Example: Weather

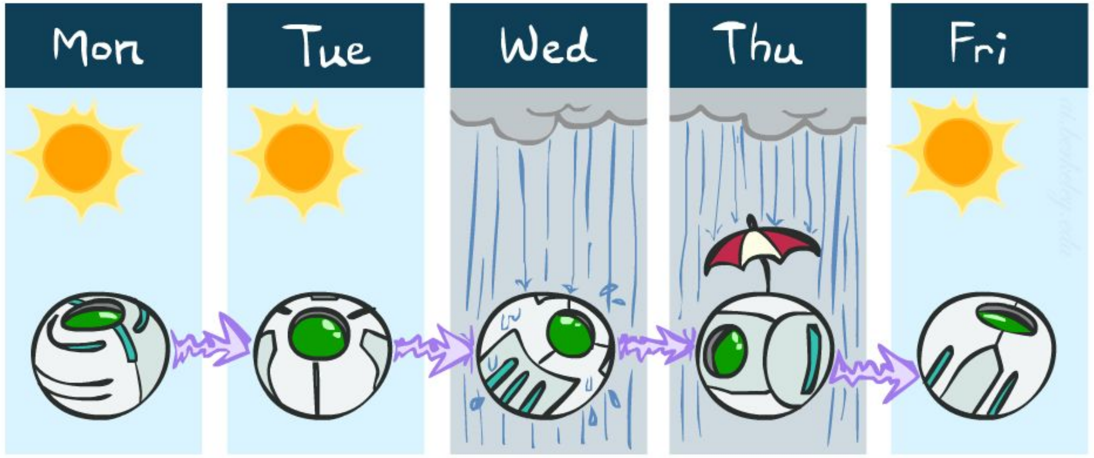
- States {rain, sun}

- Initial distribution $P(X_0)$

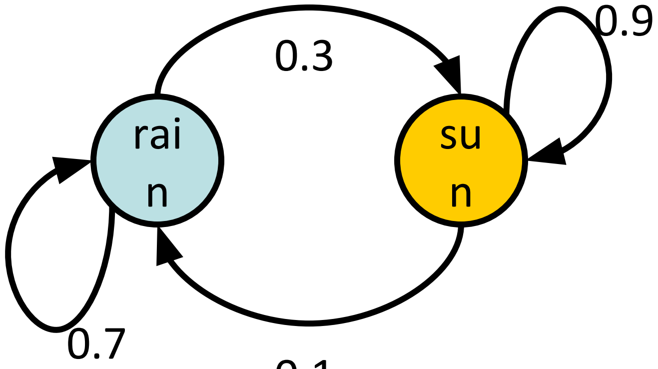
$P(X_0)$	
sun	rain
0.5	0.5

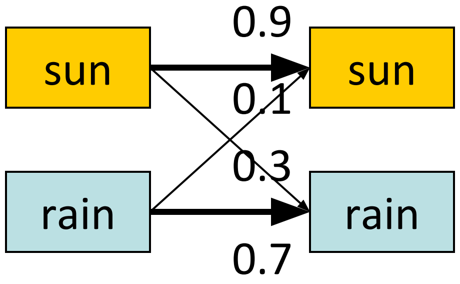
- Transition model $P(X_t \mid X_{t-1})$

X_{t-1}	$P(X_t \mid X_{t-1})$	
	sun	rain
sun	0.9	0.1
rain	0.3	0.7



Two new ways of representing the same CPT

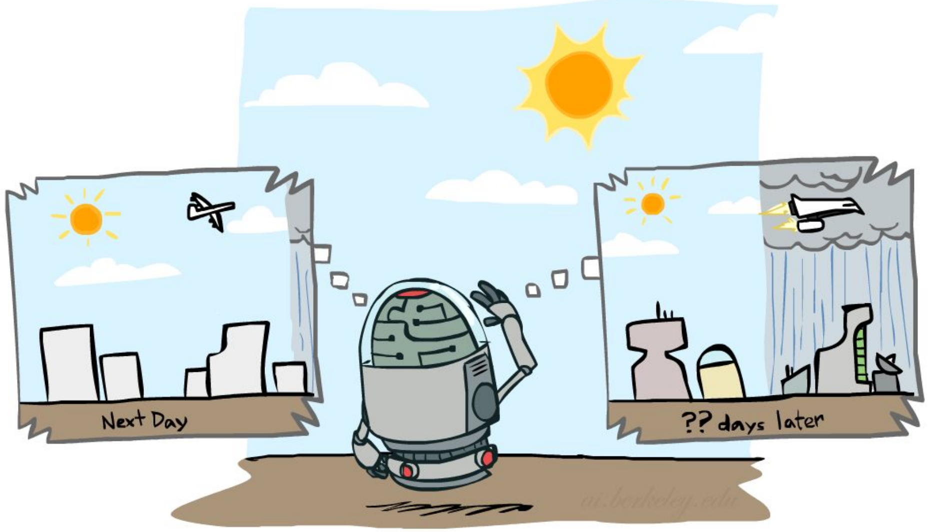




Weather prediction

- Time 0: <0.5,0.5>

X_{t-1}	$P(X_t \mid X_{t-1})$	
	sun	rain
sun	0.9	0.1
rain	0.3	0.7

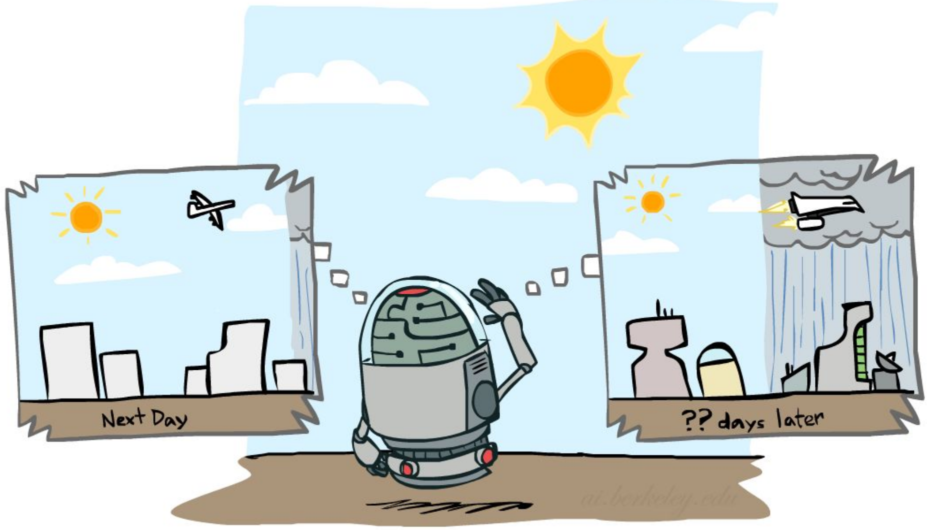


- What is the weather like at time 1?
  - $P(X_1) = \sum_{x0} P(X_1, X_0 = x_0)$
  - $\quad = \sum_{x0} P(X_0 = x_0)\, P(X_1 \mid X_0 = x_0)$
  - $\quad = 0.5<0.9,0.1> + 0.5<0.3,0.7> = <0.6,0.4>$

Weather prediction, contd.

