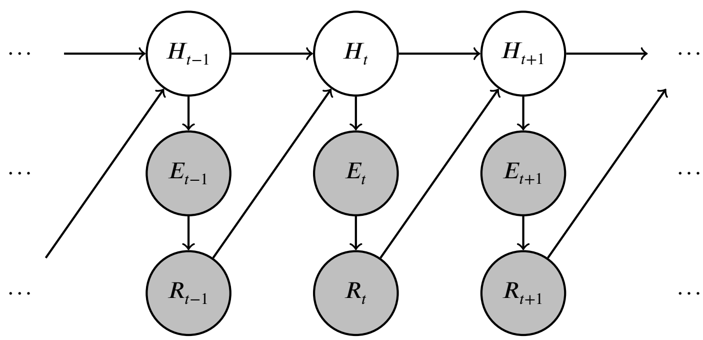
Q1. HMM: Human-Robot Interaction

In the near future, autonomous robots would live among us. Therefore, it is important for the robots to know how to properly act in the presence of humans. In this question, we are exploring a simplified model of this interaction. Here, we are assuming that we can observe the robot's actions at time t, R_t, and an evidence observation, E_t, directly caused by the human action, H_t. Human's actions and Robot's actions from the past time-step affect the Human's and Robot's actions in the next time-step. In this problem, we will remain consistent with the convention that capital letters (H_t) refer to random variables and lowercase letters (h_t) refer to a particular value the random variable can take. The structure is given below:



You are supplied with the following probability tables: $P(R_t \mid E_t)$, $P(H_t \mid H_{t-1}, R_{t-1})$, $P(H_0)$, $P(E_t \mid H_t)$.

Let us derive the forward algorithm for this model. We will split our computation into two components, a **time-elapse update** expression and a **observe update** expression.

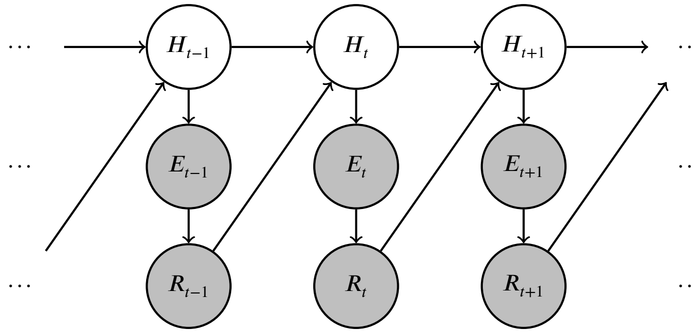
(a) We would like to incorporate the evidence that we observe at time t. Using the time-lapse update expression we will derive separately, we would like to find the **observe update** expression:

$$O(H_t) = P(H_t|e_{0:t}, r_{0:t})$$

In other words, we would like to compute the distribution of potential human states at time t given all observations up to and including time t. In addition to the conditional probability tables associated with the network's nodes, we are given $T(H_t) = P(H_t \mid e_{0:t-1}, r_{0:t-1})$, which we will assume is correctly computed in the time-elapse update that we will derive in the next part. From the options below, select *all* the options that **both** make valid independence assumptions and would evaluate to the observe update expression.

- ☐ $\frac{P(H_t|e_{0:t-1},r_{0:t-1})P(e_t|H_t)P(r_t|e_t)}{\sum_{h_t} P(h_t|e_{0:t-1},r_{0:t-1})P(e_t|h_t)P(r_t|e_t)}$
- ☐ $\frac{P(H_t|e_{0:t-1},r_{0:t-1})P(e_t|H_t)}{\sum_{h_t} P(h_t|e_{0:t-1},r_{0:t-1})P(e_t|h_t)}$
- ☐ $\frac{\sum_{e_t} P(H_t|e_{0:t-1},r_{0:t-1})P(e_t|H_t)}{\sum_{h_t} P(h_t|e_{0:t-1},r_{0:t-1})P(e_t|r_{t-1},H_{t-1})}$
- ☐ $\sum_{r_{t-1}} P(H_t \mid e_{0:t-1}, r_{0:t-1})P(r_{t-1} \mid e_{t-1})$
- ☐ $\sum_{r_t} P(H_t \mid e_{0:t-1}, r_{0:t-1})P(r_t \mid r_{t-1}, e_t)$
- ☐ $\sum_{h_{t+1}} P(H_t \mid e_{0:t-1}, r_{0:t-1})P(h_{t+1} \mid r_t)$

The structure below is identical to the one in the beginning of the question and is repeated for your convenience.



(b) We are interested in predicting what the state of human is at time t (H_t), given all the observations through $t-1$. Therefore, the **time-elapse update** expression has the following form:

$$T(H_t) = P(H_t | e_{0:t-1}, r_{0:t-1})$$

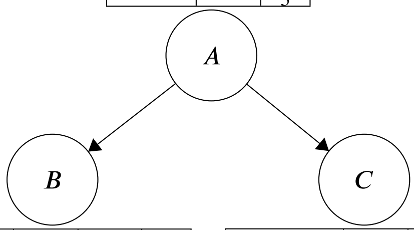
Derive an expression for the given time-elapse update above using the probability tables provided in the question and the observe update expression, $O(H_{t-1}) = P(H_{t-1} | e_{0:t-1}, r_{0:t-1})$. Write your final expression in the space provided at below. You may use the function O in your solution if you prefer.

$P(H_t | e_{0:t-1}, r_{0:t-1}) =$ ____

Q2. Sampling

Consider the following Bayes net. The joint distribution is not given, but it may be helpful to fill in the table before answering the following questions.

$P(A)$		
	$+a$	$\frac{2}{3}$
	$-a$	$\frac{1}{3}$



$P(B\|A)$			
	$+a$	$+b$	$\frac{1}{2}$
	$+a$	$-b$	$\frac{1}{2}$
	$-a$	$+b$	$\frac{1}{4}$
	$-a$	$-b$	$\frac{3}{4}$

$P(C\|A)$			
	$+a$	$+c$	$\frac{1}{2}$
	$+a$	$-c$	$\frac{1}{2}$
	$-a$	$+c$	$\frac{2}{3}$
	$-a$	$-c$	$\frac{1}{3}$

$P(A, B, C)$			
$+a$	$+b$	$+c$	
$+a$	$+b$	$-c$	
$+a$	$-b$	$+c$	
$+a$	$-b$	$-c$	
$-a$	$+b$	$+c$	
$-a$	$+b$	$-c$	
$-a$	$-b$	$+c$	
$-a$	$-b$	$-c$	

We are going to use sampling to approximate the query $P(C \mid +b)$. Consider the following samples:

Sample 1	Sample 2	Sample 3
$(+a, +b, +c)$	$(+a, -b, -c)$	$(-a, +b, +c)$

(a) Fill in the following table with the probabilities of drawing each respective sample given that we are using each of the following sampling techniques.

P(sample | method)	Sample 1	Sample 2
Prior Sampling		
Rejection Sampling		
Likelihood Weighting		

Lastly, we want to figure out the probability of getting Sample 3 by Gibbs sampling. We'll initialize the sample to $(+a, +b, +c)$, and resample A then C.

(b) What is the probability the sample equals $(-a, +b, +c)$ after resampling A?

(c) What is the probability the sample equals $(-a, +b, +c)$ after resampling C, given that the sample equals $(-a, +b, +c)$ after resampling A?

(d) What is the probability of drawing Sample 3, $(-a, +b, +c)$, using Gibbs sampling in this way?