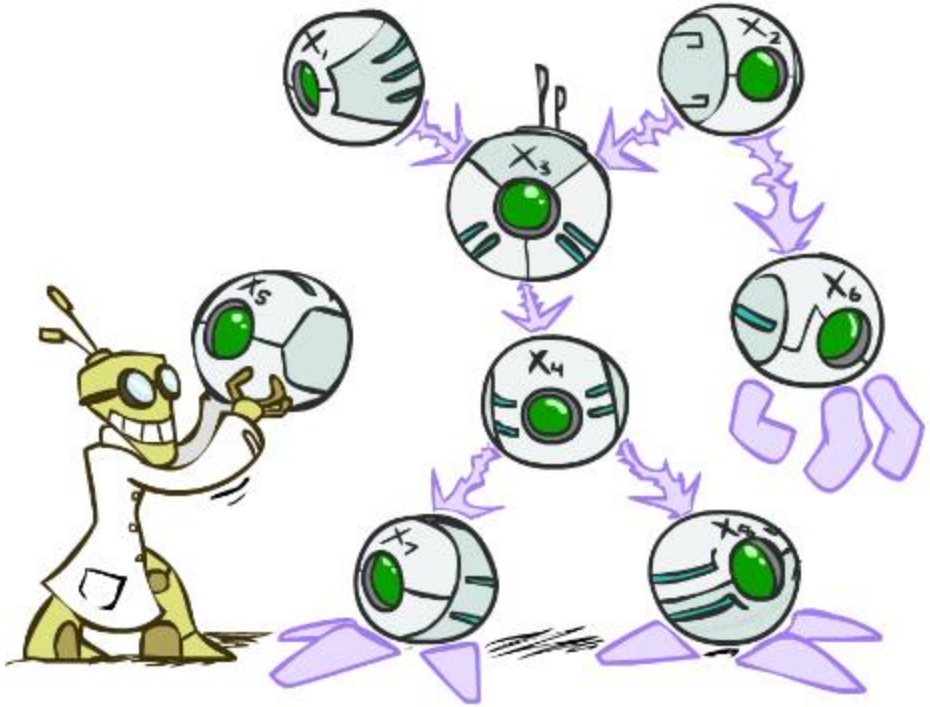
CS 188: Artificial Intelligence

Bayes' Nets



[These slides were created by Dan Klein and Pieter Abbeel for CS188 Intro to AI at UC Berkeley. All CS188 materials are available at http://ai.berkeley.edu.]

Announcements

- Midterm
  - Wednesday March 19, 7-9pm
  - Check Ed and Calendar for more midterm logistics/prep sessions, and see exam logistics page near top of course web site for more info.
- HW5
  - Due on Wednesday 3/5/25 at 11:59 PT
- Project 3
  - Due on Friday 3/7/25 at 11:59 PT

Recap: Probabilistic Inference

- *Probabilistic inference:* compute a desired probability from other known probabilities (e.g. conditional from joint)

- We generally compute conditional probabilities
  - P(on time | no reported accidents)
  - P(on time | no accidents, 5 a.m.)
  - …
  - These represent the agent's *beliefs* given the evidence
  - Observing new evidence causes *beliefs to be updated*

- Saw *Inference by Enumeration* as our first algorithm to do this

Recap: Probability Distributions

- Joint Distribution: $P(X, Y, \dots)$

Example:

$$P(T, W)$$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

- Marginal Distribution $P(X)$:

$$P(x) = \sum_y P(x, y)$$

- Conditional Distribution $P(X|y)$:
  - $P(X|Y)$ denotes a collection of distributions for each value y

$$P(x|y) = \frac{P(x, y)}{P(y)}$$

Recap: Probability Rules

- Product Rule:

$$P(y)P(x|y) = P(x,y)$$

- Chain Rule:

$$P(x_1, x_2, x_3) = P(x_1)P(x_2|x_1)P(x_3|x_1, x_2)$$

$$P(x_1, x_2, \ldots x_n) = \prod_i P(x_i|x_1 \ldots x_{i-1})$$

- Bayes Rule:

$$P(x|y) = \frac{P(y|x)}{P(y)}P(x) \qquad P(\text{cause}|\text{effect}) = \frac{P(\text{effect}|\text{cause})P(\text{cause})}{P(\text{effect})}$$

Example: Inference with Bayes' Rule

- Example: Diagnostic probability from causal probability:

$$P(\text{cause}|\text{effect}) = \frac{P(\text{effect}|\text{cause})P(\text{cause})}{P(\text{effect})}$$

- Example:
  - M: meningitis, S: stiff neck

$$\left.\begin{aligned} P(+m) &= 0.0001 \\ P(+s|+m) &= 0.8 \\ P(+s|-m) &= 0.01 \end{aligned}\right\} \text{Example givens}$$

$$P(+m|+s) = \frac{P(+s|+m)P(+m)}{P(+s)} = \frac{P(+s|+m)P(+m)}{P(+s|+m)P(+m) + P(+s|-m)P(-m)} = \frac{0.8 \times 0.0001}{0.8 \times 0.0001 + 0.01 \times 0.999}$$

P(+m | +s) ≅ 0.008

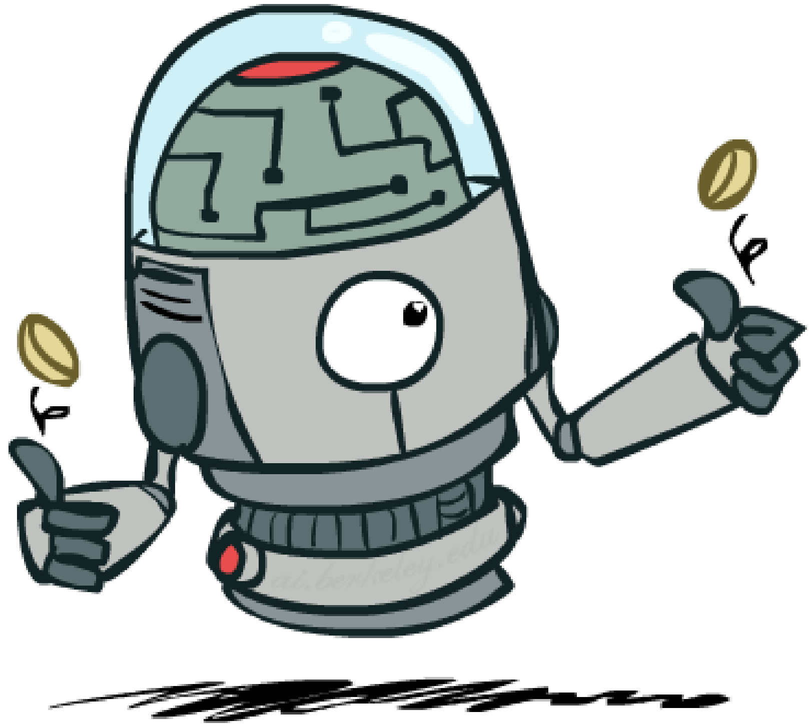
Probabilistic Models

- Models describe how (a portion of) the world works

- Models are always simplifications
  - May not account for every variable
  - May not account for all interactions between variables
  - "All models are wrong; but some are useful."
    – George E. P. Box

- What do we do with probabilistic models?
  - We (or our agents) need to reason about unknown variables, given evidence
  - Example: explanation (diagnostic reasoning)
  - Example: prediction (causal reasoning)
  - Example: value of information

Independence



Independence

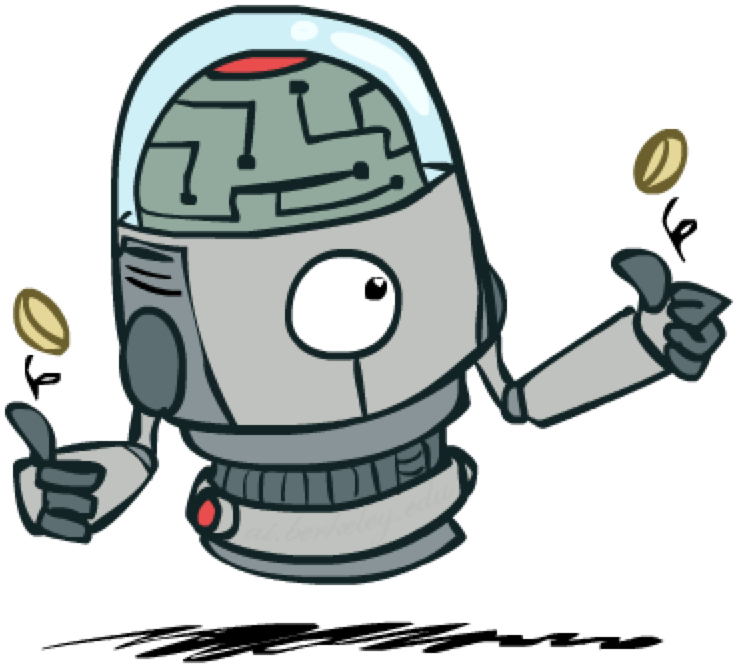
- Two variables are *independent* if:

$$\forall x, y : P(x, y) = P(x)P(y)$$

  - This says that their joint distribution *factors* into a product two simpler distributions
  - Another form:

$$\forall\, x, y : P(x|y) = P(x)$$

  - We write: $X \perp\!\!\!\perp Y$



- Independence is a simplifying *modeling assumption*
  - *Empirical* joint distributions: at best "close" to independent
  - What could we assume for {Weather, Traffic, Cavity, Toothache}?

Example: Independence?