- Time 1: <0.6,0.4>

X_{t-1}	$P(X_t \mid X_{t-1})$	
	sun	rain
sun	0.9	0.1
rain	0.3	0.7

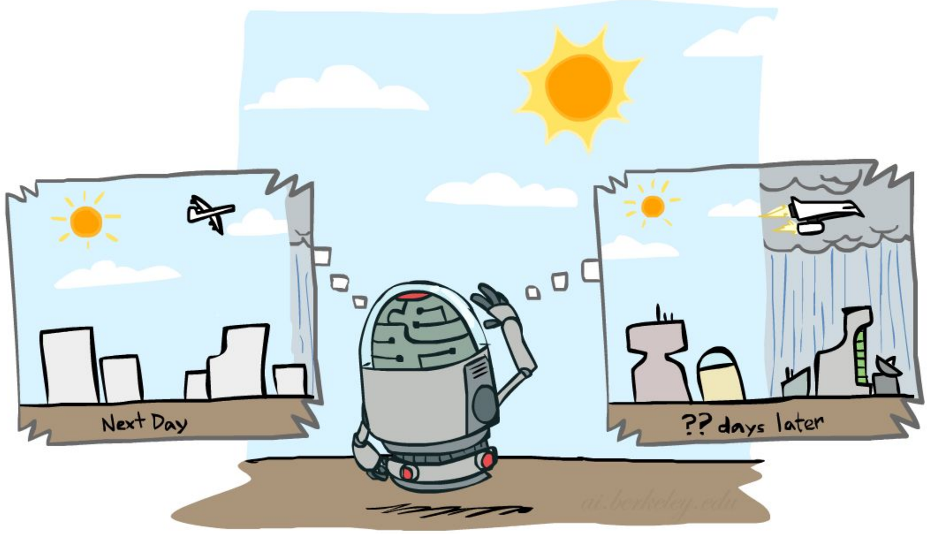


- What is the weather like at time 2?
  - $P(X_2) = \sum_{x1} P(X_2, X_1 = x_1)$
  - $= \sum_{x1} P(X_1 = x_1)\, P(X_2 \mid X_1 = x_1)$
  - $= 0.6\langle 0.9, 0.1\rangle + 0.4\langle 0.3, 0.7\rangle = \langle 0.66, 0.34\rangle$

Weather prediction, contd.

- Time 2: <0.66,0.34>

X_{t-1}	$P(X_t \mid X_{t-1})$	
	sun	rain
sun	0.9	0.1
rain	0.3	0.7



- What is the weather like at time 3?
  - $P(X_3) = \sum_{x2} P(X_3, X_2{=}x_2)$
  - $\qquad = \sum_{x2} P(X_2{=}x_2)\, P(X_3 \mid X_2{=}x_2)$
  - $\qquad = 0.66\langle 0.9,0.1\rangle + 0.34\langle 0.3,0.7\rangle = \langle 0.696,0.304\rangle$

Forward algorithm (simple form)

Probability from previous iteration

Transition model

- What is the state at time t?
  - $P(X_t) = \sum_{x_{t-1}} P(X_t, X_{t-1}=x_{t-1})$
  - $\quad = \sum_{x_{t-1}} P(X_{t-1}=x_{t-1})\, P(X_t \mid X_{t-1}=x_{t-1})$
- Iterate this update starting at $t=0$
  - This is called a ***recursive*** update: $P_t = g(P_{t-1}) = g(g(g(g(\ldots P_0))))$

And the same thing in linear algebra

- What is the weather like at time 2?
  - $P(X_2) = 0.6\langle 0.9,0.1\rangle + 0.4\langle 0.3,0.7\rangle = \langle 0.66,0.34\rangle$
- In matrix-vector form:
  - $P(X_2) = \begin{pmatrix} 0.9 & 0.3 \\ 0.1 & 0.7 \end{pmatrix} \begin{pmatrix} 0.6 \\ 0.4 \end{pmatrix} = \begin{pmatrix} 0.66 \\ 0.34 \end{pmatrix}$

X_{t-1}	$P(X_t \mid X_{t-1})$	
	sun	rain
sun	0.9	0.1
rain	0.3	0.7

- I.e., multiply by T^T, transpose of transition matrix

Stationary Distributions

- The limiting distribution is called the ***stationary distribution*** P_∞ of the chain
- It satisfies $P_\infty = P_{\infty+1} = T^T P_\infty$
- Solving for P_∞ in the example:

$$\begin{pmatrix} 0.9 & 0.3 \\ 0.1 & 0.7 \end{pmatrix} \begin{pmatrix} p \\ 1-p \end{pmatrix} = \begin{pmatrix} p \\ 1-p \end{pmatrix}$$

$0.9p + 0.3(1-p) = p$

$p = 0.75$

Stationary distribution is <0.75,0.25> ***regardless of starting distribution***

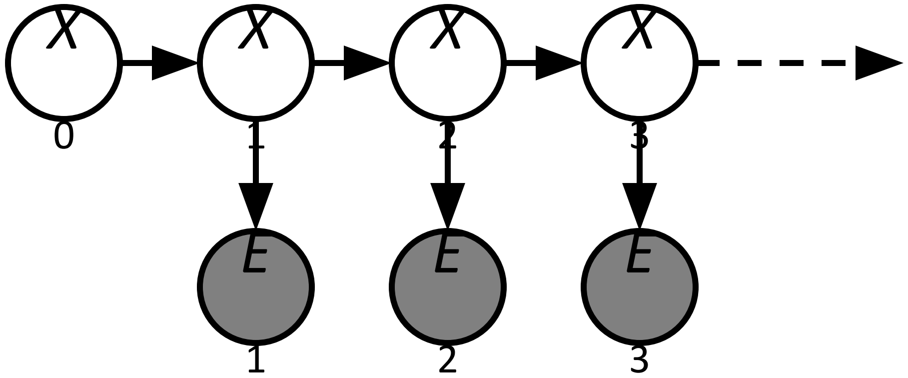
Hidden Markov Models

Hidden Markov Models

- Usually the true state is not observed directly

- Hidden Markov models (HMMs)
  - Underlying Markov chain over states *X*
  - You observe evidence *E* at each time step

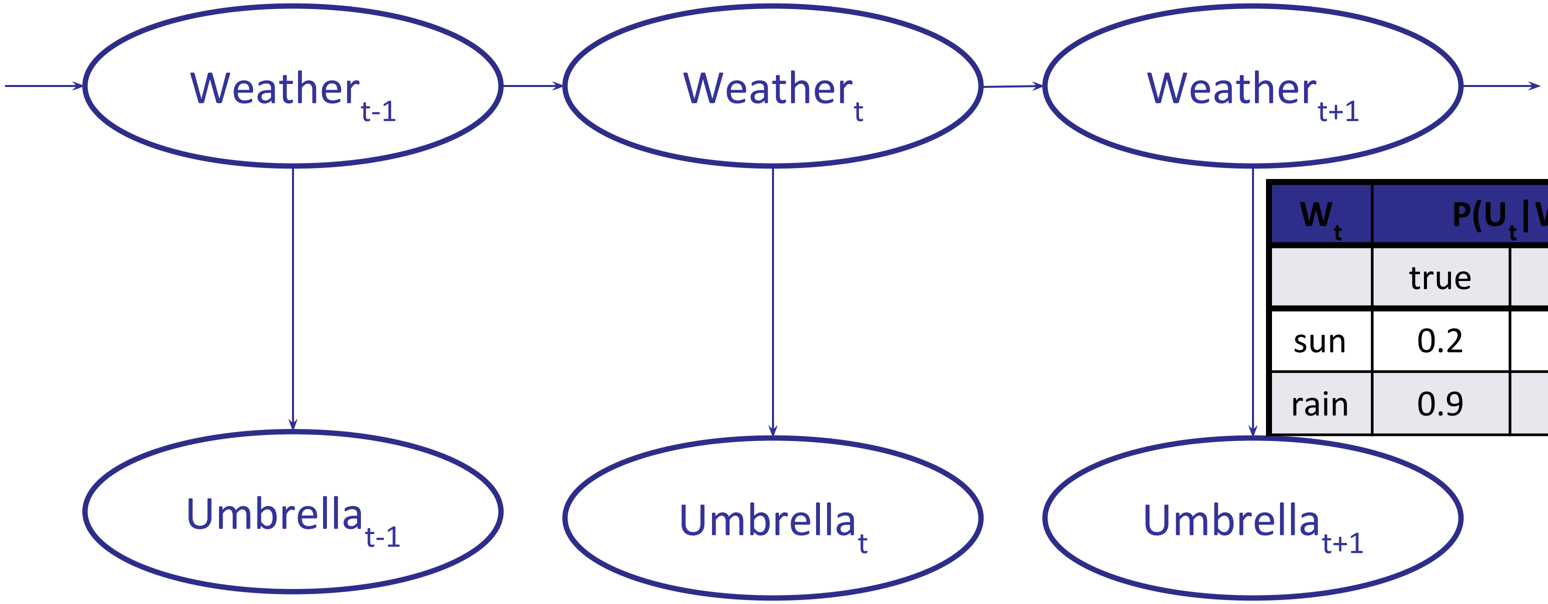
  - X_t is a single discrete variable; E_t may be continuous and may consist of several variables



