CS 188
Spring 2021

Regular Discussion 4 Solutions

Q1. Propositional Logic

(a) Provide justification for whether each of the following are correct or incorrect.

(i) $(X \vee Y) \models Y$

Incorrect: $(X \vee Y) \models Y$ if and only if $(X \vee Y) \wedge \neg Y$ is unsatisfiable; however, the latter is satisfied by $X = true$ and $Y = false$.

(ii) $\neg X \vee (Y \wedge Z) \models (X \implies Y)$

Correct: Via the same reasoning as the previous part, we can attempt to show that $(\neg X \vee (Y \wedge Z)) \wedge \neg(X \implies Y)$ is unsatisfiable as follows:

- $(\neg X \vee (Y \wedge Z)) \wedge \neg(X \implies Y)$
- $(\neg X \vee (Y \wedge Z)) \wedge \neg(\neg X \vee Y)$
- $(\neg X \vee (Y \wedge Z)) \wedge (X \wedge \neg Y)$

Its clear that for the RHS to evaluate to true, $X = true$ and $Y = false$. However, setting that automatically makes the LHS evaluate to false. Thus, the whole thing is unsatisfiable, so the original must be correct.

(iii) $(X \vee Y) \wedge (Z \vee \neg Y) \models (X \vee Z)$

Correct: In general, $A \models B$ if and only if $A \implies B$ is valid. To show that this works for this problem, we can write it as an implication and prove by counterexample that the statement is always valid. Consider $(X \vee Y) \wedge (Z \vee \neg Y) \implies (X \vee Z)$. In general, $A \implies B$ only evaluates to false if $A = true$ and $B = false$. In order for the RHS to evaluate to false, $X = false$ and $Z = false$. Plugging those in, the LHS evaluates to $(false \vee Y) \wedge (false \vee \neg Y)$, which can never evaluate to true. Therefore, that case never holds, so the implication is valid.

(b) Consider the following sentence:

$$[(Food \implies Party) \vee (Drinks \implies Party)] \implies [(Food \wedge Drinks) \implies Party] \ .$$

(i) Determine, using enumeration, whether this sentence is valid, satisfiable (but not valid), or unsatisfiable.

A simple truth table has eight rows, and shows that the sentence is true for all models and hence valid.

(ii) Convert the left-hand and right-hand sides of the main implication into CNF.

For the left-hand side we have:

- $(Food \implies Party) \vee (Drinks \implies Party)$
- $(\neg Food \vee Party) \vee (\neg Drinks \vee Party)$
- $(\neg Food \vee Party \vee \neg Drinks \vee Party)$
- $(\neg Food \vee \neg Drinks \vee Party)$

For the right-hand side we have:

- $(Food \wedge Drinks) \implies Party$
- $\neg(Food \wedge Drinks) \vee Party$
- $(\neg Food \vee \neg Drinks) \vee Party$
- $(\neg Food \vee \neg Drinks \vee Party)$

(iii) What do you observe about the LHS and RHS after converting to CNF? Explain how your results prove the answer to part b.i.

The two sides are identical in CNF, and hence the original sentence is of the form $P \implies P$, which is valid for any P.

Q2. Encrypted Knowledge Base

We have a propositional logic knowledge base as shown below, and we are trying to find a satisfying assignment for the variables A, B, C, D, and E. Each line corresponds to a valid propositional logic sentence:

$$\neg\ A$$
$$B \Rightarrow A$$
$$D$$
$$C\ \vee\ B$$
$$D\ \vee\ E$$

(a) Your buddy Albert runs his solver, and hands you the model $M = \{A = False, B = False, C = True, D = True, E = True\}$ that causes all of the knowledge base sentences to be true. We have a query sentence α specified as $(A \vee C) \Rightarrow E$. Our model M also causes α to be true. Can we say that the knowledge base entails α? Explain briefly (in one sentence) why or why not.

No, the knowledge base does not entail α. There are other models for which the knowledge base could be true and the query be false. Specifically $\{A = False, B = False, C = True, D = True, E = False\}$ satisfies the knowledge base but causes the query α to be false.

(b) Now we attempt to use theorem-proving methods to see whether our knowledge base entails a query sentence. To use these methods, it is useful to convert our knowledge base to conjunctive normal form (CNF), which satisfies:

- The sentence is a conjunction of (one or more) clauses.
- Each clause is a disjunction of literals.
- Each literal is a symbol or a negated symbol.

(i) Which sentences in the knowledge base are not already in conjunctive normal form? Convert them to CNF.

$B \Rightarrow A$ is converted to $\neg B \vee A$

(ii) Write the entire knowledge base as a single sentence in CNF.

After taking the conjunction of all sentences, we get:

$$\neg A \wedge (\neg B \vee A) \wedge D \wedge (C \vee B) \wedge (D \vee E)$$

(iii) Describe the steps necessary for converting $(A \wedge B) \vee (C \wedge D)$ to CNF.

We can distribute $\vee$ over $\wedge$ in the following way: $(\alpha \vee (\beta \wedge \gamma)) \equiv ((\alpha \vee \beta) \wedge (\alpha \vee \gamma)$.
So from $(A \wedge B) \vee (C \wedge D)$, we can take $\alpha = (A \wedge B), \beta = C, \gamma = D$. This gives us:
$((A \wedge B) \vee C) \wedge ((A \wedge B) \vee D))$.
Applying the same distributive property above a second time, we get:
$(((C \vee A) \wedge (C \vee B)) \wedge ((D \vee A) \wedge (D \vee B)))$
Since $\wedge$ is associative, we can rewrite:
$(C \vee A) \wedge (C \vee B) \wedge (D \vee A) \wedge (D \vee B)$
Finally by commutativity of $\vee$ we have:
$(A \vee C) \wedge (B \vee C) \wedge (A \vee D) \wedge (B \vee D)$
(You could have stopped at the previous step without applying commutativity, and that would have also been a perfectly valid CNF form.)