$P(T)$

T	P
hot	0.5
cold	0.5

$P_1(T, W)$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

$P_2(T, W)$

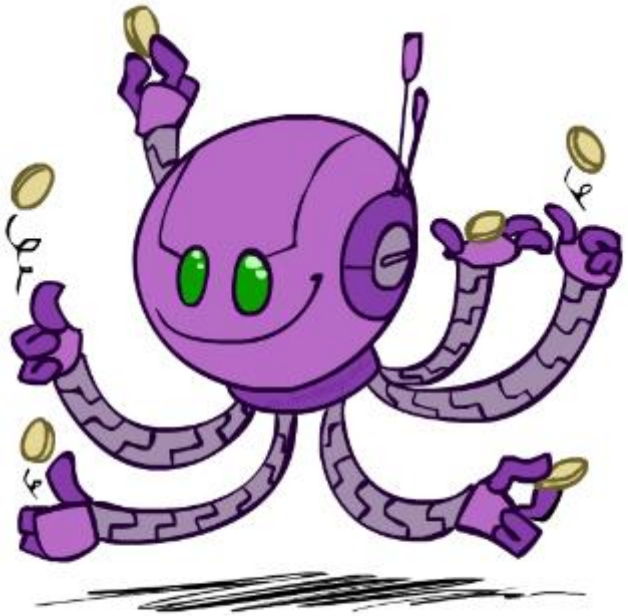
T	W	P
hot	sun	0.3
hot	rain	0.2
cold	sun	0.3
cold	rain	0.2

$P(W)$

W	P
sun	0.6
rain	0.4

Example: Independence

- N fair, independent coin flips:



$P(X_1)$

H	0.5
T	0.5

$P(X_2)$

H	0.5
T	0.5

. . .

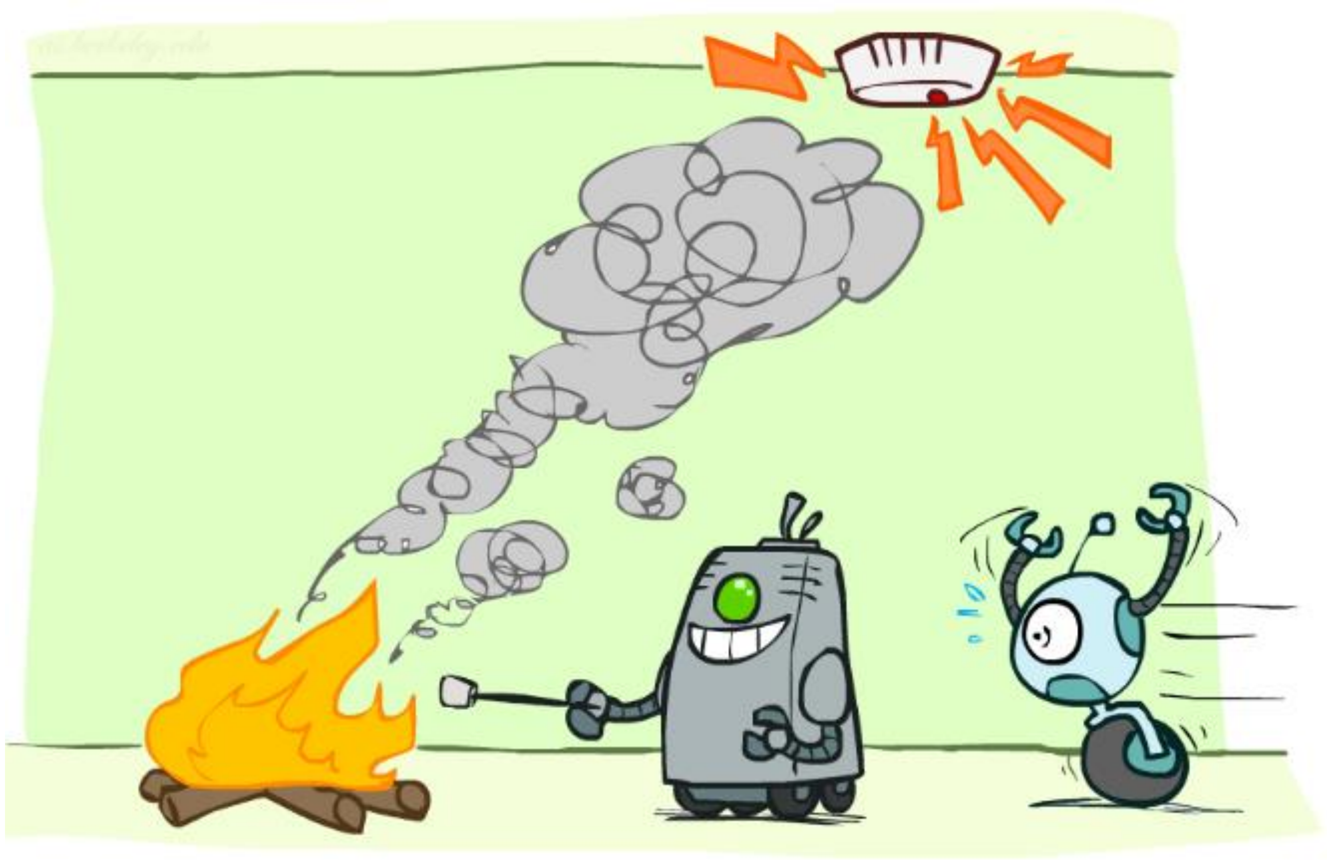
$P(X_n)$

H	0.5
T	0.5

$P(X_1, X_2, \ldots, X_n)$

2^n { ⋮ }

Conditional Independence



Conditional Independence

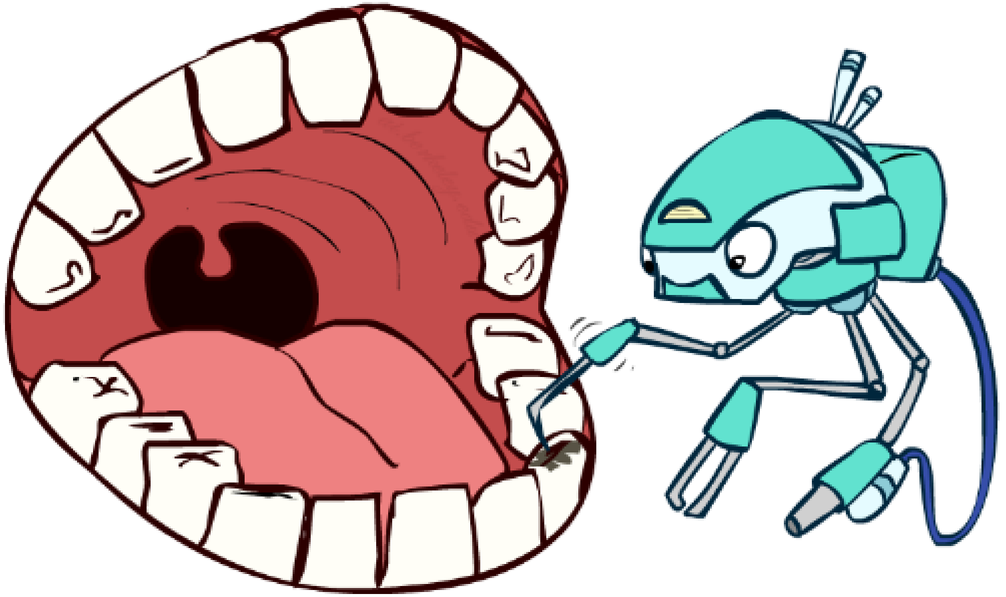
- P(Toothache, Cavity, Catch)

- If I have a cavity, the probability that the probe catches in it doesn't depend on whether I have a toothache:
  - P(+catch | +toothache, +cavity) = P(+catch | +cavity)

- The same independence holds if I don't have a cavity:
  - P(+catch | +toothache, -cavity) = P(+catch| -cavity)

- Catch is *conditionally independent* of Toothache given Cavity:
  - P(Catch | Toothache, Cavity) = P(Catch | Cavity)

- Equivalent statements:
  - P(Toothache | Catch, Cavity) = P(Toothache | Cavity)
  - P(Toothache, Catch | Cavity) = P(Toothache | Cavity) P(Catch | Cavity)
  - One can be derived from the other easily



Conditional Independence

- Unconditional (absolute) independence very rare between variables in the same system (why?)
- *Conditional independence* is our most basic and robust form of knowledge about uncertain environments.
- X is conditionally independent of Y given Z written $X \perp\!\!\!\perp Y | Z$

  if and only if:

$$\forall\, x, y, z : P(x, y|z) = P(x|z)P(y|z)$$

  or, equivalently, if and only if

$$\forall\, x, y, z : P(x|y, z) = P(x|z)$$

  or

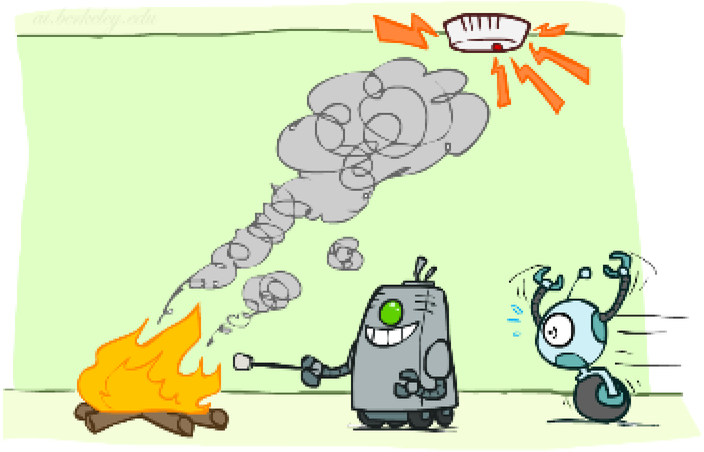
$$\forall\, x, y, z : P(y|x, z) = P(y|z)$$

