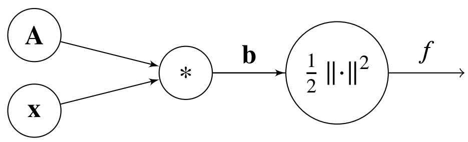
CS 188 Spring 2025 | Introduction to Artificial Intelligence | Exam Prep 11 Solutions

Q1. Backpropagation

In this question we will perform the backward pass algorithm on the formula

$$f = \frac{1}{2}\left\|\mathbf{A}\mathbf{x}\right\|^2$$

Here, $\mathbf{A} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\mathbf{b} = \mathbf{A}\mathbf{x} = \begin{bmatrix} A_{11}x_1 + A_{12}x_2 \\ A_{21}x_1 + A_{22}x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$, and $f = \frac{1}{2}\left\|\mathbf{b}\right\|^2 = \frac{1}{2}\left(b_1^2 + b_2^2\right)$ is a scalar.



(a) Calculate the following partial derivatives of f.

(i) Find $\frac{\partial f}{\partial \mathbf{b}} = \begin{bmatrix} \frac{\partial f}{\partial b_1} \\ \frac{\partial f}{\partial b_2} \end{bmatrix}$.

○ $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ ● $\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ ○ $\begin{bmatrix} b_2 \\ b_1 \end{bmatrix}$ ○ $\begin{bmatrix} f(b_1) \\ f(b_2) \end{bmatrix}$ ○ $\begin{bmatrix} A_{11} \\ A_{22} \end{bmatrix}$ ○ $\begin{bmatrix} b_1 + b_2 \\ b_1 - b_2 \end{bmatrix}$

(b) Calculate the following partial derivatives of b_1.

(i) $\left(\frac{\partial b_1}{\partial A_{11}}, \frac{\partial b_1}{\partial A_{12}}\right)$

○ (A_{11}, A_{12}) ○ $(0, 0)$ ○ (x_2, x_1) ○ $(A_{11}x_1, A_{12}x_2)$ ● (x_1, x_2)

(ii) $\left(\frac{\partial b_1}{\partial A_{21}}, \frac{\partial b_1}{\partial A_{22}}\right)$

○ (A_{21}, A_{22}) ○ (x_1, x_2) ○ $(1, 1)$ ● $(0, 0)$ ○ $(A_{21}x_1, A_{22}x_2)$

(iii) $\left(\frac{\partial b_1}{\partial x_1}, \frac{\partial b_1}{\partial x_2}\right)$

● (A_{11}, A_{12}) ○ (A_{21}, A_{22}) ○ $(0, 0)$ ○ (b_1, b_2) ○ $(A_{21}x_1, A_{22}x_2)$

(c) Calculate the following partial derivatives of f.

(i) $\left(\frac{\partial f}{\partial A_{11}}, \frac{\partial f}{\partial A_{12}}\right)$

○ (A_{11}, A_{12}) ○ $(A_{11}b_1, A_{12}b_2)$ ○ $(A_{11}x_1, A_{12}x_2)$
● (x_1b_1, x_2b_1) ○ (x_1b_2, x_2b_2) ○ (x_1b_1, x_2b_2)

(ii) $\left(\frac{\partial f}{\partial A_{21}}, \frac{\partial f}{\partial A_{22}}\right)$

○ (A_{21}, A_{22}) ○ $(A_{21}b_1, A_{22}b_2)$ ○ $(A_{21}x_1, A_{22}x_2)$
○ (x_1b_1, x_2b_1) ● (x_1b_2, x_2b_2) ○ (x_1b_1, x_2b_2)

(iii) $\left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}\right)$

○ $(A_{11}b_1 + A_{12}b_2, A_{21}b_1 + A_{22}b_2)$ ● $(A_{11}b_1 + A_{21}b_2, A_{12}b_1 + A_{22}b_2)$
○ $(A_{11}b_1 + A_{12}b_1, A_{21}b_2 + A_{22}b_2)$ ○ $(A_{11}b_1 + A_{21}b_1, A_{12}b_2 + A_{22}b_2)$

(d) Now we consider the general case where $\mathbf{A}$ is an $n \times d$ matrix, and $\mathbf{x}$ is a $d \times 1$ vector. As before, $f = \frac{1}{2} \|\mathbf{A}\mathbf{x}\|^2$.