Example: Weather HMM

- An HMM is defined by:
  - Initial distribution: $P(X_0)$
  - Transition model: $P(X_t \mid X_{t-1})$
  - Sensor model: $P(E_t \mid X_t)$

W_{t-1}	$P(W_t \mid W_{t-1})$	
	sun	rain
sun	0.9	0.1
rain	0.3	0.7

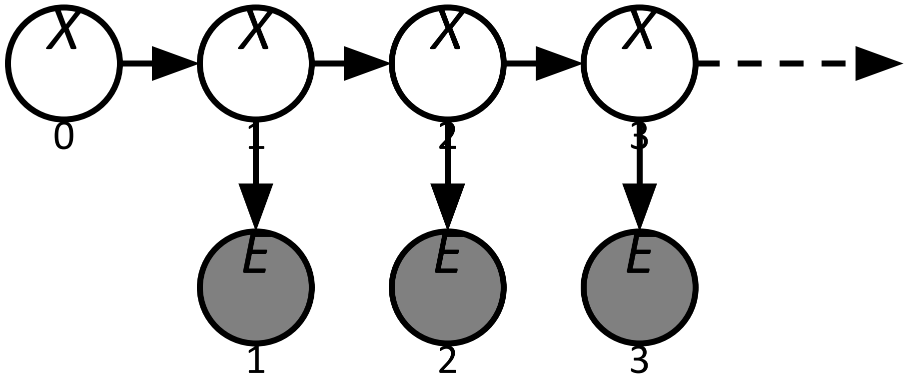


W_t	$P(U_t \mid W_t)$	
	true	false
sun	0.2	0.8
rain	0.9	0.1

HMM as probability model

- Joint distribution for Markov model: $P(X_0,\ldots, X_T) = P(X_0) \prod_{t=1:T} P(X_t \mid X_{t-1})$
- Joint distribution for hidden Markov model:

  $P(X_0,X_1,\ldots, X_T,E_T) = P(X_0) \prod_{t=1:T} P(X_t \mid X_{t-1})\, P(E_t \mid X_t)$
- Future states are independent of the past given the present
- Current evidence is independent of everything else given the current state
- Are evidence variables independent of each other?



Useful notation:

$X_{a:b} = X_a , X_{a+1}, \ldots, X_b$

Real HMM Examples

- Speech recognition HMMs:
  - Observations are acoustic signals (continuous valued)
  - States are specific positions in specific words (so, tens of thousands)

- Machine translation HMMs:
  - Observations are words (tens of thousands)
  - States are translation options

- Robot tracking:
  - Observations are range readings (continuous)
  - States are positions on a map (continuous)

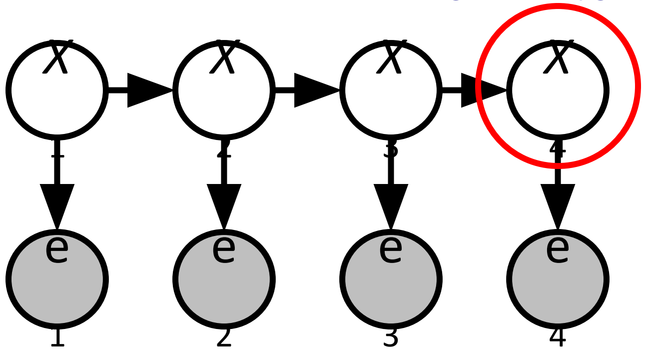
- Molecular biology:
  - Observations are nucleotides ACGT
  - States are coding/non-coding/start/stop/splice-site etc.

Inference tasks

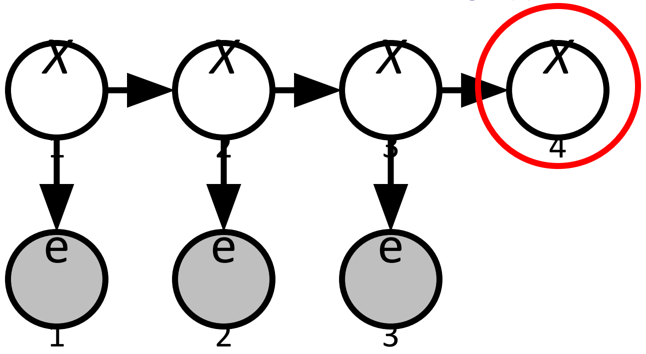
- ***Filtering***: $P(X_t | e_{1:t})$
  - ***belief state***—input to the decision process of a rational agent
- ***Prediction***: $P(X_{t+k} | e_{1:t})$ for $k > 0$
  - evaluation of possible action sequences; like filtering without the evidence
- ***Smoothing***: $P(X_k | e_{1:t})$ for $0 \leq k < t$
  - better estimate of past states, essential for learning
- ***Most likely explanation***: $\arg\max_{x_{1:t}} P(x_{1:t} \mid e_{1:t})$
  - speech recognition, decoding with a noisy channel

