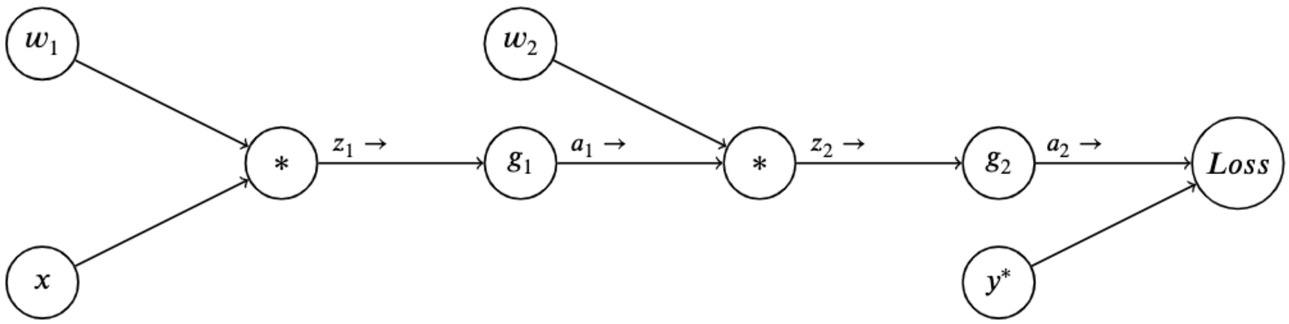
Q1. Neural Nets

Consider the following computation graph for a simple neural network for binary classification. Here x is a single real-valued input feature with an associated class $y^\star$ (0 or 1). There are two weight parameters w_1 and w_2, and non-linearity functions g_1 and g_2 (to be defined later, below). The network will output a value a_2 between 0 and 1, representing the probability of being in class 1. We will be using a loss function $Loss$ (to be defined later, below), to compare the prediction a_2 with the true class $y^\star$.



1. Perform the forward pass on this network, writing the output values for each node z_1, a_1, z_2 and a_2 in terms of the node's input values:

$$z_1 = x * w_1$$
$$a_1 = g_1(z_1)$$
$$z_2 = a_1 * w_2$$
$$a_2 = g_2(z_2)$$

2. Compute the loss $Loss(a_2, y^\star)$ in terms of the input x, weights w_i, and activation functions g_i:

   Recursively substituting the values computed above, we have:

$$Loss(a_2, y^\star) = Loss(g_2(w_2 * g_1(w_1 * x)), y^\star)$$

3. Now we will work through parts of the backward pass, incrementally. Use the chain rule to derive $\frac{\partial Loss}{\partial w_2}$. Write your expression as a product of partial derivatives at each node: i.e. the partial derivative of the node's output with respect to its inputs. (Hint: the series of expressions you wrote in part 1 will be helpful; you may use any of those variables.)

$$\frac{\partial Loss}{\partial w_2} = \frac{\partial Loss}{\partial a_2}\frac{\partial a_2}{\partial z_2}\frac{\partial z_2}{\partial w_2}$$

4. Suppose the loss function is quadratic, $Loss(a_2, y^\star) = \frac{1}{2}(a_2 - y^\star)^2$, and g_1 and g_2 are both sigmoid functions $g(z) = \frac{1}{1+e^{-z}}$ (note: it's typically better to use a different type of loss, *cross-entropy*, for classification problems, but we'll use this to make the math easier).

   Using the chain rule from Part 3, and the fact that $\frac{\partial g(z)}{\partial z} = g(z)(1 - g(z))$ for the sigmoid function, write $\frac{\partial Loss}{\partial w_2}$ in terms of the values from the forward pass, $y^\star$, a_1, and a_2:

   First we'll compute the partial derivatives at each node:

$$\frac{\partial Loss}{\partial a_2} = (a_2 - y^\star)$$
$$\frac{\partial a_2}{\partial z_2} = \frac{\partial g_2(z_2)}{\partial z_2} = g_2(z_2)(1 - g_2(z_2)) = a_2(1 - a_2)$$
$$\frac{\partial z_2}{\partial w_2} = a_1$$

   Now we can plug into the chain rule from part 3:

$$\frac{\partial Loss}{\partial w_2} = \frac{\partial Loss}{\partial a_2}\frac{\partial a_2}{\partial z_2}\frac{\partial z_2}{\partial w_2}$$
$$= (a_2 - y^\star) * a_2(1 - a_2) * a_1$$

5. Now use the chain rule to derive $\frac{\partial Loss}{\partial w_1}$ as a product of partial derivatives at each node used in the chain rule:

$$\frac{\partial Loss}{\partial w_1} = \frac{\partial Loss}{\partial a_2}\frac{\partial a_2}{\partial z_2}\frac{\partial z_2}{\partial a_1}\frac{\partial a_1}{\partial z_1}\frac{\partial z_1}{\partial w_1}$$

6. Finally, write $\frac{\partial Loss}{\partial w_1}$ in terms of $x, y^\star, w_i, a_i, z_i$: The partial derivatives at each node (in addition to the ones we computed in Part 4) are:

$$\frac{\partial z_2}{\partial a_1} = w_2$$
$$\frac{\partial a_1}{\partial z_1} = \frac{\partial g_1(z_1)}{\partial z_1} = g_1(z_1)(1 - g_1(z_1)) = a_1(1 - a_1)$$
$$\frac{\partial z_1}{\partial a_1} = x$$

   Plugging into the chain rule from Part 5 gives:

