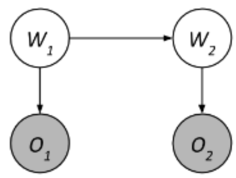
MT Review: HMMs Solutions

HMMs

State variables W_t and observation (evidence) variables (O_t), which are supposed to be shaded below. Transition model $P(W_{t+1}|W_t)$. Sensor model $P(O_t|W_t)$. The joint distribution of the HMM can be factorized as

$$P(W_1, ..., W_T, O_1, ...O_T) = P(W_1) \prod_{t=1}^{T-1} P(W_{t+1}|W_t) \prod_{t=1}^{T} P(O_t|W_t) \quad (1)$$



Define the following belief distribution

- $B(W_t) = P(W_t|O_1, ..., O_t)$: Belief about state W_t given all the observations up until (and including) timestep t.
- $B'(W_t) = P(W_t|O_1, ..., O_{t-1})$: Belief about state W_t given all the observations up until (but not including) timestep t.

Forward Algorithm

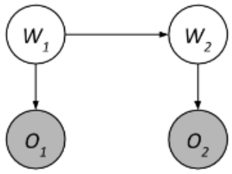
- For each state at time t, keep track of the total probability of all paths to it.
- *Prediction update*: $B'(W_{t+1}) = \sum_{w_t} P(W_{t+1}|w_t)B(w_t)$
- *Observation update*: $B(W_{t+1}) \propto P(O_{t+1}|W_{t+1})B'(W_{t+1})$

Viterbi Algorithm

- For each state at time t, keep track of the maximum probability of any path to it.
- Solve for $argmax_{w_{1:t}} P(w_{1:t}|o_{1:t})$ via forward and backward pass.

1 HMMs

Consider the following Hidden Markov Model. O_1 and O_2 are supposed to be shaded.



W_1	$P(W_1)$
0	0.3
1	0.7

W_t	W_{t+1}	$P(W_{t+1}\|W_t)$
0	0	0.4
0	1	0.6
1	0	0.8
1	1	0.2

W_t	O_t	$P(O_t\|W_t)$
0	a	0.9
0	b	0.1
1	a	0.5
1	b	0.5

Suppose that we observe $O_1 = a$ and $O_2 = b$.
Using the forward algorithm, compute the probability distribution $P(W_2|O_1 = a, O_2 = b)$ one step at a time.

(a) Compute $P(W_1, O_1 = a)$.

$P(W_1, O_1 = a) = P(W_1)P(O_1 = a|W_1)$
$P(W_1 = 0, O_1 = a) = (0.3)(0.9) = 0.27$
$P(W_1 = 1, O_1 = a) = (0.7)(0.5) = 0.35$

(b) Using the previous calculation, compute $P(W_2, O_1 = a)$.

$P(W_2, O_1 = a) = \sum_{w_1} P(w_1, O_1 = a)P(W_2|w_1)$
$P(W_2 = 0, O_1 = a) = (0.27)(0.4) + (0.35)(0.8) = 0.388$
$P(W_2 = 1, O_1 = a) = (0.27)(0.6) + (0.35)(0.2) = 0.232$

(c) Using the previous calculation, compute $P(W_2, O_1 = a, O_2 = b)$.

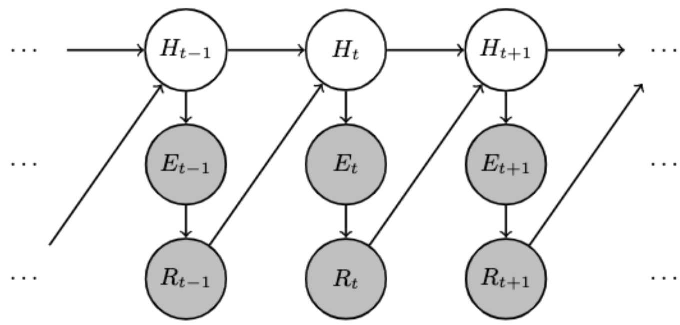
$P(W_2, O_1 = a, O_2 = b) = P(W_2, O_1 = a)P(O_2 = b|W_2)$
$P(W_2 = 0, O_1 = a, O_2 = b) = (0.388)(0.1) = 0.0388$
$P(W_2 = 1, O_1 = a, O_2 = b) = (0.232)(0.5) = 0.116$

(d) Finally, compute $P(W_2|O_1 = a, O_2 = b)$.

Renormalizing the distribution above, we have
$P(W_2 = 0|O_1 = a, O_2 = b) = 0.0388/(0.0388 + 0.116) \approx 0.25$
$P(W_2 = 1|O_1 = a, O_2 = b) = 0.116/(0.0388 + 0.116) \approx 0.75$

Q2. HMM: Human-Robot Interaction

In the near future, autonomous robots would live among us. Therefore, it is important for the robots to know how to properly act in the presence of humans. In this question, we are exploring a simplified model of this interaction. Here, we are assuming that we can observe the robot's actions at time t, R_t, and an evidence observation, E_t, directly caused by the human action, H_t. Human's actions and Robot's actions from the past time-step affect the Human's and Robot's actions in the next time-step. In this problem, we will remain consistent with the convention that capital letters (H_t) refer to random variables and lowercase letters (h_t) refer to a particular value the random variable can take. The structure is given below:



You are supplied with the following probability tables: $P(R_t \mid E_t)$, $P(H_t \mid H_{t-1}, R_{t-1})$, $P(H_0)$, $P(E_t \mid H_t)$.