Inference tasks

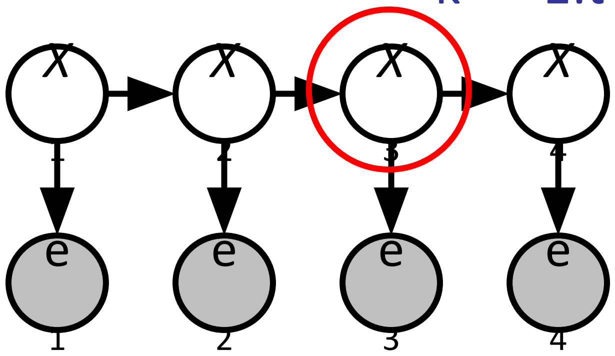
Filtering: $P(X_t | e_{1:t})$



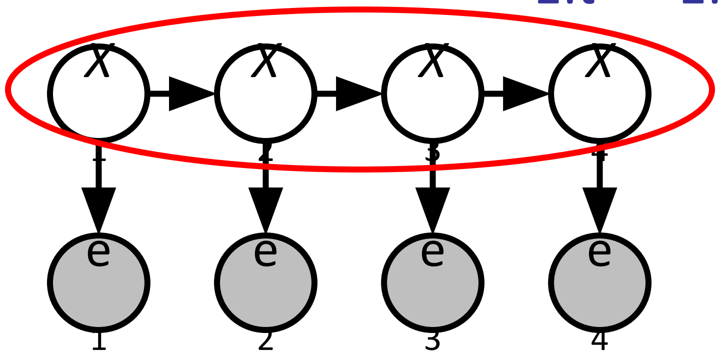
Prediction: $P(X_{t+k} | e_{1:t})$



Smoothing: $P(X_k | e_{1:t})$, $k<t$



Explanation: $P(X_{1:t} | e_{1:t})$

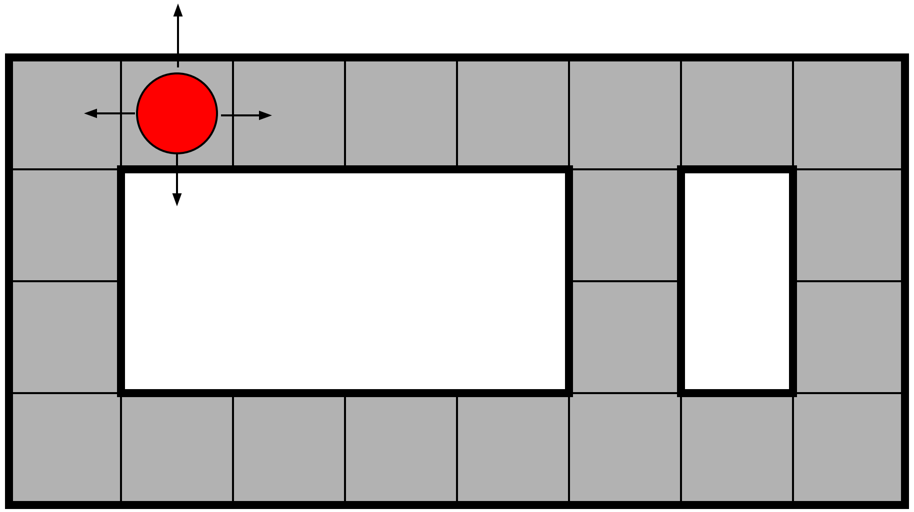


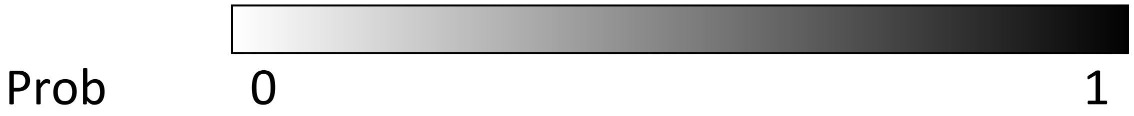
Filtering / Monitoring

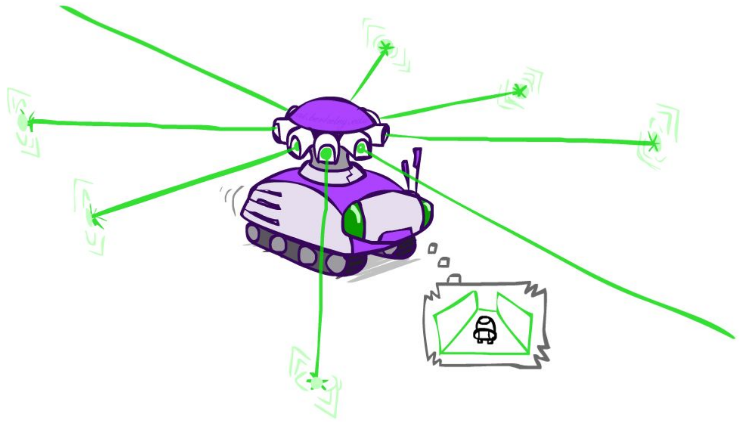
- Filtering, or monitoring, or state estimation, is the task of maintaining the distribution $\boldsymbol{f}_{1:t} = P(X_t | e_{1:t})$ over time
- We start with $\boldsymbol{f}_0$ in an initial setting, usually uniform
- Filtering is a fundamental task in engineering and science
- The Kalman filter (continuous variables, linear dynamics, Gaussian noise) was invented in 1960 and used for trajectory estimation in the Apollo program; core ideas used by Gauss for planetary observations; ***788,000*** papers on Google Scholar

Example: Robot Localization

Example from
Michael Pfeiffer





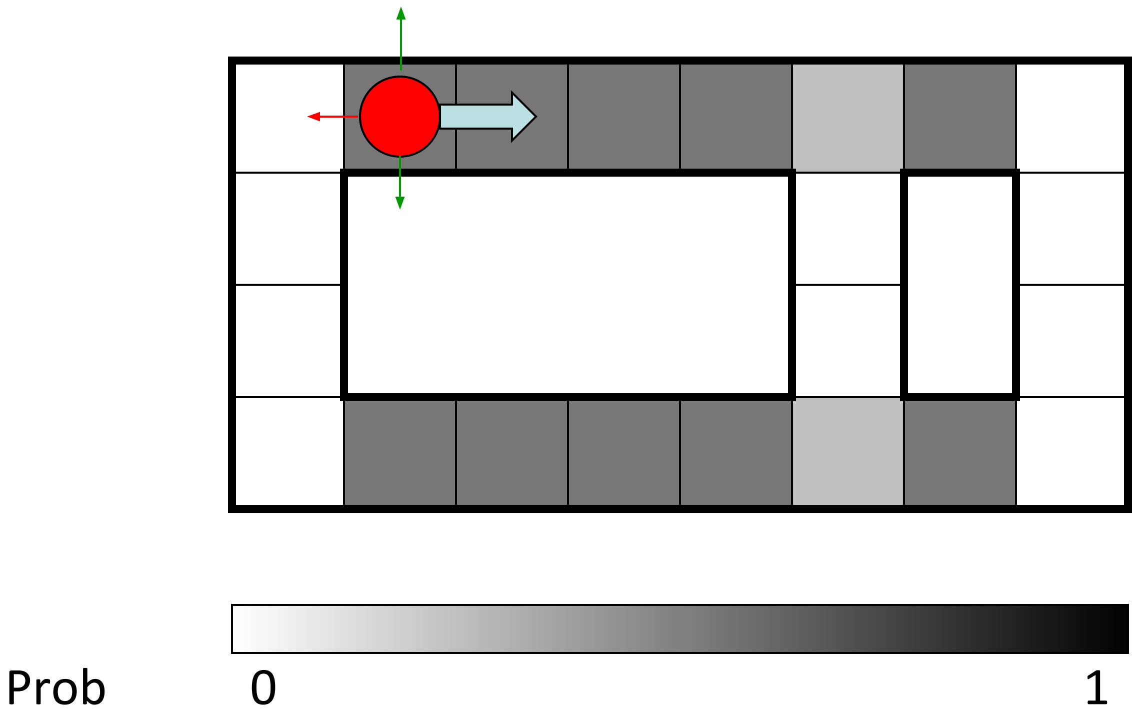


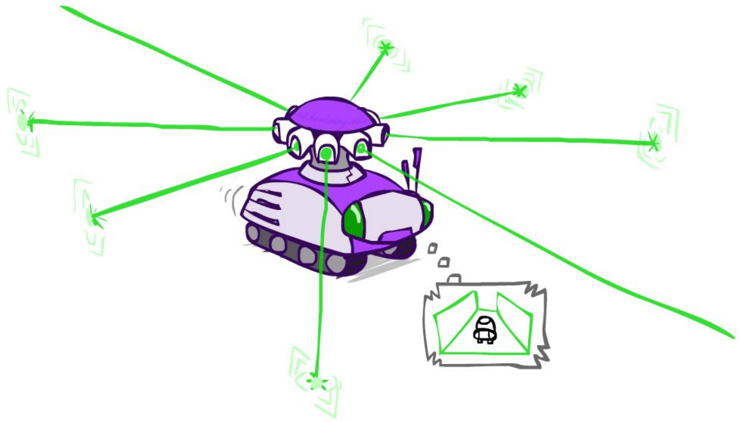
t=0

Sensor model: four bits for wall/no-wall in each direction, never more than 1 mistake

Transition model: action may fail with small prob.

