Announcements

- HW2 is due **Friday, February 10, 11:59 PM PT**
- Project 2 is due **Tuesday, February 14, 11:59 PM PT**
Today

- Efficient Solution of CSPs
- Iterative Improvement
Review: CSPs

- CSPs:
  - Variables
  - Domains
  - Constraints
    - Implicit (provide code to compute)
    - Explicit (provide a list of the legal tuples)
    - Unary / Binary / N-ary

- Goals:
  - Here: find any solution
  - Also: find all, find best, etc.
K-Consistency
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- Increasing degrees of consistency
  - 1-Consistency (Node Consistency): Each single node’s domain has a value which meets that node’s unary constraints
  - 2-Consistency (Arc Consistency): For each pair of nodes, any consistent assignment to one can be extended to the other
  - K-Consistency: For each k nodes, any consistent assignment to k-1 can be extended to the k\textsuperscript{th} node.

- Higher k more expensive to compute

- (You need to know the k=2 case: arc consistency)
Strong K-Consistency

- Strong k-consistency: also k-1, k-2, ... 1 consistent
- Claim: strong n-consistency means we can solve without backtracking!
- Why?
  - Choose any assignment to any variable
  - Choose a new variable
  - By 2-consistency, there is a choice consistent with the first
  - Choose a new variable
  - By 3-consistency, there is a choice consistent with the first 2
  - ...

- Lots of middle ground between arc consistency and n-consistency! (e.g. k=3, called path consistency)
Ordering
Ordering: Minimum Remaining Values

- **Variable Ordering: Minimum remaining values (MRV):**
  - Choose the variable with the fewest legal left values in its domain

- Why min rather than max?
- Also called “most constrained variable”
- “Fail-fast” ordering
Value Ordering: Least Constraining Value

- Given a choice of variable, choose the *least constraining value*
- I.e., the one that rules out the fewest values in the remaining variables
- Note that it may take some computation to determine this! (E.g., rerunning filtering)

Why least rather than most?

Combining these ordering ideas makes 1000 queens feasible

[Demo: coloring – backtracking + AC + ordering]
Structure
Problem Structure

- Extreme case: independent subproblems
  - Example: Tasmania and mainland do not interact

- Independent subproblems are identifiable as connected components of constraint graph

- Suppose a graph of n variables can be broken into subproblems of only c variables:
  - Worst-case solution cost is $O((n/c)(d^c))$, linear in n
  - E.g., $n = 80$, $d = 2$, $c = 20$
  - $2^{80} = 4$ billion years at 10 million nodes/sec
  - $(4)(2^{20}) = 0.4$ seconds at 10 million nodes/sec
Tree-Structured CSPs

- Theorem: if the constraint graph has no loops, the CSP can be solved in $O(n d^2)$ time
  - Compare to general CSPs, where worst-case time is $O(d^n)$

- This property also applies to probabilistic reasoning (later): an example of the relation between syntactic restrictions and the complexity of reasoning
Tree-Structured CSPs

- **Algorithm for tree-structured CSPs:**
  - **Order:** Choose a root variable, order variables so that parents precede children
  - **Remove backward:** For $i = n : 2$, apply $\text{RemoveInconsistent}($Parent($X_i$),$X_i$)$
  - **Assign forward:** For $i = 1 : n$, assign $X_i$ consistently with Parent($X_i$)

- **Runtime:** $O(n d^2)$ (why?)
Claim 1: After backward pass, all root-to-leaf arcs are consistent
Proof: Each $X \rightarrow Y$ was made consistent at one point and $Y$’s domain could not have been reduced thereafter (because $Y$’s children were processed before $Y$)

Claim 2: If root-to-leaf arcs are consistent, forward assignment will not backtrack
Proof: Induction on position

Why doesn’t this algorithm work with cycles in the constraint graph?

Note: we’ll see this basic idea again with Bayes’ nets
Improving Structure
Conditioning: instantiate a variable, prune its neighbors' domains

Cutset conditioning: instantiate (in all ways) a set of variables such that the remaining constraint graph is a tree

Cutset size c gives runtime $O\left( (d^c) (n-c) d^2 \right)$, very fast for small c
Choose a cutset

Instantiate the cutset (all possible ways)

Compute residual CSP for each assignment

Solve the residual CSPs (tree structured)
Find the smallest cutset for the graph below.
Tree Decomposition*

- Idea: create a tree-structured graph of mega-variables
- Each mega-variable encodes part of the original CSP
- Subproblems overlap to ensure consistent solutions

\[
\begin{align*}
\text{Agree on shared vars} & \quad \left\{ (\text{WA}=r, \text{SA}=g, \text{NT}=b), \\
& \quad (\text{WA}=b, \text{SA}=r, \text{NT}=g), \\
& \quad \ldots \right\}
\end{align*}
\]

\[
\begin{align*}
\text{Agree on shared vars} & \quad \left\{ (\text{NT}=r, \text{SA}=g, \text{Q}=b), \\
& \quad (\text{NT}=b, \text{SA}=g, \text{Q}=r), \\
& \quad \ldots \right\}
\end{align*}
\]

\[
\begin{align*}
\text{Agree: } (\text{M1}, \text{M2}) & \in \\
& \left\{ ((\text{WA}=g, \text{SA}=g, \text{NT}=g), (\text{NT}=g, \text{SA}=g, \text{Q}=g)), \ldots \right\}
\end{align*}
\]
Iterative Improvement
Iterative Algorithms for CSPs

- Local search methods typically work with “complete” states, i.e., all variables assigned

- To apply to CSPs:
  - Take an assignment with unsatisfied constraints
  - Operators *reassign* variable values
  - No fringe! Live on the edge.

- Algorithm: While not solved,
  - Variable selection: randomly select any conflicted variable
  - Value selection: min-conflicts heuristic:
    - Choose a value that violates the fewest constraints
    - I.e., hill climb with \( h(n) = \) total number of violated constraints
Example: 4-Queens

- States: 4 queens in 4 columns ($4^4 = 256$ states)
- Operators: move queen in column
- Goal test: no attacks
- Evaluation: $c(n) = $ number of attacks

[Demo: n-queens – iterative improvement (L5D1)]
[Demo: coloring – iterative improvement]
Performance of Min-Conflicts

- Given random initial state, can solve n-queens in almost constant time for arbitrary n with high probability (e.g., n = 10,000,000)!

- The same appears to be true for any randomly-generated CSP except in a narrow range of the ratio

\[ R = \frac{\text{number of constraints}}{\text{number of variables}} \]
Summary: CSPs

- CSPs are a special kind of search problem:
  - States are partial assignments
  - Goal test defined by constraints

- Basic solution: backtracking search

- Speed-ups:
  - Ordering
  - Filtering
  - Structure

- Iterative min-conflicts is often effective in practice
Next Time: Adversarial Search!