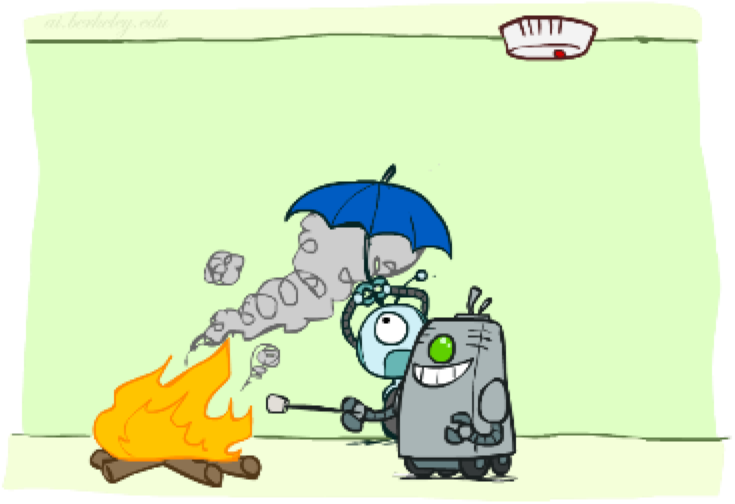
Conditional Independence

- What about this domain:
  - Traffic
  - Umbrella
  - Raining

Conditional Independence

- What about this domain:
  - Fire
  - Smoke
  - Alarm





Conditional Independence and the Chain Rule

- Chain rule: $P(X_1, X_2, \ldots, X_n) = P(X_1)P(X_2|X_1)P(X_3|X_1, X_2) \ldots$

- Standard decomposition:

$$P(\text{Traffic}, \text{Rain}, \text{Umbrella}) = P(\text{Rain})P(\text{Traffic}|\text{Rain})P(\text{Umbrella}|\text{Rain}, \text{Traffic})$$

- With assumption of conditional independence:

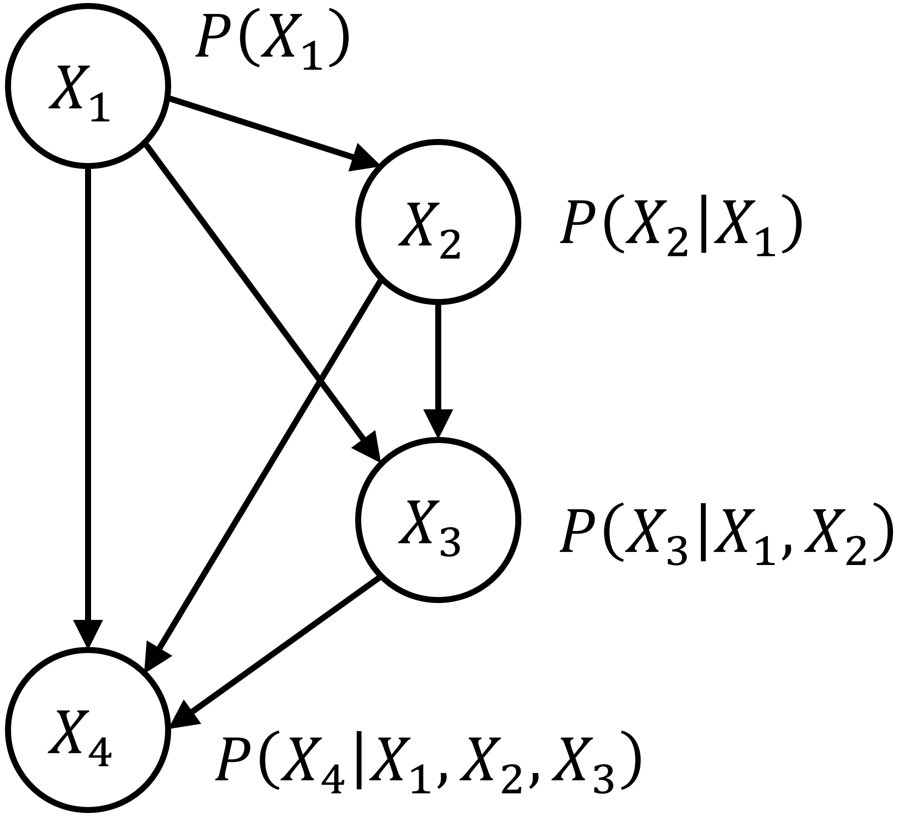
$$P(\text{Traffic}, \text{Rain}, \text{Umbrella}) = P(\text{Rain})P(\text{Traffic}|\text{Rain})P(\text{Umbrella}|\text{Rain})$$

- Bayes'nets / graphical models help us express conditional independence assumptions

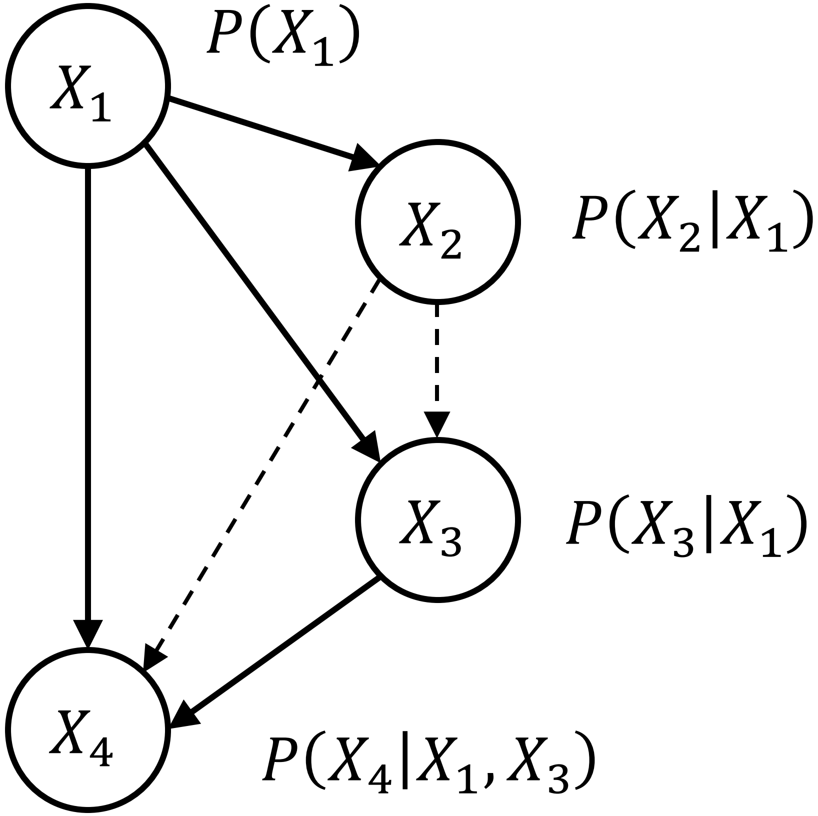
Conditional Independence and the Chain Rule

- Chain rule: $P(X_1, X_2, \ldots, X_n) = P(X_1)P(X_2|X_1)P(X_3|X_1, X_2)\ldots$ works in any order of X_i

Compete Dependency graph:



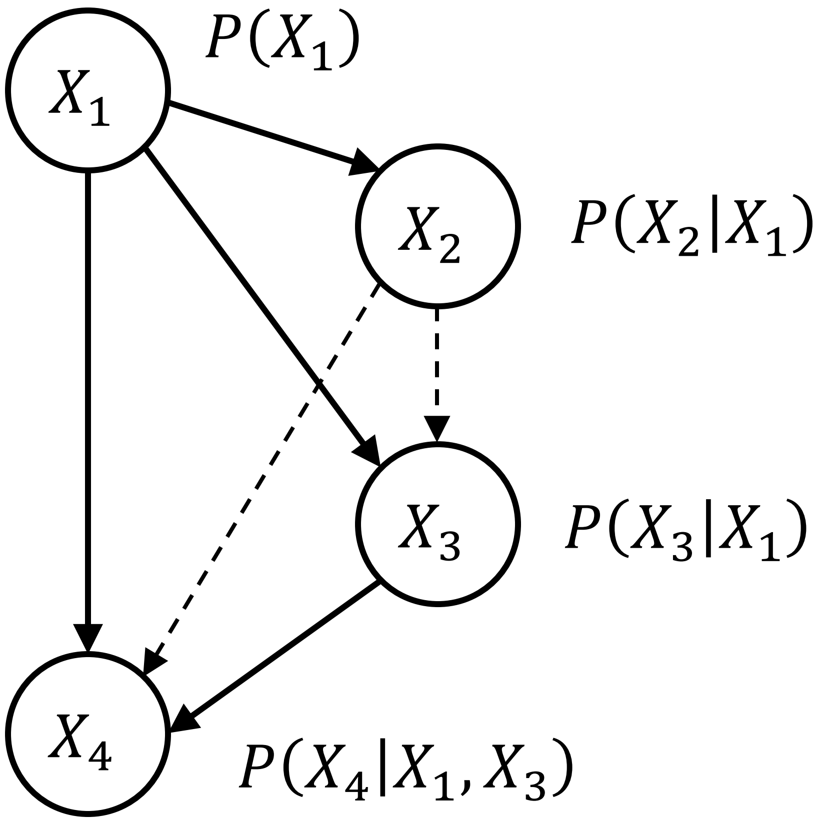
Partial Dependency Graph:



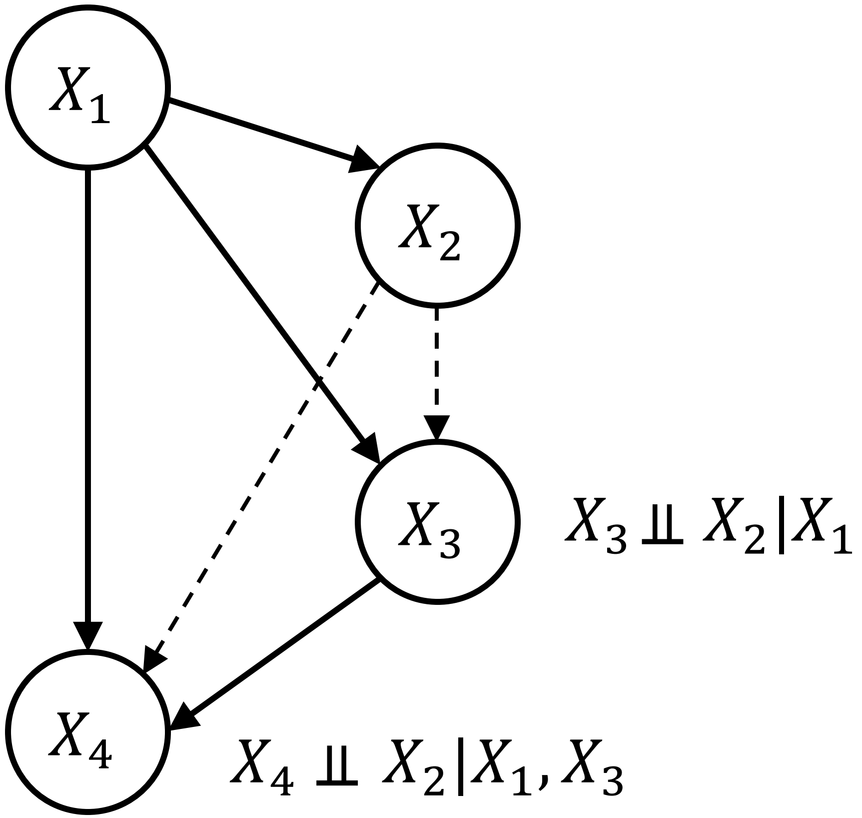
Conditional Independence and the Chain Rule