Example: Robot Localization



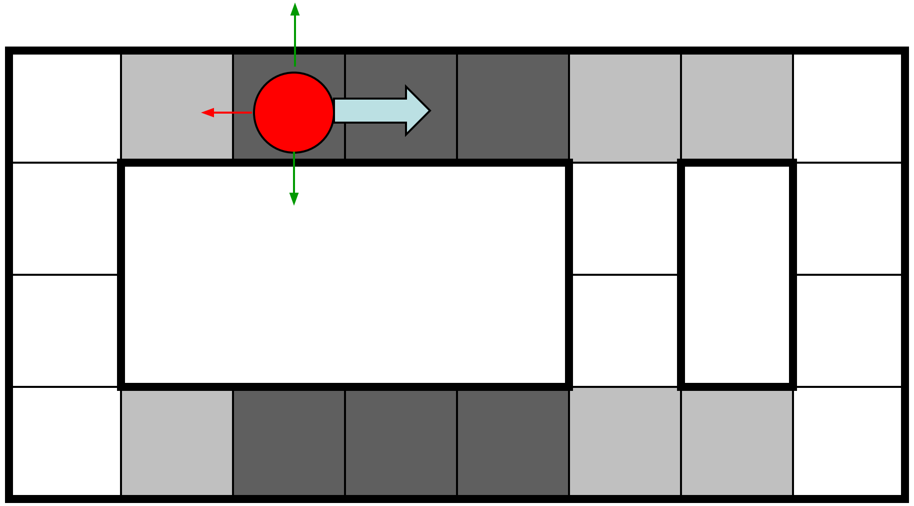


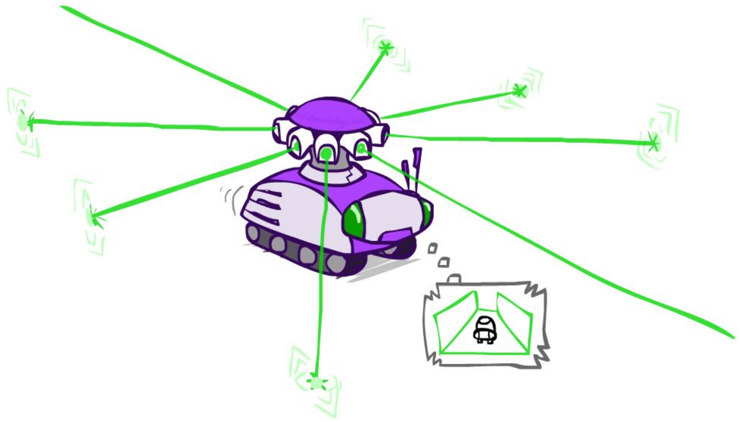
t=1

Lighter grey: was ***possible*** to get the reading,

but ***less likely*** (required 1 mistake)