Let us derive the forward algorithm for this model. We will split our computation into two components, a **time-elapse update** expression and a **observe update** expression.

(a) We would like to incorporate the evidence that we observe at time t. Using the time-elapse update expression we will derive separately, we would like to find the **observe update** expression:

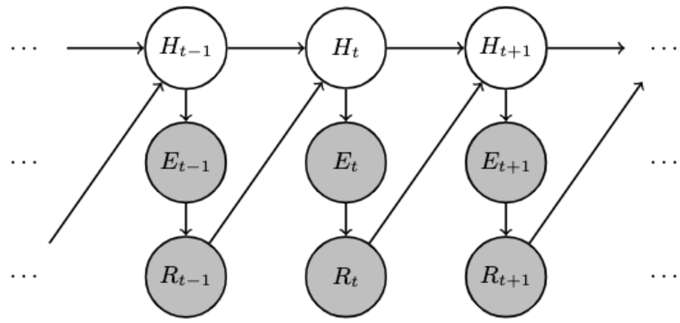
$$O(H_t) = P(H_t|e_{0:t}, r_{0:t})$$

In other words, we would like to compute the distribution of potential human states at time t given all observations up to and including time t. In addition to the conditional probability tables associated with the network's nodes, we are given $T(H_t) = P(H_t \mid e_{0:t-1}, r_{0:t-1})$, which we will assume is correctly computed in the time-elapse update that we will derive in the next part. From the options below, select *all* the options that **both** make valid independence assumptions and would evaluate to the observe update expression.

- [x] $\frac{P(H_t|e_{0:t-1}, r_{0:t-1})P(e_t|H_t)P(r_t|e_t)}{\sum_{h_t} P(h_t|e_{0:t-1}, r_{0:t-1})P(e_t|h_t)P(r_t|e_t)}$
- [x] $\frac{P(H_t|e_{0:t-1}, r_{0:t-1})P(e_t|H_t)}{\sum_{h_t} P(h_t|e_{0:t-1}, r_{0:t-1})P(e_t|h_t)}$
- [ ] $\frac{\sum_{e_t} P(H_t|e_{0:t-1}, r_{0:t-1})P(e_t|H_t)}{\sum_{h_t} P(h_t|e_{0:t-1}, r_{0:t-1})P(e_t|r_{t-1}, H_{t-1})}$
- [ ] $\sum_{r_{t-1}} P(H_t \mid e_{0:t-1}, r_{0:t-1})P(r_{t-1} \mid e_{t-1})$
- [ ] $\sum_{r_t} P(H_t \mid e_{0:t-1}, r_{0:t-1})P(r_t \mid r_{t-1}, e_t)$
- [ ] $\sum_{h_{t+1}} P(H_t \mid e_{0:t-1}, r_{0:t-1})P(h_{t+1} \mid r_t)$

$$P(H_t|e_{0:t}, r_{0:t}) = \frac{P(H_t, e_{0:t}, r_{0:t})}{\sum_{h_t} P(h_t, e_{0:t}, r_{0:t})} = \frac{P(H_t \mid e_{0:t-1}, r_{0:t-1})P(e_t \mid H_t)P(r_t \mid e_t)}{P(r_t \mid e_t)\sum_{h_t} P(h_t \mid e_{0:t-1}, r_{0:t-1})P(e_t \mid h_t)}$$

The structure below is identical to the one in the beginning of the question and is repeated for your convenience.



(b) We are interested in predicting what the state of human is at time t (H_t), given all the observations through $t-1$. Therefore, the **time-elapse update** expression has the following form:

$$T(H_t) = P(H_t|e_{0:t-1}, r_{0:t-1})$$

Derive an expression for the given time-elapse update above using the probability tables provided in the question and the observe update expression, $O(H_{t-1}) = P(H_{t-1}|e_{0:t-1}, r_{0:t-1})$. Write your final expression in the space provided at below. You may use the function O in your solution if you prefer.

The derivation of the time-elapse update for this setup is similar to the one we have seen in lecture; however, here, we have additional observations and dependencies.

$$\begin{aligned} P(H_t|e_{0:t-1}, r_{0:t-1}) &= \sum_{h_{t-1}} P(H_t, h_{t-1} \mid e_{0:t-1}, r_{0:t-1}) \\ &= \sum_{h_{t-1}} P(H_t \mid h_{t-1}, r_{t-1}) P(h_{t-1} \mid e_{0:t-1}, r_{0:t-1}) \end{aligned}$$

$P(H_t|e_{0:t-1}, r_{0:t-1}) =$ $\sum_{h_{t-1}} P(H_t \mid h_{t-1}, r_{t-1}) P(h_{t-1} \mid e_{0:t-1}, r_{0:t-1})$