CS 188
Spring 2023

Final Review: HMMs Solutions

Q1. HMMs

Consider a process where there are transitions among a finite set of states $s_1, \cdots, s_k$ over time steps $i = 1, \cdots, N$. Let the random variables $X_1, \cdots, X_N$ represent the state of the system at each time step and be generated as follows:

- Sample the initial state s from an initial distribution $P_1(X_1)$, and set $i = 1$
- Repeat the following:
  1. Sample a duration d from a duration distribution P_D over the integers $\{1, \cdots, M\}$, where M is the maximum duration.
  2. Remain in the current state s for the next d time steps, i.e., set
  $$x_i = x_{i+1} = \cdots = x_{i+d-1} = s \tag{1}$$
  3. Sample a successor state s' from a transition distribution $P_T(X_t|X_{t-1} = s)$ over the other states $s' \neq s$ (so there are no self transitions)
  4. Assign $i = i + d$ and $s = s'$.

This process continues indefinitely, but we only observe the first N time steps.

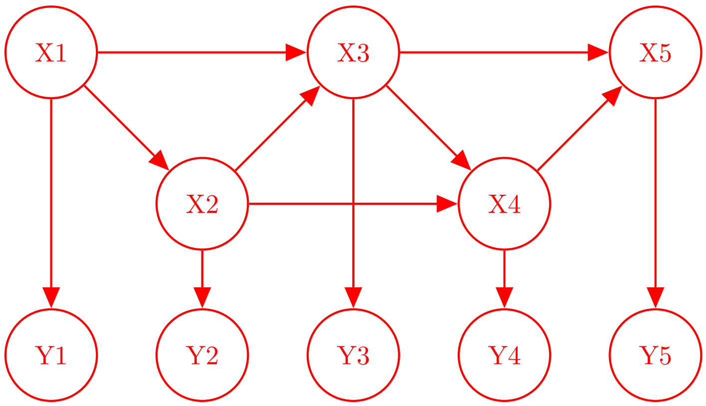
(a) Assuming that all three states s_1, s_2, s_3 are different, what is the probability of the sample sequence $s_1, s_1, s_2, s_2, s_2, s_3, s_3$? Write an algebraic expression. Assume $M \geq 3$.

$$p_1(s_1)p_D(2)p_T(s_2|s_1)p_D(3)p(s_3|s_2)(1 - p_D(1)) \tag{2}$$

At each time step i we observe a noisy version of the state X_i that we denote Y_i and is produced via a conditional distribution $P_E(Y_i|X_i)$.

(b) Only in this subquestion assume that $N > M$. Let $X_1, \cdots, X_N$ and $Y_1, \cdots, Y_N$ random variables defined as above. What is the maximum index $i \leq N - 1$ so that $X_1 \perp\!\!\!\perp X_N | X_i, X_{i+1}, \cdots, X_{N-1}$ is guaranteed?
$i = N - M$

(c) Only in this subquestion, assume the max duration $M = 2$, and P_D uniform over $\{1, 2\}$ and each x_i is in an alphabet $\{a, b\}$. For $(X_1, X_2, X_3, X_4, X_5, Y_1, Y_2, Y_3, Y_4, Y_5)$ draw a Bayes Net over these 10 random variables with the property that removing any of the edges would yield a Bayes net inconsistent with the given distribution.



(d) In this part we will explore how to write the described process as an HMM with an extended state space. Write the states $z = (s, t)$ where s is a state of the original system and t represents the time elapsed in that state. For example, the state sequence $s_1, s_1, s_1, s_2, s_3, s_3$ would be represented as $(s_1, 1), (s_1, 2), (s_1, 3), (s_2, 1), (s_3, 1), (s_3, 2)$.

Answer all of the following in terms of the parameters $P_1(X_1), P_D(d), P_T(X_{j+1}|X_j), P_E(Y_i|X_i), k$ (total number of possible states), N, and M (max duration).

