes across different classes):

$$P_{LAP,k}(x|y) = \frac{\mathit{count}(x,y) + k}{\mathit{count}(y) + k|X|}.$$

There are two particularly notable cases for Laplace smoothing. The first is when $k=0$, then $P_{LAP,0}(x) = P_{MLE}(x)$. The second is the case where $k=\infty$. Observing a very large, infinite number of each outcome makes the results of your actual sample inconsequential and so your Laplace estimates imply that each outcome is equally likely. Indeed:

$$P_{LAP,\infty}(x) = \frac{1}{|X|}$$

# Q1. Utilities

Davis is on his way to a final exam planning meeting. He is already running late (the meeting is starting now) and he's trying to determine whether he should wait for the bus or just walk.

It takes 20 minutes to get to Cory Hall by walking, and only 5 minutes to get to there by bus. The bus will either come in 10, 20, or 30 minutes, each with probability 1/3.

**(a)** Davis hates being late; his utility for being late as a function of $t$, the number of minutes late he is, is

$$U_D(t) = \begin{cases} 0 & : t \leq 0 \\ -2^{t/5} & : t > 0 \end{cases}$$

What is the expected utility of each action? Should he wait for the bus or walk?

$$\begin{aligned} EU(walk) &= -2^{20/5} = -16 \\ EU(bus) &= \frac{1}{3}(-2^{(10+5)/5} - 2^{(20+5)/5} - 2^{(30+5)/5}) \\ &= \frac{1}{3}(-8 - 32 - 128) = -168/3 = -56 \end{aligned}$$

Davis should walk.

**(b)** Pat is running late too. However, Pat reasons that once he's late, it doesn't matter how late he is. Therefore, his utility function is

$$U_P(t) = \begin{cases} 0 & : t \leq 0 \\ -10 & : t > 0 \end{cases}$$

Moreover, Pat prefers riding the bus because it is more comfortable, so riding the bus incurs a utility bonus of 5.

If Pat is deciding whether to take the bus or walk when the meeting is just starting, what are his expected utilities for each action? Should he take the bus or walk?

$$\begin{aligned} EU(walk) &= -10 \\ EU(bus) &= -10 + 5 = -5 \end{aligned}$$

Pat should take the bus.

**(c)** Give an example of a decreasing utility function in terms of time such that it will favor decisions that always minimize expected time to get to the meeting.

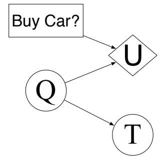
$U(t) = -t$. Any decreasing linear function of $t$ is correct.

**(d)** Give an example of a decreasing utility function in terms of time such that it will be risk-seeking; that is, a lottery with expected time of arrival $t$ will be preferred to a guarantee of arrival time $t$.

$U(t) = -\sqrt{t}$. Any decreasing function with a positive second derivative (concave up) is correct.

# 2 Decision Networks and VPI

A used car buyer can decide to carry out various tests with various costs (e.g., kick the tires, take the car to a qualified mechanic) and then, depending on the outcome of the tests, decide which car to buy. We will assume that the buyer is deciding whether to buy car $c$ and that there is time to carry out at most one test which costs \$50 and which can help to figure out the quality of the car. A car can be in good shape (of good quality $Q = +q$) or in bad shape (of bad quality Q=¬q), and the test might help to indicate what shape the car is in. There are only two outcomes for the test T: pass (T=pass) or fail (T=fail). Car $c$ costs \$1,500, and its market value is \$2,000 if it is in good shape; if not, \$700 in repairs will be needed to make it in good shape. The buyer's estimate is that $c$ has 70% chance of being in good shape. The Decision Network is shown below.




**(a)** Calculate the expected net gain from buying car c, given no test.

$$\begin{aligned} EU(\text{buy}) &= P(Q=+q)\cdot U(+q,\text{buy}) + P(Q=\neg q)\cdot U(\neg q, buy) \\ &= .7\cdot 500 + 0.3\cdot -200 = 290 \end{aligned}$$

**(b)** Tests can be described by the probability that the car will pass or fail the test given that the car is in good or bad shape. We have the following information:

$$P(T=\text{pass}|Q=+q) = 0.9$$
$$P(T=\text{pass}|Q=\neg q) = 0.2$$

Calculate the probability that the car will pass (or fail) its test, and then the probability that it is in good (or bad) shape given each possible test outcome.