- Chain rule: $P(X_1, X_2, \dots, X_n) = P(X_1)P(X_2|X_1)P(X_3|X_1, X_2) \dots$

  Partial Dependency graph:

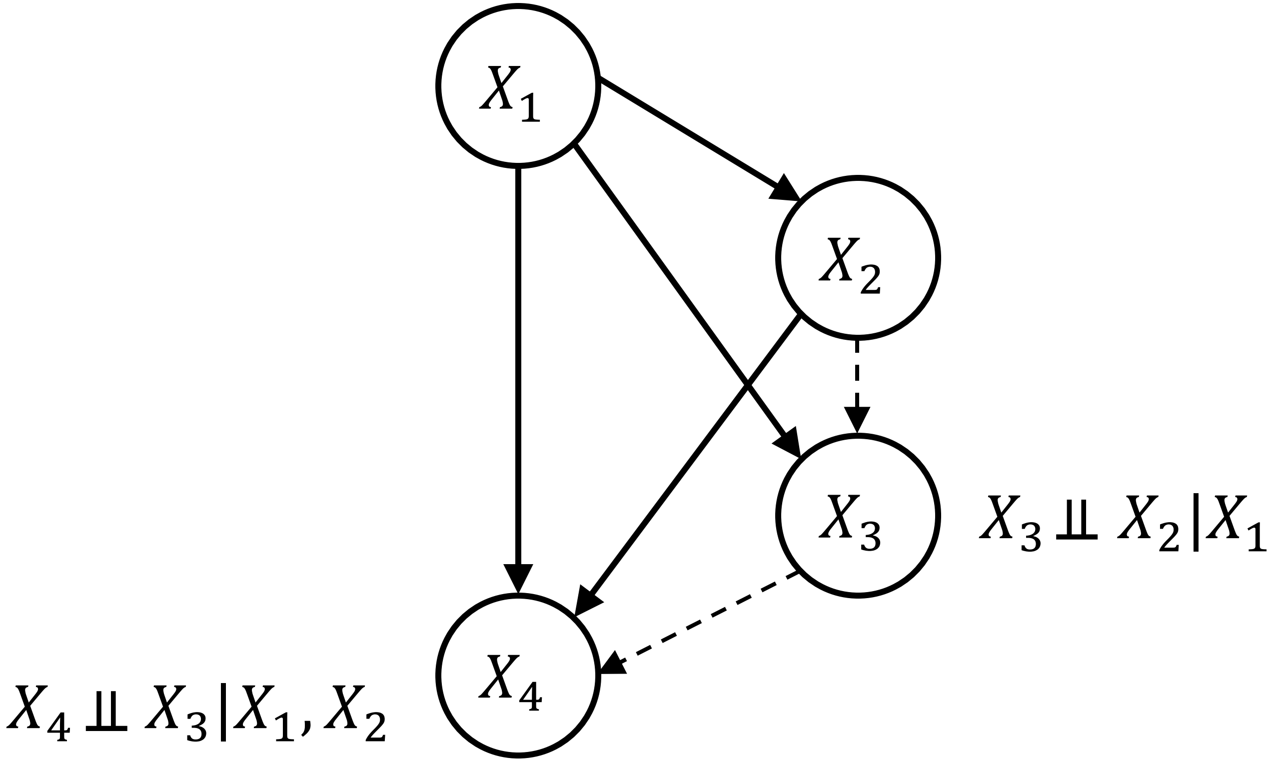


  Independences from the Graph:



Conditional Independence and the Chain Rule

- In general a node is conditionally independent from all its non-descendents in the graph, given its parents.



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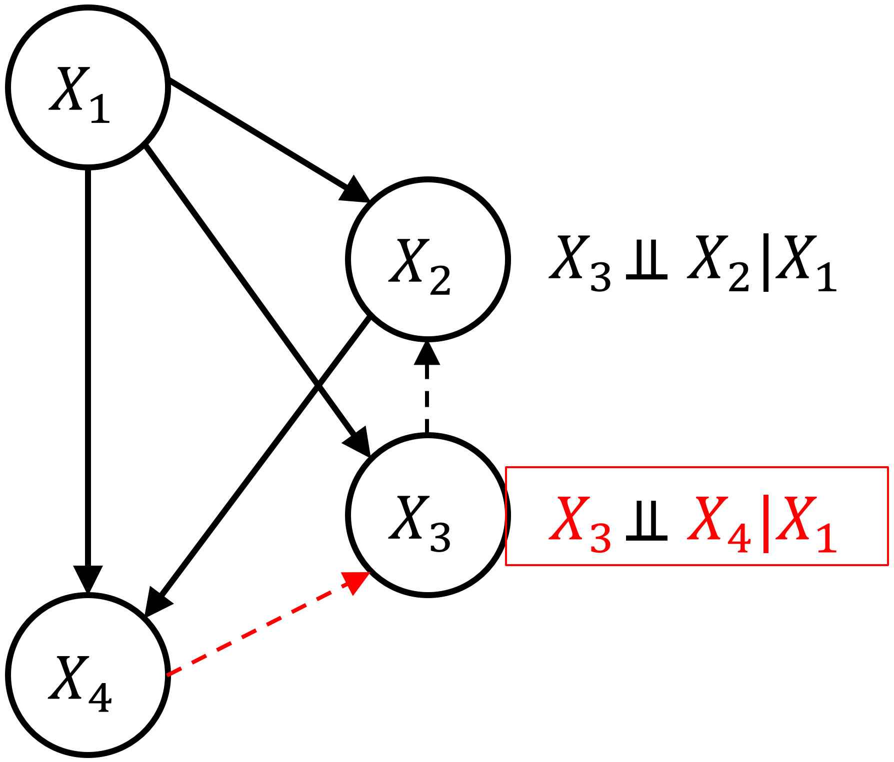