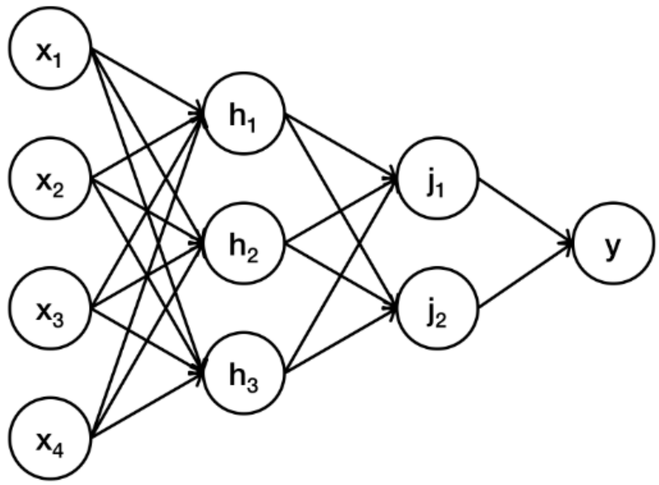
$$\frac{\partial Loss}{\partial w_1} = \frac{\partial Loss}{\partial a_2}\frac{\partial a_2}{\partial z_2}\frac{\partial z_2}{\partial a_1}\frac{\partial a_1}{\partial z_1}\frac{\partial z_1}{\partial w_1}$$
$$= (a_2 - y^\star) * a_2(1 - a_2) * w_2 * a_1(1 - a_1) * x$$

7. What is the gradient descent update for w_1 with step-size α in terms of the values computed above?

$$w_1 \leftarrow w_1 - \alpha(a_2 - y^\star) * a_2(1 - a_2) * w_2 * a_1(1 - a_1) * x$$

Q2. Neural Network Data Sufficiency

The next few problems use the below neural network as a reference. Neurons h_{1-3} and j_{1-2} all use ReLU activation functions. Neuron y uses the identity activation function: $f(x) = x$. In the questions below, let $w_{a,b}$ denote the weight that connects neurons a and b. Also, let o_a denote the value that neuron a outputs to its next layer.



Given this network, in the following few problems, you have to decide whether the data given are sufficient for answering the question.

(a) Given the above neural network, what is the value of o_y?

Data item 1: the values of all weights in the network and the values $o_{h_1}, o_{h_2}, o_{h_3}$
Data item 2: the values of all weights in the network and the values o_{j_1}, o_{j_2}

○ Data item (1) alone is sufficient, but data item (2) alone is not sufficient to answer the question.
○ Data item (2) alone is sufficient, but data item (1) alone is not sufficient to answer the question.
○ Both statements taken together are sufficient, but neither data item alone is sufficient.
● Each data item alone is sufficient to answer the question.
○ Statements (1) and (2) together are not sufficient, and additional data is needed to answer the question.

(b) Given the above neural network, what is the value of o_{h_1}?

Data item 1: the neuron input values, i.e., o_{x_1} through o_{x_4}
Data item 2: the values o_{j_1}, o_{j_2}

○ Data item (1) alone is sufficient, but data item (2) alone is not sufficient to answer the question.
○ Data item (2) alone is sufficient, but data item (1) alone is not sufficient to answer the question.
○ Both statements taken together are sufficient, but neither data item alone is sufficient.
○ Each data item alone is sufficient to answer the question.
● Statements (1) and (2) together are not sufficient, and additional data is needed to answer the question.

(c) Given the above neural network, what is the value of o_{j_1}?

Data item 1: the values of all weights connecting neurons h_1, h_2, h_3 to j_1, j_2
Data item 2: the values $o_{h_1}, o_{h_2}, o_{h_3}$

○ Data item (1) alone is sufficient, but data item (2) alone is not sufficient to answer the question.
○ Data item (2) alone is sufficient, but data item (1) alone is not sufficient to answer the question.
● Both statements taken together are sufficient, but neither data item alone is sufficient.
○ Each data item alone is sufficient to answer the question.
○ Statements (1) and (2) together are not sufficient, and additional data is needed to answer the question.

(d) Given the above neural network, what is the value of $\partial o_y / \partial w_{j_2,y}$?

Data item 1: the value of o_{j_2}
Data item 2: all weights in the network and the neuron input values, i.e., o_{x_1} through o_{x_4}

- ○ Data item (1) alone is sufficient, but data item (2) alone is not sufficient to answer the question.
- ○ Data item (2) alone is sufficient, but data item (1) alone is not sufficient to answer the question.
- ○ Both statements taken together are sufficient, but neither data item alone is sufficient.
- ● Each data item alone is sufficient to answer the question.
- ○ Statements (1) and (2) together are not sufficient, and additional data is needed to answer the question.

(e) Given the above neural network, what is the value of $\partial o_y / \partial w_{h_2,j_2}$?

Data item 1: the value of $w_{j_2,y}$
Data item 2: the value of $\partial o_{j_2} / \partial w_{h_2,j_2}$

- ○ Data item (1) alone is sufficient, but data item (2) alone is not sufficient to answer the question.
- ○ Data item (2) alone is sufficient, but data item (1) alone is not sufficient to answer the question.
- ● Both statements taken together are sufficient, but neither data item alone is sufficient.
- ○ Each data item alone is sufficient to answer the question.
- ○ Statements (1) and (2) together are not sufficient, and additional data is needed to answer the question.

(f) Given the above neural network, what is the value of $\partial o_y / \partial w_{x_1,h_3}$?

Data item 1: the value of all weights in the network and the neuron input values, i.e., o_{x_1} through o_{x_4}
Data item 2: the value of w_{x_1,h_3}

- ● Data item (1) alone is sufficient, but data item (2) alone is not sufficient to answer the question.
- ○ Data item (2) alone is sufficient, but data item (1) alone is not sufficient to answer the question.
- ○ Both statements taken together are sufficient, but neither data item alone is sufficient.
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