(i) Find $\frac{\partial f}{\partial \mathbf{A}}$ in terms of $\mathbf{A}$ and $\mathbf{x}$ only.

○ $\mathbf{x}^\top \mathbf{A}^\top \mathbf{A}\mathbf{x}$ ● $\mathbf{A}\mathbf{x}\mathbf{x}^\top$ ○ $\mathbf{A}\left(\mathbf{A}^\top \mathbf{A}\right)^{-1}$ ○ $\mathbf{A}\mathbf{A}^\top \mathbf{A}\mathbf{x}$ ○ $\mathbf{A}$

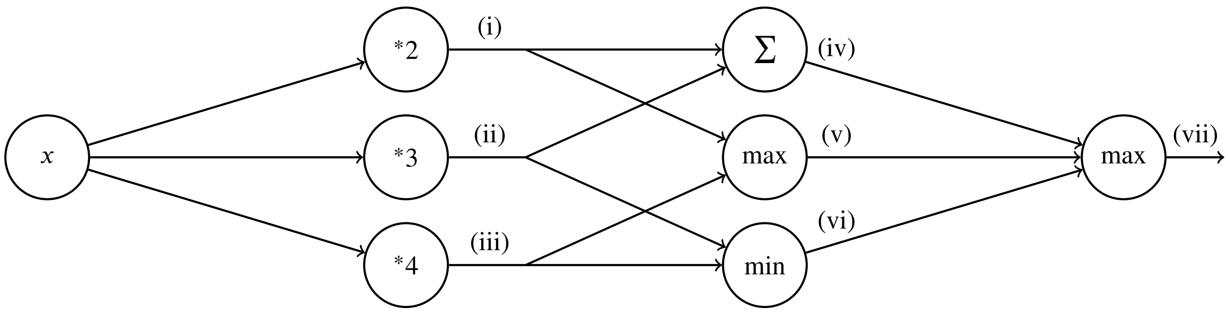
(ii) Find $\frac{\partial f}{\partial \mathbf{x}}$ in terms of $\mathbf{A}$ and $\mathbf{x}$ only.

○ $\mathbf{x}$ ○ $\left(\mathbf{A}^\top \mathbf{A}\right)^{-1} \mathbf{x}$ ○ $\mathbf{x}\mathbf{x}^\top \mathbf{x}$ ○ $\mathbf{x}^\top \mathbf{A}^\top \mathbf{A}\mathbf{x}$ ● $\mathbf{A}^\top \mathbf{A}\mathbf{x}$

Q2. Deep Learning

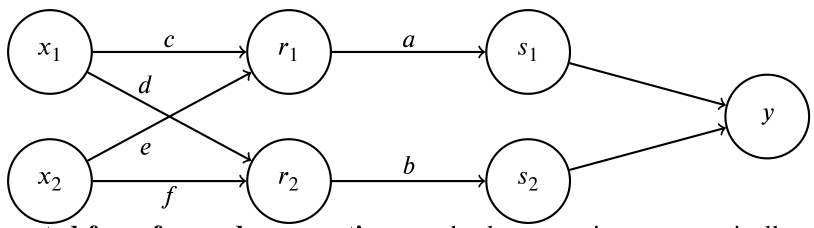
(a) Perform forward propagation on the neural network below for $x = 1$ by filling in the values in the table. Note that (i), ..., (vii) are outputs after performing the appropriate operation as indicated in the node.