Example: Robot Localization

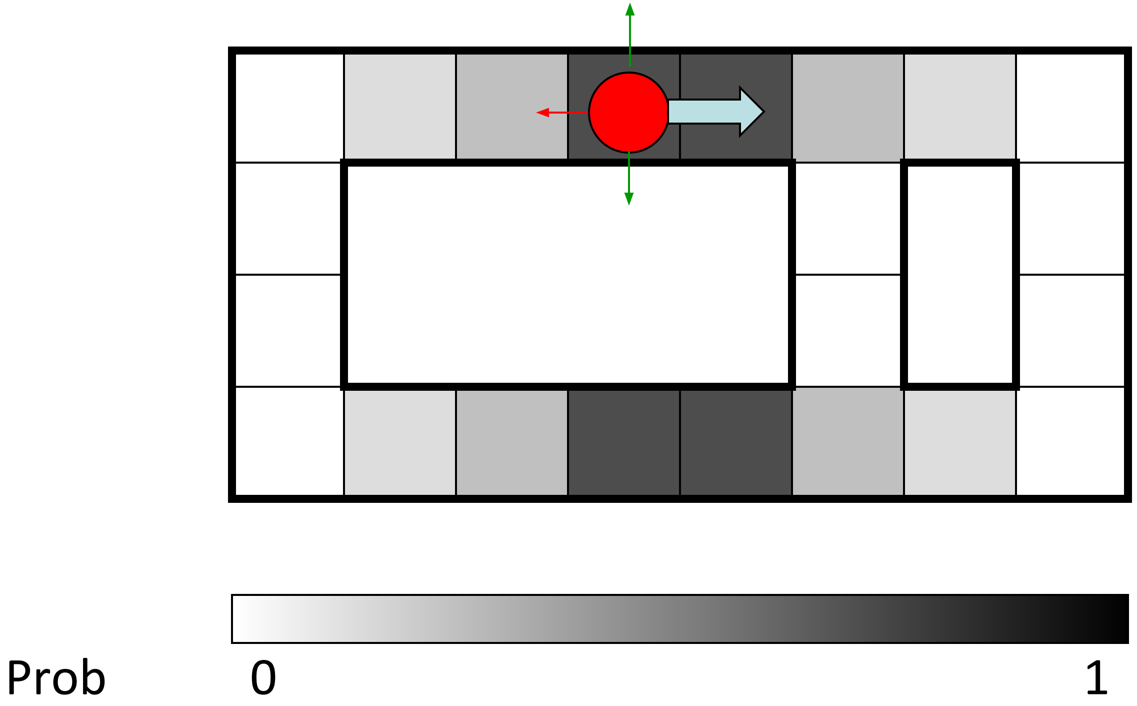


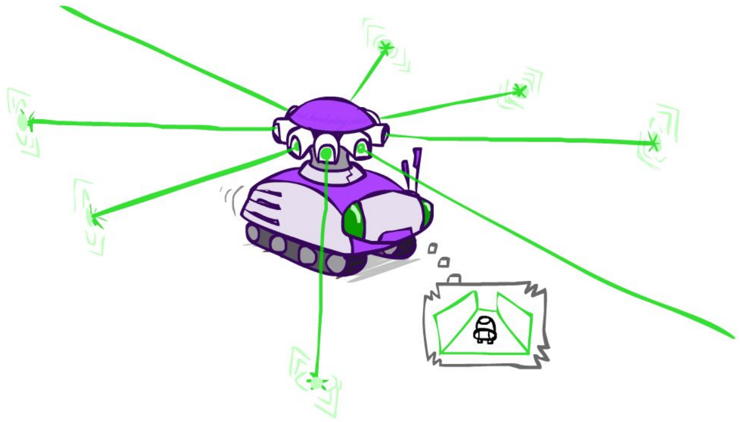


Prob 0 1

t=2

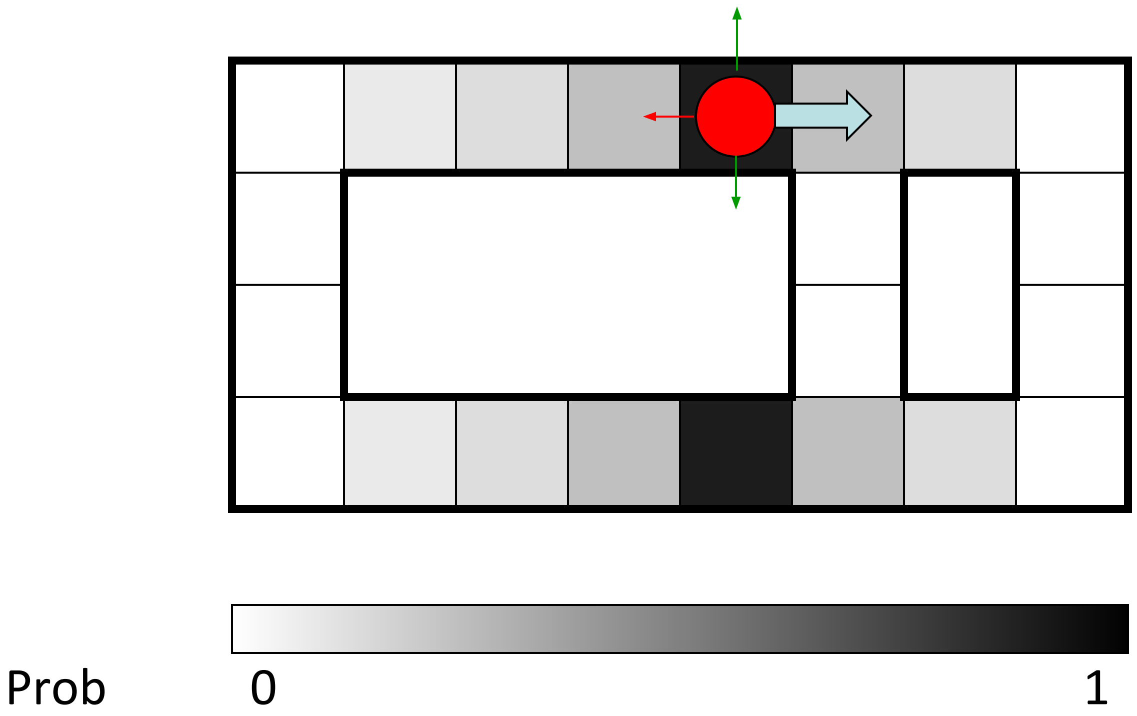
Example: Robot Localization

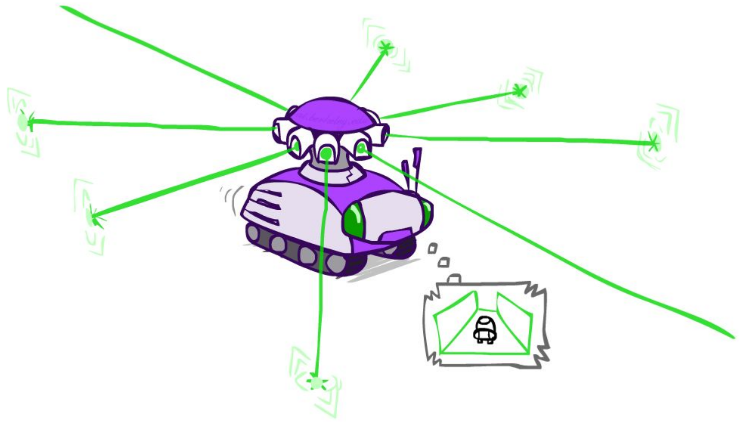




t=3

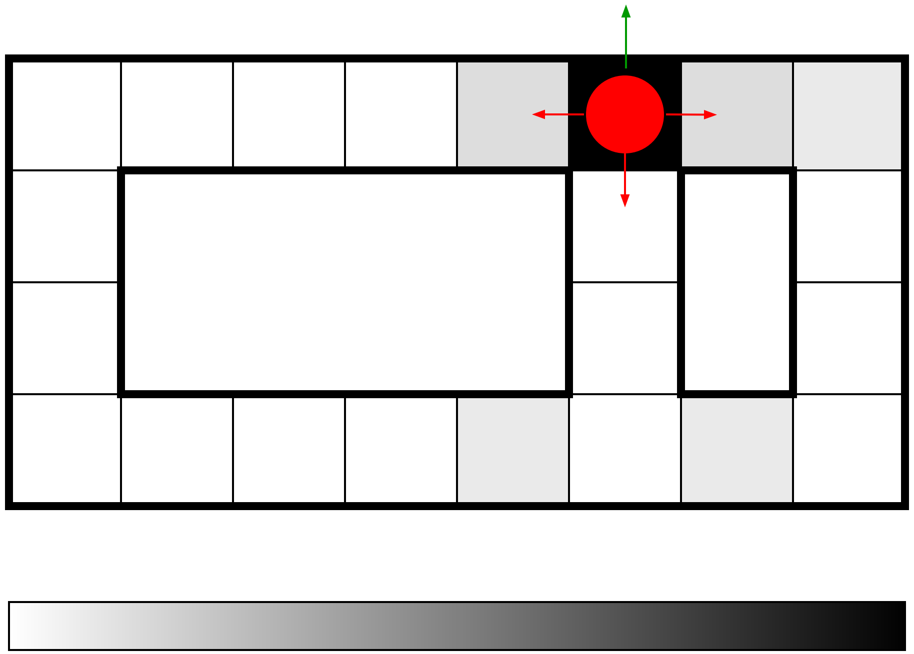
Example: Robot Localization



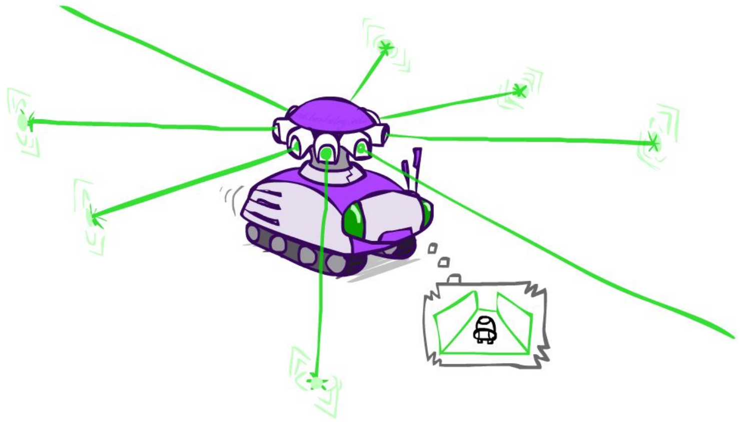


t=4

Example: Robot Localization



Prob 0 1



t=5

Filtering algorithm

- Aim: devise a ***recursive filtering*** algorithm of the form
  - $P(X_{t+1} | e_{1:t+1}) = g(e_{t+1}, P(X_t | e_{1:t}))$

- $P(X_{t+1} | e_{1:t+1}) =$

Filtering algorithm

- Aim: devise a ***recursive filtering*** algorithm of the form
  - $P(X_{t+1} | e_{1:t+1}) = g(e_{t+1}, P(X_t | e_{1:t}))$

- $P(X_{t+1} | e_{1:t+1}) = P(X_{t+1} | e_{1:t}, e_{t+1})$
- $\quad = \alpha\, P(e_{t+1} | X_{t+1}, e_{1:t})\, P(X_{t+1} | e_{1:t})$
- $\quad = \alpha\, P(e_{t+1} | X_{t+1})\, P(X_{t+1} | e_{1:t})$
- $\quad = \alpha\, P(e_{t+1} | X_{t+1}) \sum_{xt} P(x_t | e_{1:t})\, P(X_{t+1} | x_t, e_{1:t})$
- $\quad = \alpha\, P(e_{t+1} | X_{t+1}) \sum_{xt} P(x_t | e_{1:t})\, P(X_{t+1} | x_t)$

Apply Bayes' rule

Apply conditional independence

Condition on X_t

Apply conditional independence

Normalize

Update

Predict

Filtering algorithm

- $P(X_{t+1} | e_{1:t+1}) = \underbrace{\alpha}_{\text{Normalize}} \underbrace{P(e_{t+1} | X_{t+1})}_{\text{Update}} \underbrace{\sum_{x_t} P(x_t \mid e_{1:t})\, P(X_{t+1} | x_t)}_{\text{Predict}}$