$$\begin{aligned} P(T=\text{pass}) &= \sum_q P(T=\text{pass}, Q=q) \\ &= P(T=\text{pass}|Q=+q)P(Q=+q) + P(T=\text{pass}|Q=\neg q)P(Q=\neg q) \\ &= 0.69 \\ P(T=fail) &= 0.31 \\ P(Q=+q|T=\text{pass}) &= \frac{P(T=\text{pass}|Q=+q)P(Q=+q)}{P(T=\text{pass})} \\ &= \frac{0.9\cdot 0.7}{0.69} = \frac{21}{23} \approx 0.91 \\ P(Q=+q|T=\text{fail}) &= \frac{P(T=\text{fail}|Q=+q)P(Q=+q)}{P(T=\text{fail})} \\ &= \frac{0.1\cdot 0.7}{0.31} = \frac{7}{31} \approx 0.22 \end{aligned}$$

**(c)** Calculate the optimal decisions given either a pass or a fail, and their expected utilities.

$$\begin{aligned} EU(\text{buy}|T=pass) &= P(Q=+q|T=\text{pass})U(+q,\text{buy}) + P(Q=\neg q|T=\text{pass})U(\neg q,\text{buy}) \\ &\approx 0.91\cdot 500 + 0.09\cdot(-200) \approx 437 \\ EU(\text{buy}|T=\text{fail}) &= P(Q=+q|T=\text{fail})U(+q,\text{buy}) + P(Q=\neg q|T=\text{fail})U(\neg q,\text{buy}) \\ &\approx 0.22\cdot 500 + 0.78\cdot(-200) = -46 \\ EU(\neg\text{buy}|T=pass) &= 0 \\ EU(\neg\text{buy}|T=fail) &= 0 \end{aligned}$$

Therefore: $MEU(T=\text{pass}) = 437$ (with buy) and $MEU(T=\text{fail}) = 0$ (using $\neg$buy)

**(d)** Calculate the value of (perfect) information of the test. Should the buyer pay for a test?

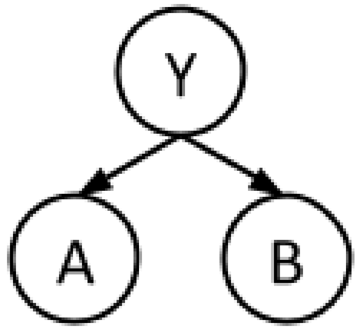
$$\begin{aligned} VPI(T) &= (\sum_t P(T=t)MEU(T=t)) - MEU(\phi) \\ &= 0.69\cdot 437 + 0.31\cdot 0 - 290 \approx 11.53 \end{aligned}$$

You shouldn't pay for it, since the cost is \$50.

# 3 Naive Bayes

In this question, we will train a Naive Bayes classifier to predict class labels $Y$ as a function of input features $A$ and $B$. $Y$, $A$, and $B$ are all binary variables, with domains 0 and 1. We are given 10 training points from which we will estimate our distribution.

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $A$ | 1 | 1 | 1 | 1 | 0 | 1 | 0 | 1 | 1 | 1 |
| $B$ | 1 | 0 | 0 | 1 | 1 | 1 | 1 | 0 | 1 | 1 |
| $Y$ | 1 | 1 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 |




**(a)** What are the maximum likelihood estimates for the tables $P(Y)$, $P(A|Y)$, and $P(B|Y)$?

| $Y$ | $P(Y)$ |
|---|---|
| 0 | 3/5 |
| 1 | 2/5 |

| $A$ | $Y$ | $P(A\|Y)$ |
|---|---|---|
| 0 | 0 | 1/6 |
| 1 | 0 | 5/6 |
| 0 | 1 | 1/4 |
| 1 | 1 | 3/4 |

| $B$ | $Y$ | $P(B\|Y)$ |
|---|---|---|
| 0 | 0 | 1/3 |
| 1 | 0 | 2/3 |
| 0 | 1 | 1/4 |
| 1 | 1 | 3/4 |

**(b)** Consider a new data point ($A = 1$, $B = 1$). What label would this classifier assign to this sample?

$$
\begin{aligned}
P(Y=0, A=1, B=1) &= P(Y=0)P(A=1|Y=0)P(B=1|Y=0) && (1)\\
&= (3/5)(5/6)(2/3) && (2)\\
&= 1/3 && (3)\\
P(Y=1, A=1, B=1) &= P(Y=1)P(A=1|Y=1)P(B=1|Y=1) && (4)\\
&= (2/5)(3/4)(3/4) && (5)\\
&= 9/40 && (6)\\
& && (7)
\end{aligned}
$$

Our classifier will predict label 0.

**(c)** Let's use Laplace Smoothing to smooth out our distribution. Compute the new distribution for $P(A|Y)$ given Laplace Smoothing with $k = 2$.

| $A$ | $Y$ | $P(A\|Y)$ |
|---|---|---|
| 0 | 0 | 3/10 |
| 1 | 0 | 7/10 |
| 0 | 1 | 3/8 |
| 1 | 1 | 5/8 |