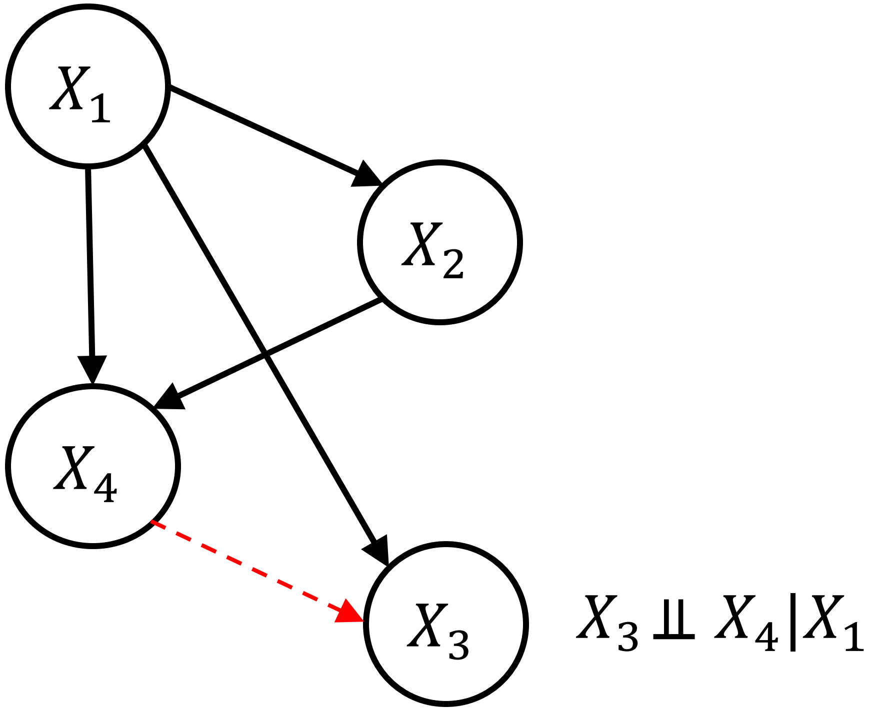

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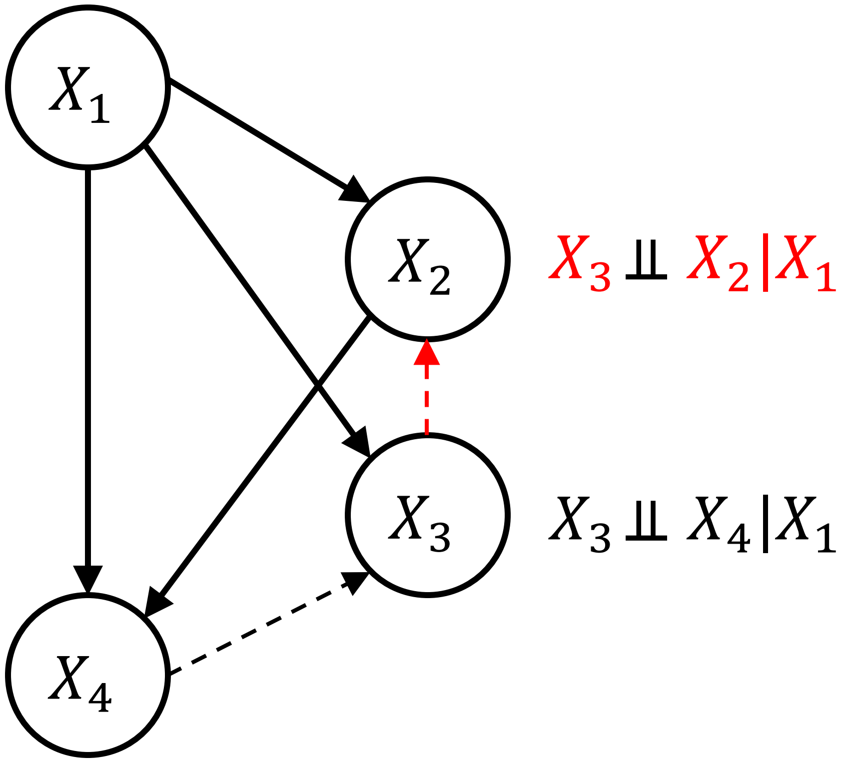
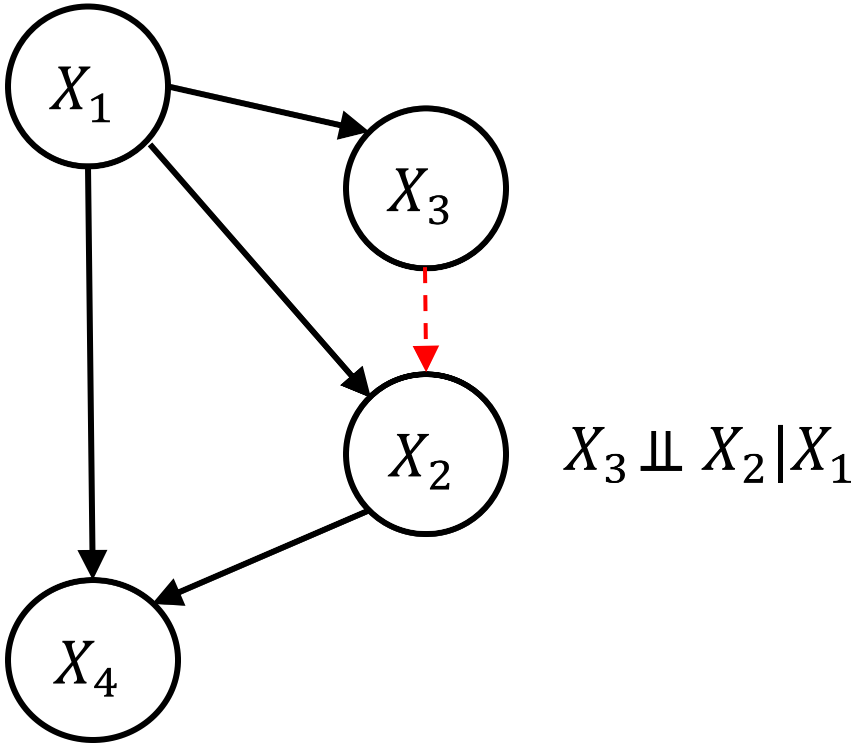
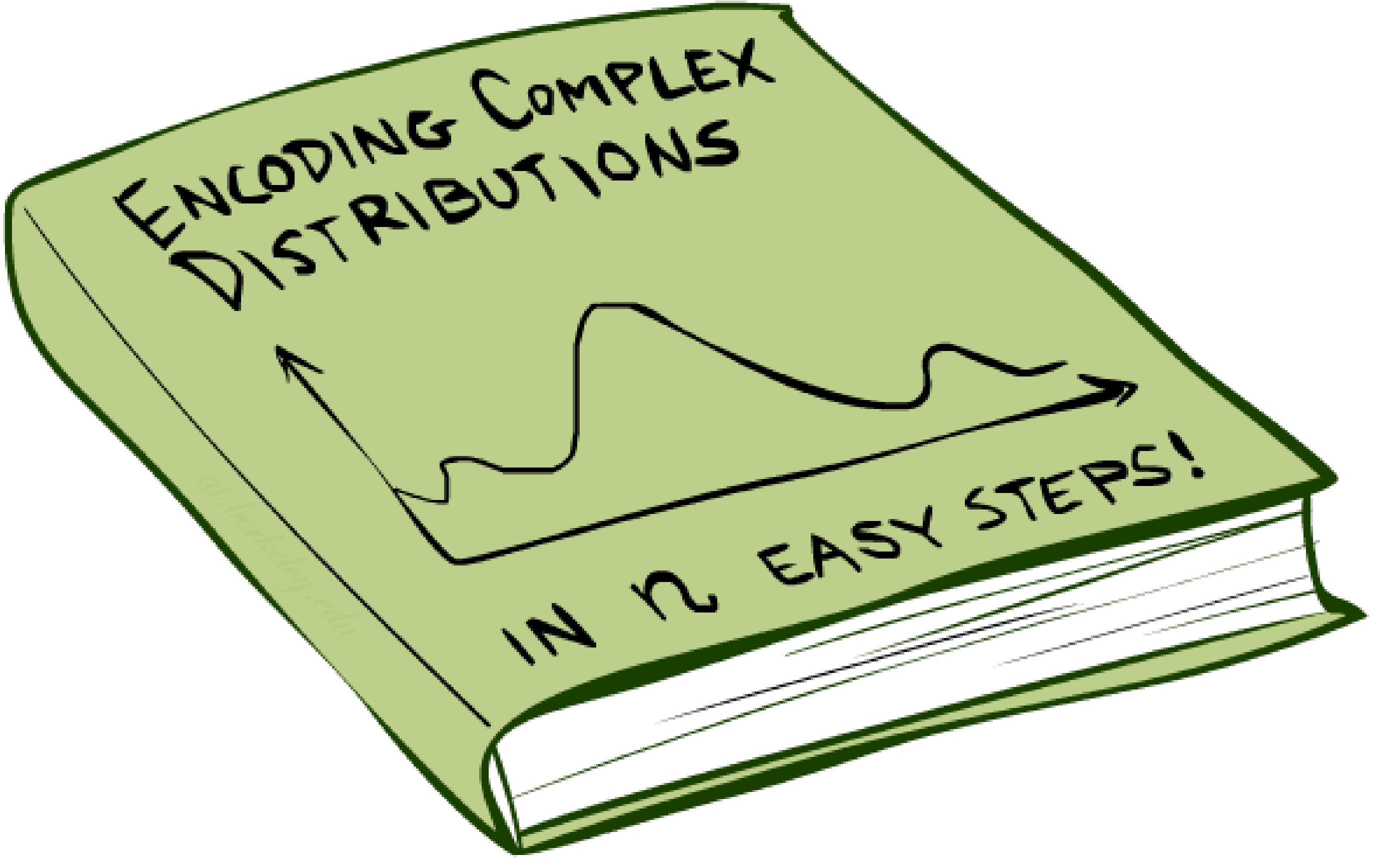




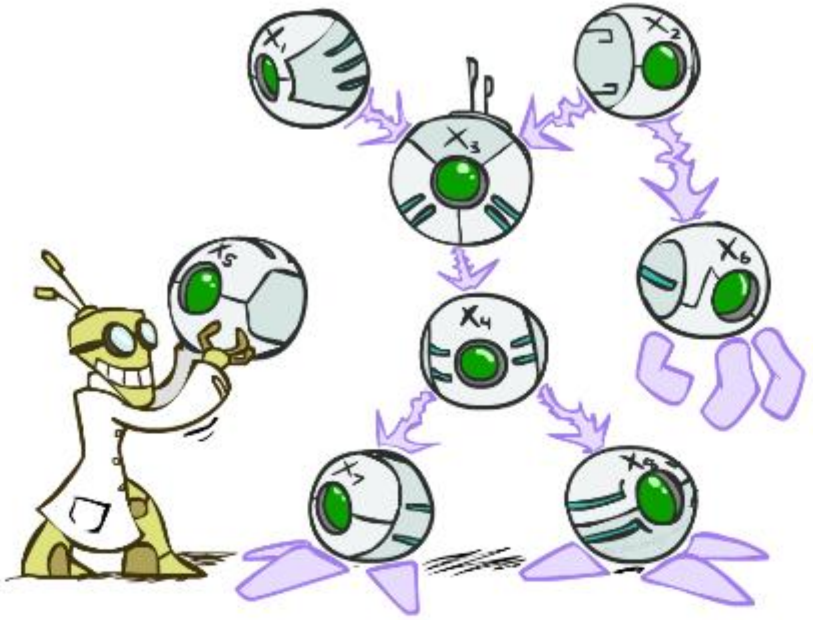
Bayes' Nets: Big Picture



Bayes' Nets: Big Picture

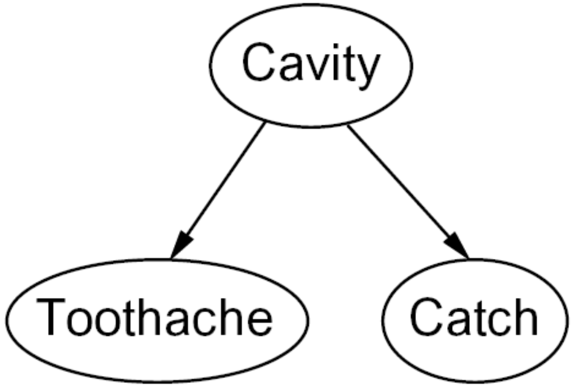
- Two problems with using full joint distribution tables as our probabilistic models:
  - Unless there are only a few variables, the joint is WAY too big to represent explicitly
  - Hard to learn (estimate) anything empirically about more than a few variables at a time

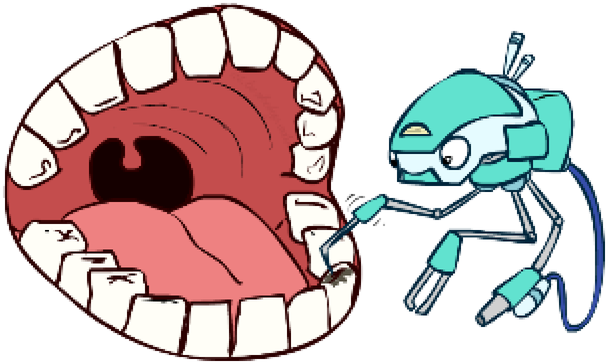
- Bayes' nets: a technique for describing complex joint distributions (models) using simple, local distributions (conditional probabilities)
  - More properly called graphical models
  - We describe how variables locally interact
  - Local interactions chain together to give global, indirect interactions
  - For about 10 min, we'll be vague about how these interactions are specified