- $\boldsymbol{f}_{1:t+1} = \text{F}\textsc{orward}(\boldsymbol{f}_{1:t}, e_{t+1})$
- Cost per time step: $O(|X|^2)$ where $|X|$ is the number of states
- Time and space costs are ***constant***, independent of t
- $O(|X|^2)$ is infeasible for models with many state variables
- We get to invent really cool approximate filtering algorithms

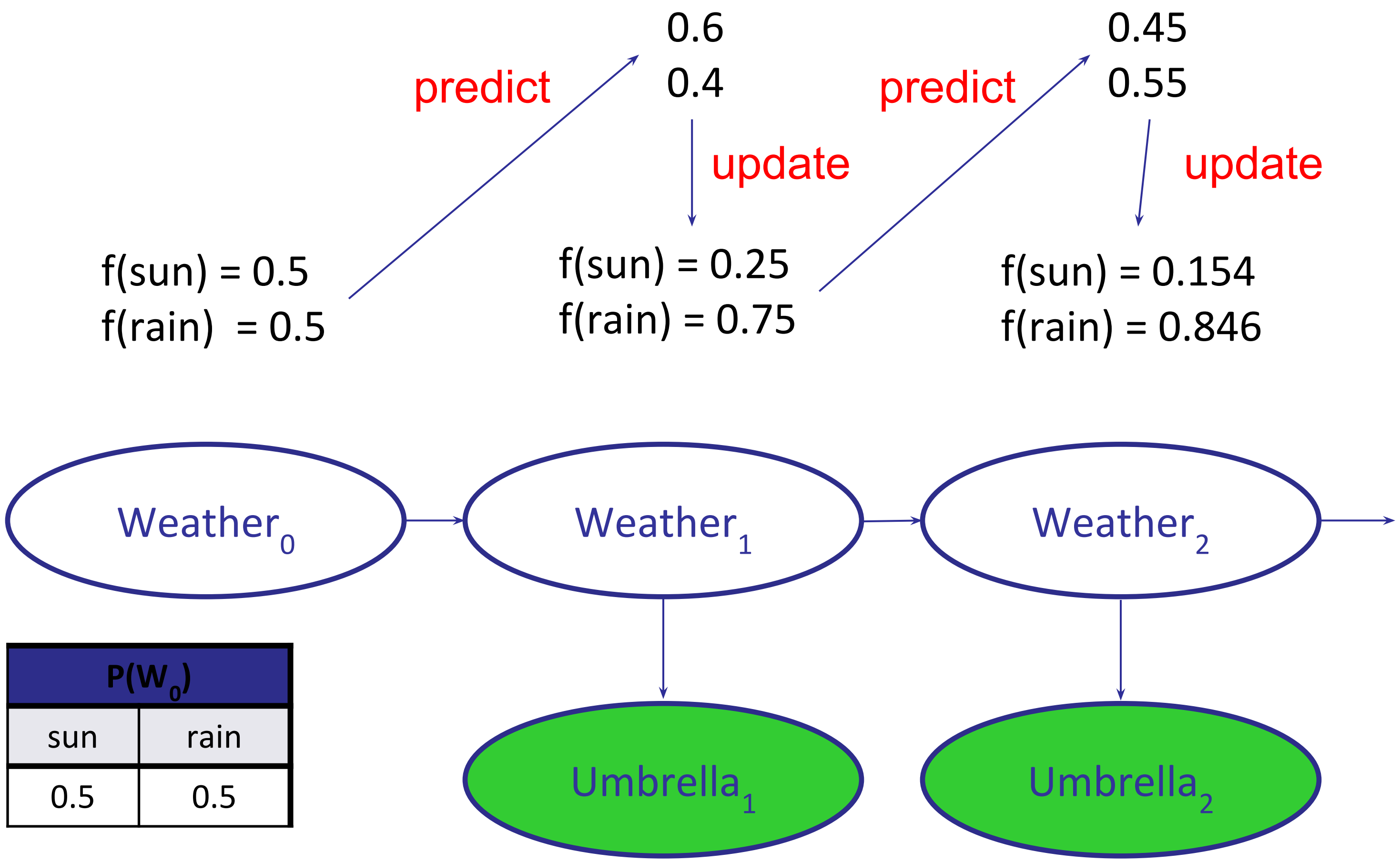
And the same thing in linear algebra

- Transition matrix T, observation matrix O_t
  - Observation matrix has state likelihoods for E_t along diagonal
  - E.g., for U_1 = true, $O_1 = \begin{pmatrix} 0.2 & 0 \\ 0 & 0.9 \end{pmatrix}$
- Filtering algorithm becomes
  - $\boldsymbol{f}_{1:t+1} = \alpha\, O_{t+1} T^{\mathrm{T}}\, \boldsymbol{f}_{1:t}$

X_{t-1}	$P(X_t \mid X_{t-1})$	
	sun	rain
sun	0.9	0.1
rain	0.3	0.7

W_t	$P(U_t \mid W_t)$	
	true	false
sun	0.2	0.8
rain	0.9	0.1

Example: Weather HMM



P(W_0)	
sun	rain
0.5	0.5

W_{t-1}	P(W_t|W_{t-1})	
	sun	rain
sun	0.9	0.1
rain	0.3	0.7

W_t	P(U_t|W_t)	
	true	false
sun	0.2	0.8
rain	0.9	0.1

Pacman – Hunting Invisible Ghosts with Sonar

[Demo: Pacman – Sonar – No Beliefs(L14D1)]

Video of Demo Pacman – Sonar

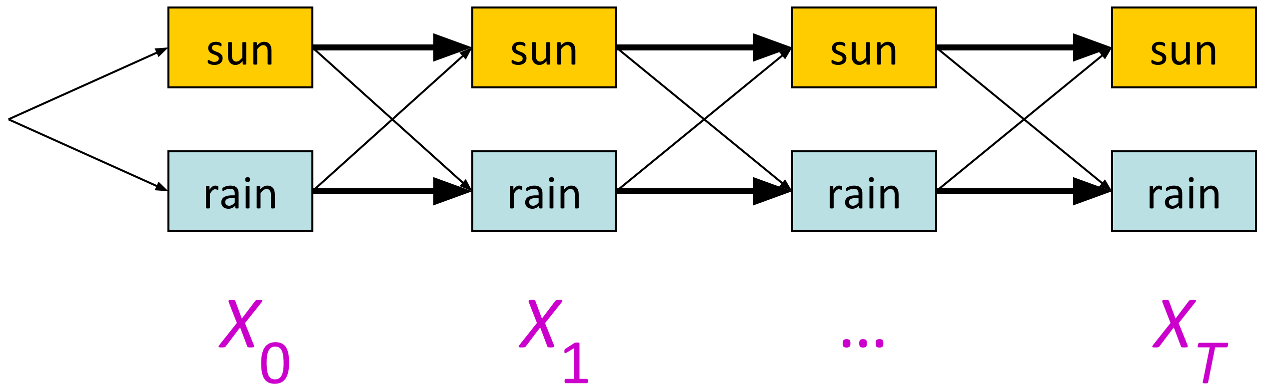
Most Likely Explanation

Inference tasks

- ***Filtering***: $P(X_t | e_{1:t})$
  - ***belief state***—input to the decision process of a rational agent
- ***Prediction***: $P(X_{t+k} | e_{1:t})$ for $k > 0$
  - evaluation of possible action sequences; like filtering without the evidence
- ***Smoothing***: $P(X_k | e_{1:t})$ for $0 \leq k < t$
  - better estimate of past states, essential for learning
- ***Most likely explanation***: $\arg\max_{x_{1:t}} P(x_{1:t} \mid e_{1:t})$
  - speech recognition, decoding with a noisy channel