(i) What is $P(Z_1)$?

$$P(x_1, t) = \begin{cases} P_1(x_1) & \text{if } t = 1 \\ 0 & \text{o.w.} \end{cases} \tag{3}$$

(ii) What is $P(Z_{i+1}|Z_i)$? Hint: You will need to break this into cases where the transition function will behave differently.

$$P(X_{i+1}, t_{i+1}|X_i, t_i) = \begin{cases} P_D(d \geq t_i + 1 | d \geq t_i) & \text{when } X_{i+1} = X_i \text{ and } t_{i+1} = t_i + 1 \text{ and } t_{i+1} \leq M \\ P_T(X_{i+1}|X_i) P_D(d = t_i | d \geq t_i) & \text{when } X_{i+1} \neq X_i \text{ and } t_{i+1} = 1 \\ 0 & \text{o.w.} \end{cases} \tag{4}$$

Where $P_D(d \geq t_i + 1 | d \geq t_i) = P_D(d \geq t_i + 1)/P_D(d \geq t_i)$.

Being in X_i, t_i, we know that d was drawn $d \geq t_i$. Conditioning on this fact, we have two choices, if $d > t_i$ then the next state is $X_{i+1} = X_i$, and if $d = t_i$ then $X_{i+1} \neq X_i$ drawn from the transition distribution and $t_{i+1} = 1$.

(iii) What is $P(Y_i|Z_i)$?

$p(Y_i|X_i, t_i) = P_E(Y_i|X_i)$

(e) In this question we explore how to write an algorithm to compute $P(X_N|y_1, \cdots, y_N)$ using the particular structure of this process.

Write $P(X_t|y_1, \cdots, y_{t-1})$ in terms of other factors. Construct an answer by checking the correct boxes below:

$$P(X_t|y_1, \cdots, y_{t-1}) = \underline{\quad \textbf{(i)} \quad} \; \underline{\quad \textbf{(ii)} \quad} \; \underline{\quad \textbf{(iii)} \quad}$$

(i) ● $\sum_{i=1}^{k} \sum_{d=1}^{M} \sum_{d'=1}^{M}$ ○ $\sum_{i=1}^{k}$

○ $\sum_{i=1}^{k} \sum_{d=1}^{M}$ ○ $\sum_{d=1}^{M}$

(ii)

○ $P(Z_t = (X_t, d) | Z_{t-1} = (s_i, d))$

○ $P(X_t | X_{t-1} = s_i)$

○ $P(X_t | X_{t-1} = s_d)$

● $P(Z_t = (X_t, d') | Z_{t-1} = (s_i, d))$

(iii) ○ $P(Z_{t-1} = (s_d, i) | y_1, \cdots, y_{t-1})$

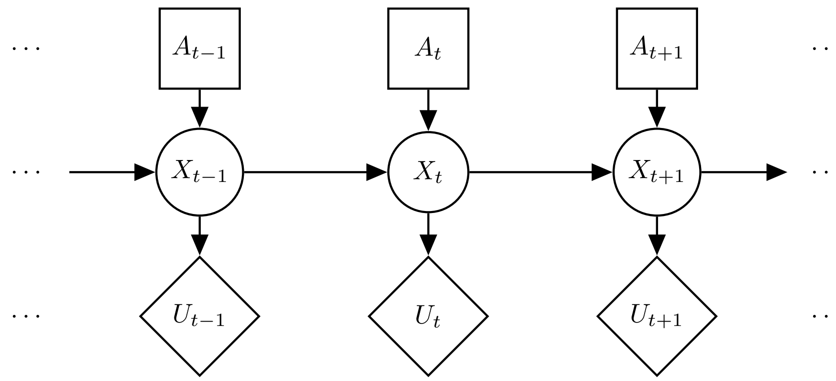
○ $P(X_{t-1} = s_d | y_1, \cdots, y_{t-1})$

● $P(Z_{t-1} = (s_i, d) | y_1, \cdots, y_{t-1})$

○ $P(X_{t-1} = s_i | y_1, \cdots, y_{t-1})$

Q2. Planning ahead with HMMs

Pacman is tired of using HMMs to estimate the location of ghosts. He wants to use HMMs to plan what actions to take in order to maximize his utility. Pacman uses the HMM (drawn to the right) of length T to model the planning problem. In the HMM, $X_{1:T}$ is the sequence of hidden states of Pacman's world, $A_{1:T}$ are actions Pacman can take, and U_t is the utility Pacman receives at the particular hidden state X_t. Notice that there are no evidence variables, and utilities are not discounted.