Graphical Model Notation

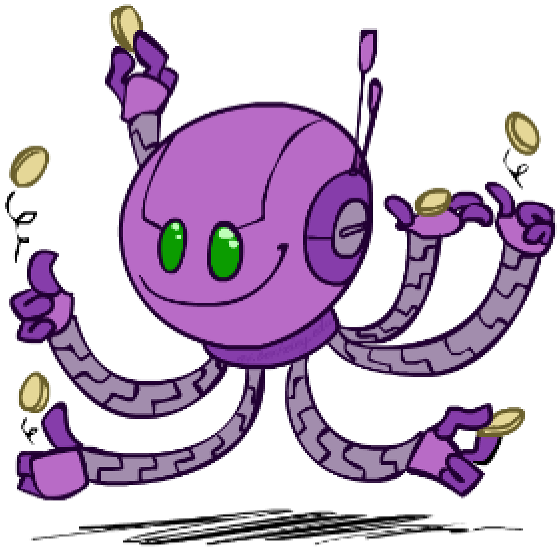
- Nodes: variables (with domains)
  - Can be assigned (observed) or unassigned (unobserved)
- Arcs: interactions
  - Similar to CSP constraints
  - Indicate "direct influence" between variables
  - Formally: encode conditional independence (more later)
- For now: imagine that arrows mean direct causation (in general, they don't!)





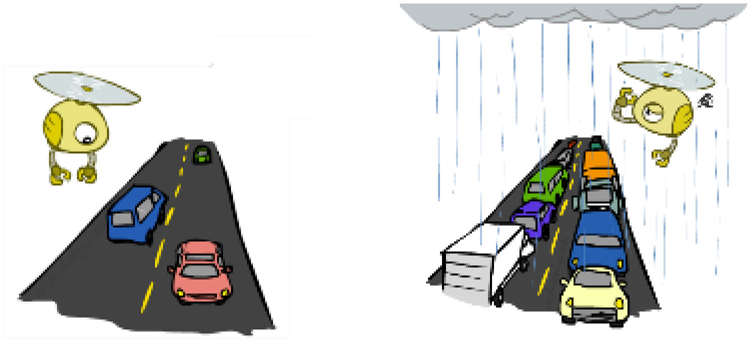
Example: Coin Flips

- N independent coin flips



- No interactions between variables: absolute independence

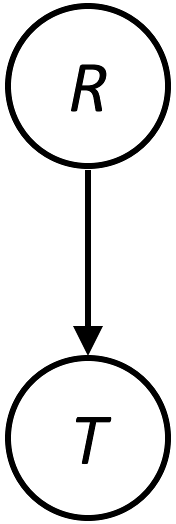
Example: Traffic



- Variables:
  - R: It rains
  - T: There is traffic

- Model 1: independence

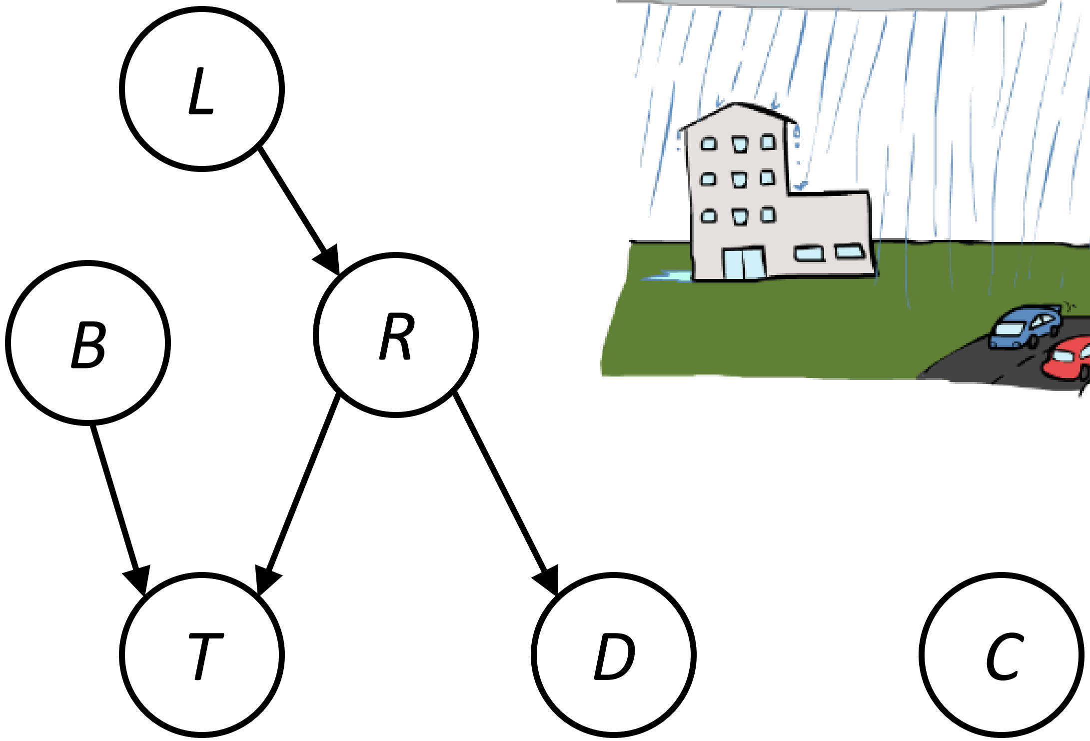
- Model 2: rain causes traffic



- Why is an agent using model 2 better?

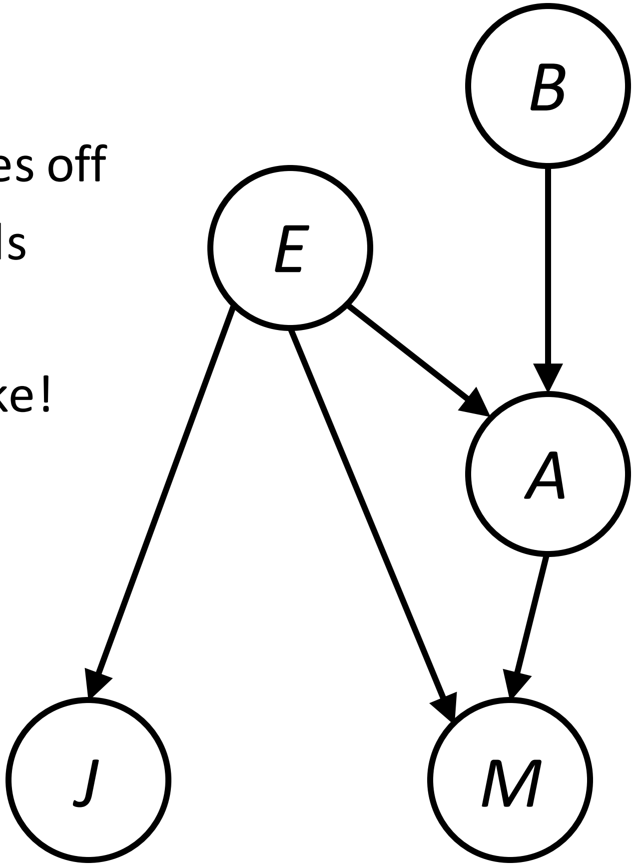
Example: Traffic II

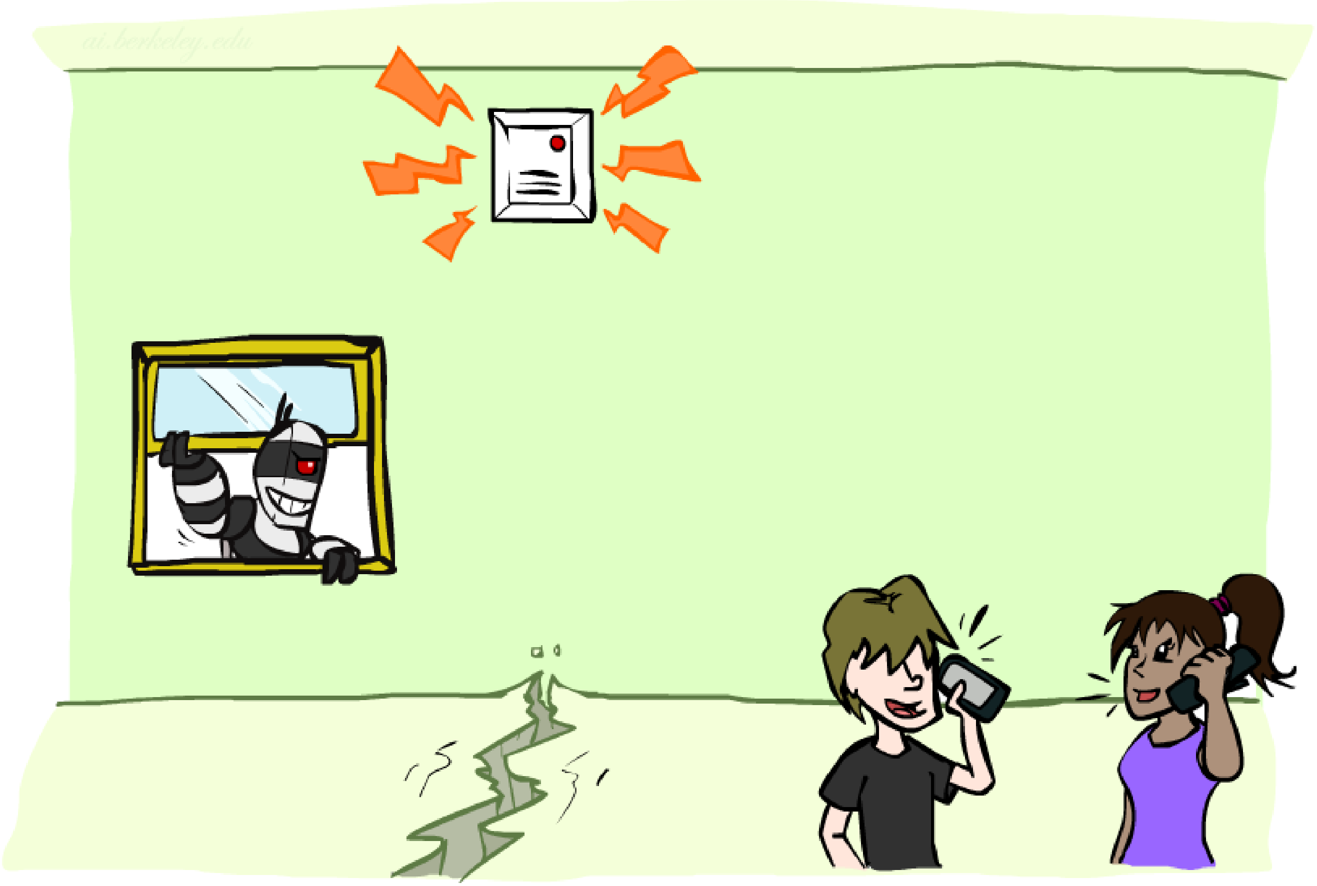
- Let's build a causal graphical model!
- Variables
  - T: Traffic
  - R: It rains
  - L: Low pressure
  - D: Roof drips
  - B: Ballgame
  - C: Cavity