(i)	(ii)	(iii)	(iv)	(v)	(vi)	(vii)
2	3	4	5	4	3	5



(b) **[Optional]** Below is a neural network with weights a, b, c, d, e, f. The inputs are x_1 and x_2.
The first hidden layer computes $r_1 = \max(c \cdot x_1 + e \cdot x_2, 0)$ and $r_2 = \max(d \cdot x_1 + f \cdot x_2, 0)$.
The second hidden layer computes $s_1 = \frac{1}{1+\exp(-a \cdot r_1)}$ and $s_2 = \frac{1}{1+\exp(-b \cdot r_2)}$.
The output layer computes $y = s_1 + s_2$. Note that the weights a, b, c, d, e, f are indicated along the edges of the neural network here.

Suppose the network has inputs $x_1 = 1, x_2 = -1$.
The weight values are $a = 1, b = 1, c = 4, d = 1, e = 2, f = 2$.
Forward propagation then computes $r_1 = 2, r_2 = 0, s_1 = 0.9, s_2 = 0.5, y = 1.4$. Note: some values are rounded.



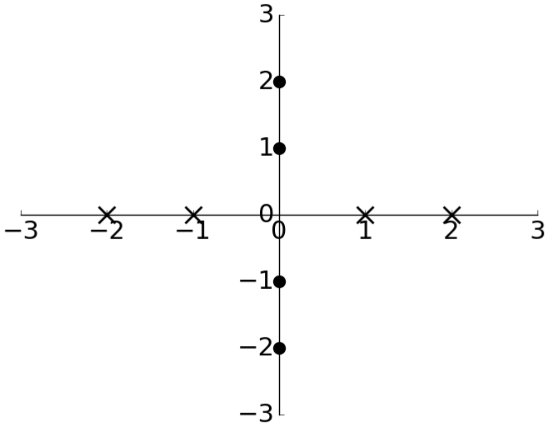
Using the values computed from forward propagation, use backpropagation to numerically calculate the following partial derivatives. Write your answers as a single number (not an expression). You do not need a calculator. Use scratch paper if needed.
Hint: For $g(z) = \frac{1}{1+\exp(-z)}$, the derivative is $\frac{\partial g}{\partial z} = g(z)(1 - g(z))$.

$\frac{\partial y}{\partial a}$	$\frac{\partial y}{\partial b}$	$\frac{\partial y}{\partial c}$	$\frac{\partial y}{\partial d}$	$\frac{\partial y}{\partial e}$	$\frac{\partial y}{\partial f}$
0.18	0	0.09	0	−0.09	0

$$
\begin{aligned}
\frac{\partial y}{\partial a} &= \frac{\partial y}{\partial s_1}\frac{\partial s_1}{\partial a} \\
&= 1 \cdot \frac{\partial g(a \cdot r_1)}{\partial a} \\
&= r_1 \cdot g(a \cdot r_1)(1 - g(a \cdot r_1)) \\
&= r_1 \cdot s_1(1 - s_1) \\
&= 2 \cdot 0.9 \cdot (1 - 0.9) \\
&= 0.18 \\
\frac{\partial y}{\partial b} &= \frac{\partial y}{\partial s_2}\frac{\partial s_2}{\partial b} \\
&= 1 \cdot \frac{\partial g(b \cdot r_2)}{\partial b} \\
&= r_2 \cdot g(b \cdot r_2)(1 - g(b \cdot r_2)) \\
&= r_2 \cdot s_2(1 - s_2) \\
&= 0 \cdot 0.5(1 - 0.5) \\
&= 0 \\
\frac{\partial y}{\partial c} &= \frac{\partial y}{\partial s_1}\frac{\partial s_1}{\partial r_1}\frac{\partial r_1}{\partial c} \\
&= 1 \cdot [a \cdot g(a \cdot r_1)(1 - g(a \cdot r_1))] \cdot x_1 \\
&= [a \cdot s_1(1 - s_1)] \cdot x_1 \\
&= [1 \cdot 0.9(1 - 0.9)] \cdot 1 \\
&= 0.09 \\
\frac{\partial y}{\partial d} &= \frac{\partial y}{\partial s_2}\frac{\partial s_2}{\partial r_2}\frac{\partial r_2}{\partial d} \\
&= \frac{\partial y}{\partial s_2}\frac{\partial s_2}{\partial r_2} \cdot 0 \\
&= 0 \\
\frac{\partial y}{\partial e} &= \frac{\partial y}{\partial s_1}\frac{\partial s_1}{\partial r_1}\frac{\partial r_1}{\partial e} \\
&= 1 \cdot [a \cdot g(a \cdot r_1)(1 - g(a \cdot r_1))] \cdot x_2 \\
&= [a \cdot s_1(1 - s_1)] \cdot x_2 \\
&= [1 \cdot 0.9(1 - 0.9)] \cdot -1 \\
&= -0.09 \\
\frac{\partial y}{\partial f} &= \frac{\partial y}{\partial s_2}\frac{\partial s_2}{\partial r_2}\frac{\partial r_2}{\partial f} \\
&= \frac{\partial y}{\partial s_2}\frac{\partial s_2}{\partial r_2} \cdot 0 \\
&= 0
\end{aligned}
$$