Most likely explanation = most probable path

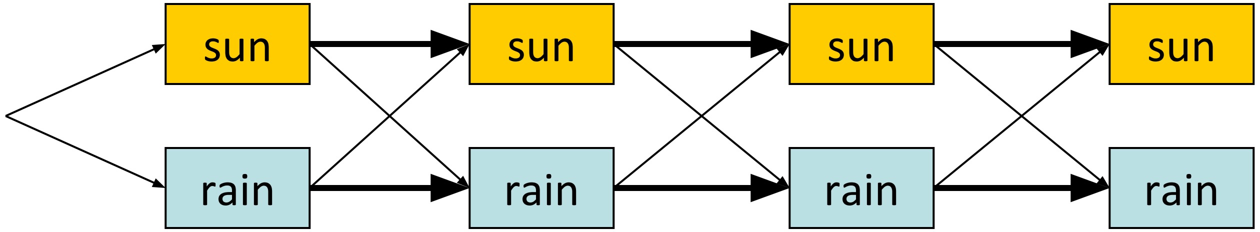
- ***State trellis***: graph of states and transitions over time



- $\arg\max_{x_{1:t}} P(x_{1:t} \mid e_{1:t})$
  $= \arg\max_{x_{1:t}} \alpha\, P(x_{1:t}, e_{1:t})$
  $= \arg\max_{x_{1:t}} P(x_{1:t}, e_{1:t})$
  $= \arg\max_{x_{1:t}} P(x_0) \prod_t P(x_t \mid x_{t-1}) P(e_t \mid x_t)$

- Each arc represents some transition $x_{t-1} \rightarrow x_t$
- Each arc has weight $P(x_t \mid x_{t-1})\ P(e_t \mid x_t)$ (arcs to initial states have weight $P(x_0)$)
- The ***product*** of weights on a path is proportional to that state sequence's probability
- Forward algorithm computes sums of paths, ***Viterbi algorithm*** computes best paths

Forward / Viterbi algorithms



X_0 X_1 … X_T

Forward Algorithm (sum)

For each state at time t, keep track of the ***total probability of all paths*** to it

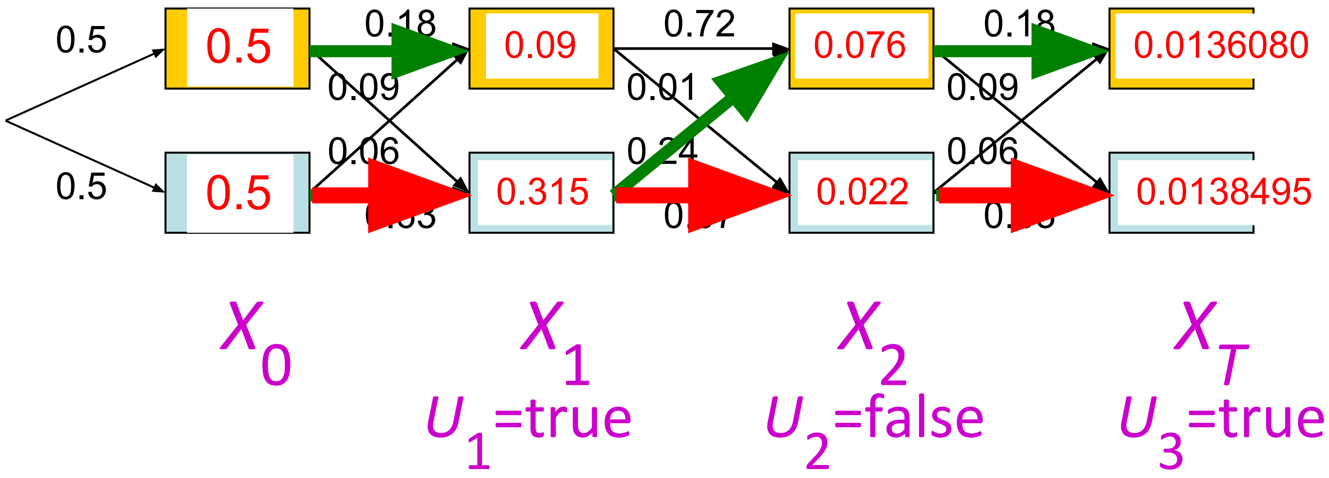
$$\mathbf{f}_{1:t+1} = \text{FORWARD}(\mathbf{f}_{1:t}, e_{t+1})$$
$$= \alpha\, P(e_{t+1}|X_{t+1}) \sum_{x_t} P(X_{t+1} \mid x_t)\, \mathbf{f}_{1:t}$$

Viterbi Algorithm (max)

For each state at time t, keep track of the ***maximum probability of any path*** to it

$$\mathbf{m}_{1:t+1} = \text{VITERBI}(\mathbf{m}_{1:t}, e_{t+1})$$
$$= P(e_{t+1}|X_{t+1}) \max_{x_t} P(X_{t+1} \mid x_t)\, \mathbf{m}_{1:t}$$

Viterbi algorithm contd.



W_{t-1}	$P(W_t \mid W_{t-1})$	
	sun	rain
sun	0.9	0.1
rain	0.3	0.7

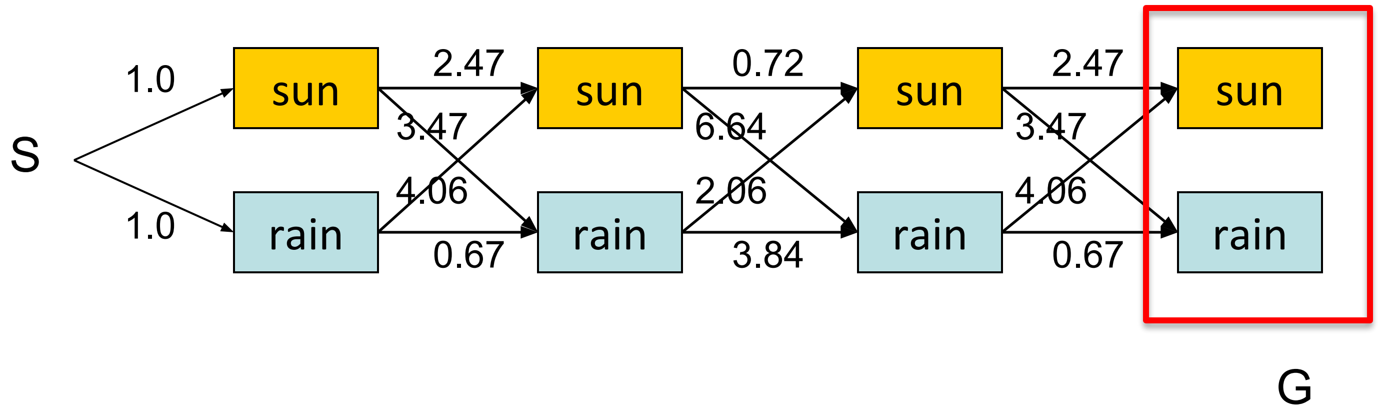
W_t	$P(U_t \mid W_t)$	
	true	false
sun	0.2	0.8
rain	0.9	0.1

Time complexity?
$O(|X|^2 T)$

Space complexity?
$O(|X| T)$

Number of paths?
$O(|X|^T)$

Viterbi in negative log space



W_{t-1}	$P(W_t \mid W_{t-1})$	
	sun	rain
sun	0.9	0.1
rain	0.3	0.7

W_t	$P(U_t \mid W_t)$	
	true	false
sun	0.2	0.8
rain	0.9	0.1

argmax of product of probabilities
= argmin of sum of negative log probabilities
= minimum-cost path

Viterbi is essentially breadth-first graph search
What about A*?