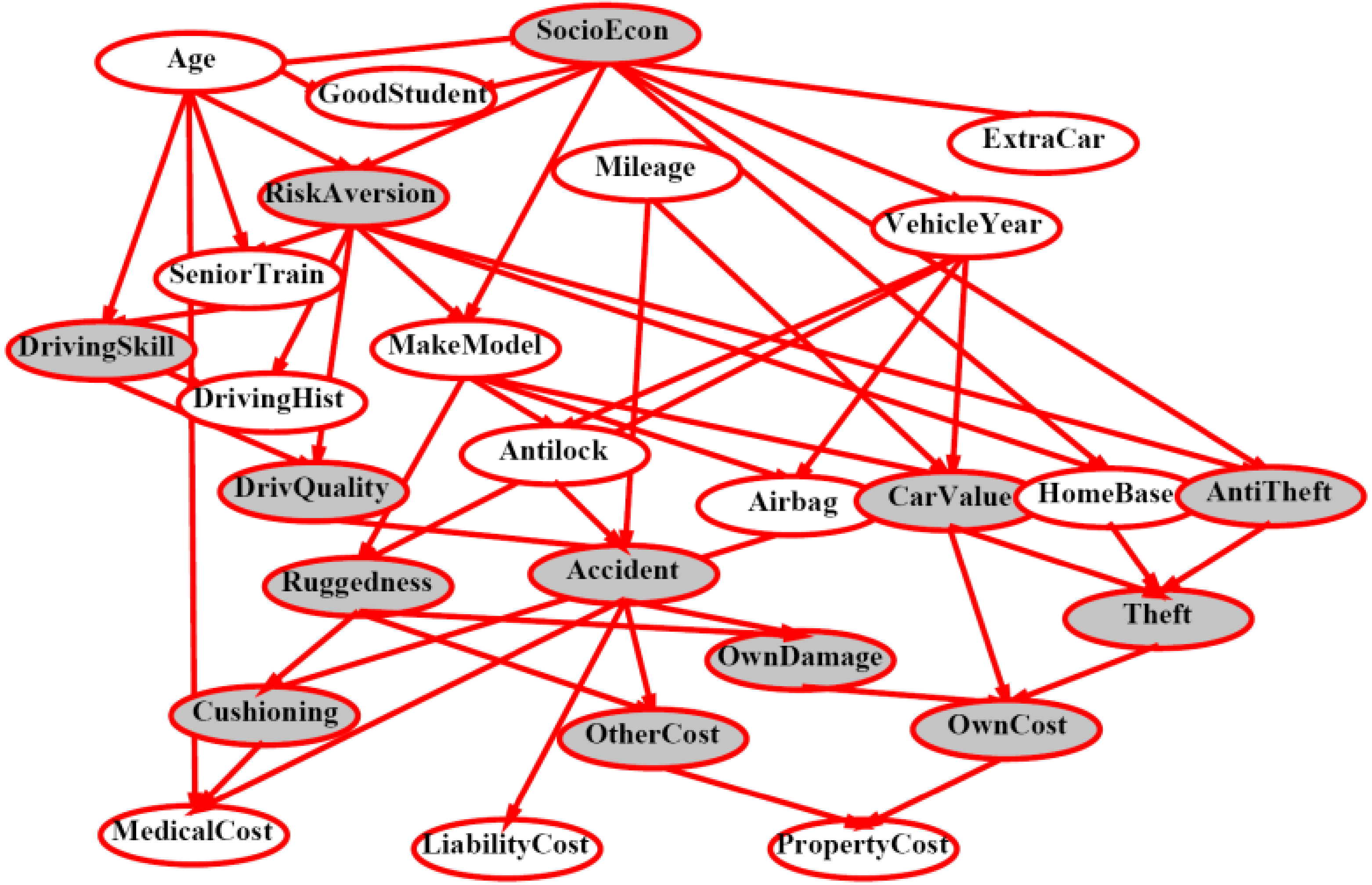
Example: Alarm Network

- Variables
  - B: Burglary
  - A: Alarm goes off
  - M: Mary calls
  - J: John calls
  - E: Earthquake!