(a) The belief at time t is defined as $B_t(X_t) = p(X_t|a_{1:t})$. The forward algorithm update has the following form:

$$B_t(X_t) = \underline{\quad \textbf{(i)} \quad} \; \underline{\quad \textbf{(ii)} \quad} \; B_{t-1}(x_{t-1}).$$

Complete the expression by choosing the option that fills in each blank.

(i)	○ $\max_{x_{t-1}}$	● $\sum_{x_{t-1}}$	○ $\max_{x_t}$	○ $\sum_{x_t}$	○ 1
(ii)	○ $p(X_t|x_{t-1})$	○ $p(X_t|x_{t-1})p(X_t|a_t)$	○ $p(X_t)$	● $p(X_t|x_{t-1}, a_t)$	○ 1

○ None of the above combinations is correct

$$\begin{aligned} B_t(X_t) &= p(X_t|a_{1:t}) \\ &= \sum_{x_{t-1}} p(X_t|x_{t-1}, a_t) p(x_{t-1}|a_{1:t-1}) \\ &= \sum_{x_{t-1}} p(X_t|x_{t-1}, a_t) B_{t-1}(x_{t-1}) \end{aligned}$$

(b) Pacman would like to take actions $A_{1:T}$ that maximizes the expected sum of utilities, which has the following form:

$$\text{MEU}_{1:T} = \underline{\ \textbf{(i)}\ } \; \underline{\ \textbf{(ii)}\ } \; \underline{\ \textbf{(iii)}\ } \; \underline{\ \textbf{(iv)}\ } \; \underline{\ \textbf{(v)}\ }$$

Complete the expression by choosing the option that fills in each blank.

(i)	● $\max_{a_{1:T}}$	○ $\max_{a_T}$	○ $\sum_{a_{1:T}}$	○ $\sum_{a_T}$	○ 1
(ii)	○ $\max_t$	○ $\prod_{t=1}^{T}$	● $\sum_{t=1}^{T}$	○ $\min_t$	○ 1
(iii)	○ $\sum_{x_t, a_t}$	● $\sum_{x_t}$	○ $\sum_{a_t}$	○ $\sum_{x_T}$	○ 1
(iv)	○ $p(x_t|x_{t-1}, a_t)$	○ $p(x_t)$	● $B_t(x_t)$	○ $B_T(x_T)$	○ 1
(v)	○ U_T	○ $\frac{1}{U_t}$	○ $\frac{1}{U_T}$	● U_t	○ 1

○ None of the above combinations is correct

$$\text{MEU}_{1:T} = \max_{a_{1:T}} \sum_{t=1}^{T} \sum_{x_t} B_t(x_t) U_t(x_t)$$

(c) A greedy ghost now offers to tell Pacman the values of some of the hidden states. Pacman needs your help to figure out if the ghost's information is useful. Assume that the transition function $p(x_t|x_{t-1}, a_t)$ is not deterministic. **With respect to the utility** U_t, mark all that can be True:

■ VPI($X_{t-1}|X_{t-2}$) > 0 □ VPI($X_{t-2}|X_{t-1}$) > 0 ■ VPI($X_{t-1}|X_{t-2}$) $= 0$ ■ VPI($X_{t-2}|X_{t-1}$) $= 0$ □ None of the above

It is always possible that VPI $= 0$. Can guarantee VPI($E|e$) is not greater than 0 if E is independent of parents(U) given e.

(d) Pacman notices that calculating the beliefs under this model is very slow using exact inference. He therefore decides to try out various particle filter methods to speed up inference. Order the following methods by how accurate their estimate of $B_T(X_T)$ is? If different methods give an equivalently accurate estimate, mark them as the same number.

	Most accurate			Least accurate
Exact inference	● 1	○ 2	○ 3	○ 4
Particle filtering with no resampling	○ 1	● 2	○ 3	○ 4
Particle filtering with resampling before every time elapse	○ 1	○ 2	○ 3	● 4
Particle filtering with resampling before every other time elapse	○ 1	○ 2	● 3	○ 4

Exact inference will always be more accurate than using a particle filter. When comparing the particle filter resampling approaches, notice that because there are no observations, each particle will have weight 1. Therefore resampling when particle weights are 1 could lead to particles being lost and hence prove bad.