(c) Below are two plots with horizontal axis x_1 and vertical axis x_2 containing data labelled $\times$ and $\bullet$. For each plot, we wish to find a function $f(x_1, x_2)$ such that $f(x_1, x_2) \geq 0$ for all data labelled $\times$ and $f(x_1, x_2) < 0$ for all data labelled $\bullet$.

Below each plot is the function $f(x_1, x_2)$ for that specific plot. Complete the expressions such that all the data is labelled correctly. If not possible, mark "No valid combination".



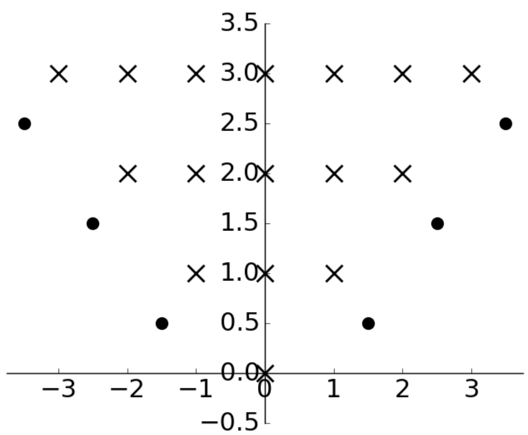
$f(x_1, x_2) = \max(\underline{\textbf{(i)}} + \underline{\textbf{(ii)}}, \underline{\textbf{(iii)}} + \underline{\textbf{(iv)}}) + \underline{\textbf{(v)}}$

(i)	● x_1	○ $-x_1$	○ 0
(ii)	○ x_2	○ $-x_2$	● 0
(iii)	○ x_1	● $-x_1$	○ 0
(iv)	○ x_2	○ $-x_2$	● 0
(v)	○ 1	● -1	○ 0

○ No valid combination

There are two possible solutions:

$$f(x_1, x_2) = \max(x_1, -x_1) - 1$$
$$f(x_1, x_2) = \max(-x_1, x_1) - 1$$



$f(x_1, x_2) = \underline{\textbf{(vi)}} - \max(\underline{\textbf{(vii)}} + \underline{\textbf{(viii)}}, \underline{\textbf{(ix)}} + \underline{\textbf{(x)}})$

(vi)	● x_2	○ $-x_2$	○ 0
(vii)	● x_1	○ $-x_1$	○ 0
(viii)	○ x_2	○ $-x_2$	● 0
(ix)	○ x_1	● $-x_1$	○ 0
(x)	○ x_2	○ $-x_2$	● 0

○ No valid combination

There are four possible solutions:

$$f(x_1, x_2) = x_2 - \max(x_1, -x_1)$$
$$f(x_1, x_2) = x_2 - \max(-x_1, x_1)$$
$$f(x_1, x_2) = -\max(x_1 - x_2, -x_1 - x_2)$$
$$f(x_2, x_2) = -\max(-x_1 - x_2, x_1 - x_2)$$