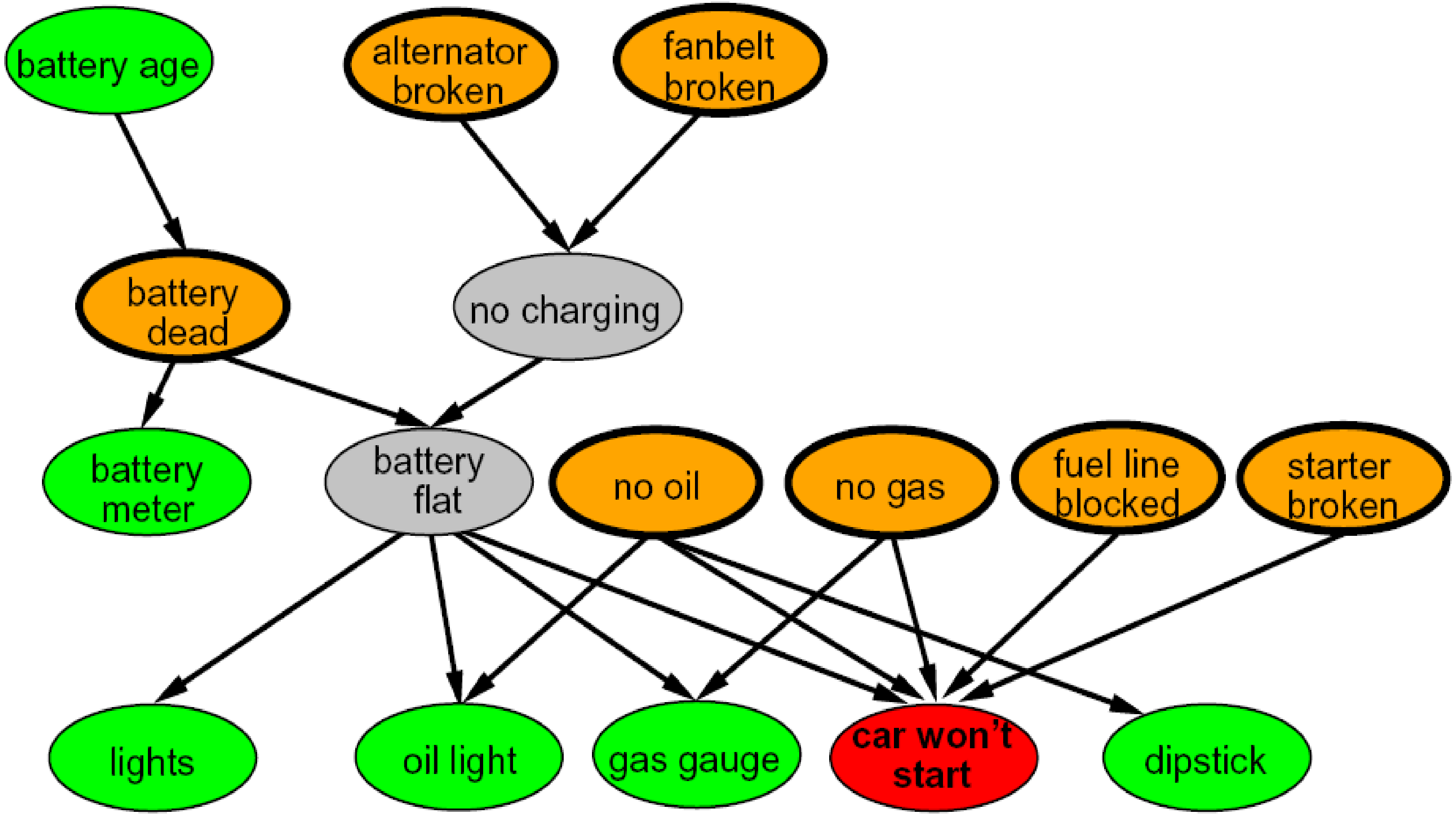




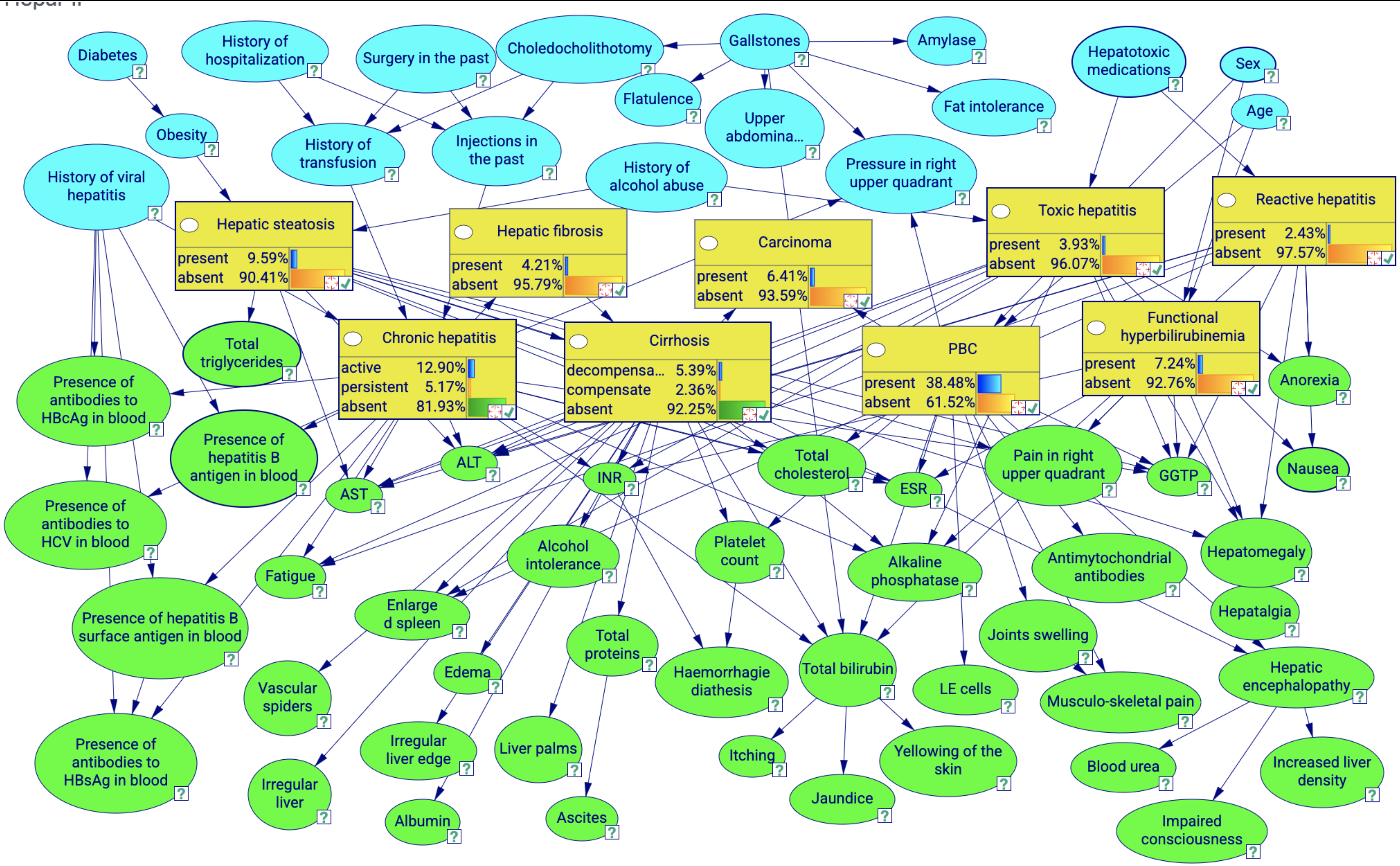
Example Bayes' Net: Insurance



Example Bayes' Net: Car



Example Bayes' Net: Medical Diagnosis



https://demo.bayesfusion.com/bayesbox.html

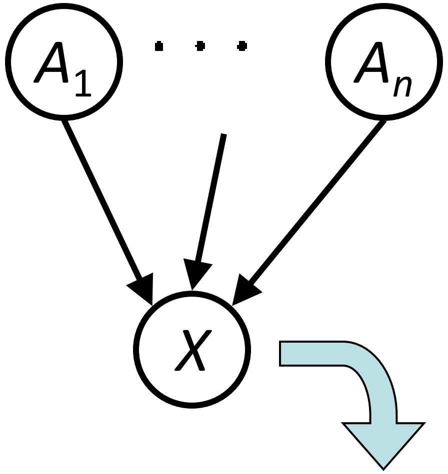
Bayes' Net Semantics

Bayes’ Net Semantics

- A set of nodes, one per variable X
- A directed, acyclic graph
- A conditional distribution for each node
  - A collection of distributions over X, one for each combination of parents’ values

    $$P(X|a_1, \dots a_n)$$

  - CPT: conditional probability table
  - Description of a noisy “causal” process

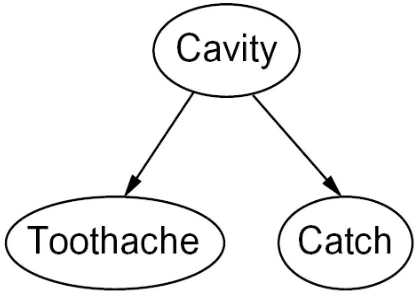


$$P(X|A_1, \dots A_n)$$

A Bayes net = Topology (graph) + Local Conditional Probabilities

Probabilities in BNs

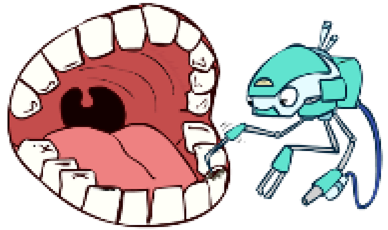
- Bayes' nets implicitly encode joint distributions
  - As a product of local conditional distributions
  - To see what probability a BN gives to a full assignment, multiply all the relevant conditionals together:



$$P(x_1, x_2, \ldots x_n) = \prod_{i=1}^{n} P(x_i | \text{parents}(X_i))$$

  - Example:

$$P(+\text{cavity}, +\text{catch}, -\text{toothache})$$



Probabilities in BNs

- Why are we guaranteed that setting

$$P(x_1, x_2, \ldots x_n) = \prod_{i=1}^{n} P(x_i | \text{parents}(X_i))$$

results in a proper joint distribution?

- Chain rule (valid for all distributions): $P(x_1, x_2, \ldots x_n) = \prod_{i=1}^{n} P(x_i | x_1 \ldots x_{i-1})$
- Assume conditional independences: $P(x_i | x_1 \ldots x_{i-1}) = P(x_i | \text{parents}(X_i))$

→ Consequence: Not every BN can represent every joint distribution

- The topology enforces certain conditional independencies

Example: Coin Flips

X_1 X_2 X_3 X_4

$P(X_1)$

h	0.5
t	0.5

$P(X_2)$

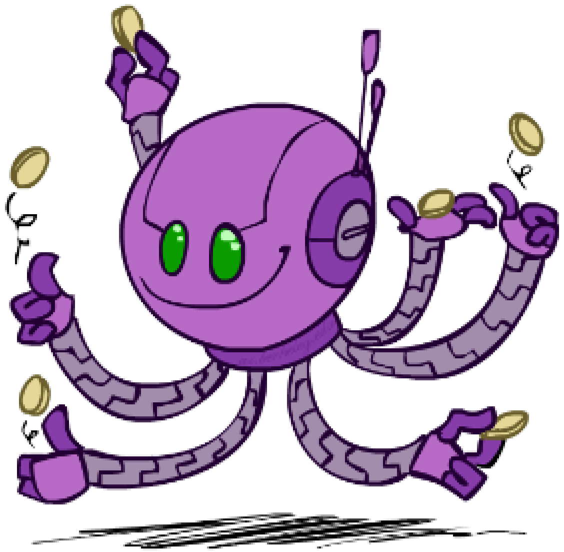
h	0.5
t	0.5

$P(X_3)$

h	0.5
t	0.5

$P(X_4)$

h	0.5
t	0.5

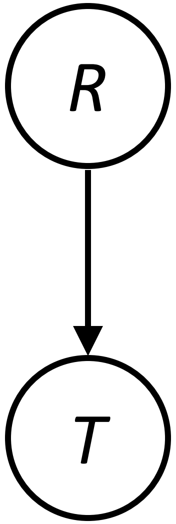


$P(h, h, t, h) = 0.5^4$

$P(x_1, x_2, \ldots x_n) = \prod_{i=1}^{n} P(x_i|\text{parents}(X_i))$

Only distributions whose variables are absolutely independent can be represented by a Bayes' net with no arcs.

Example: Traffic



$$P(x_1, x_2, \ldots x_n) = \prod_{i=1}^{n} P(x_i | \text{parents}(X_i))$$

$P(R)$

+r	1/4
-r	3/4

$P(T|R)$

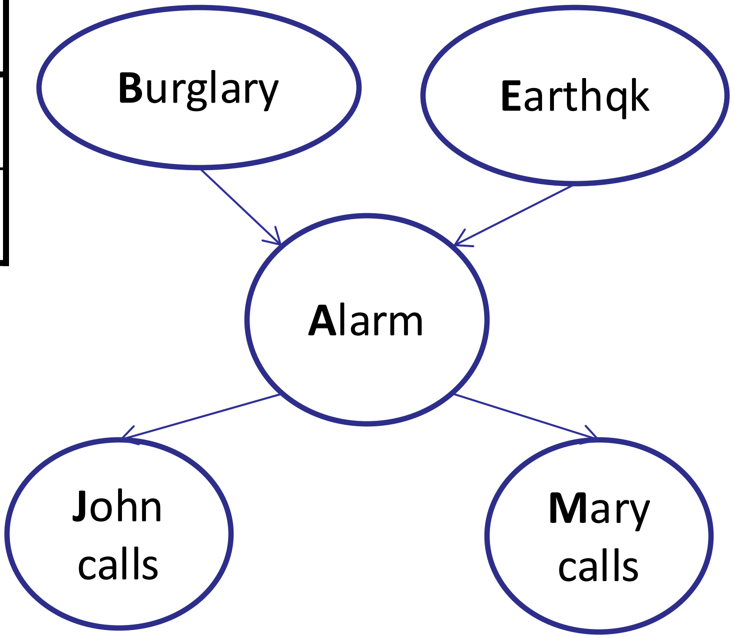
+r	+t	3/4
	-t	1/4
-r	+t	1/2
	-t	1/2

$$P(+r, -t) = P(+r)P(-t| + r)$$

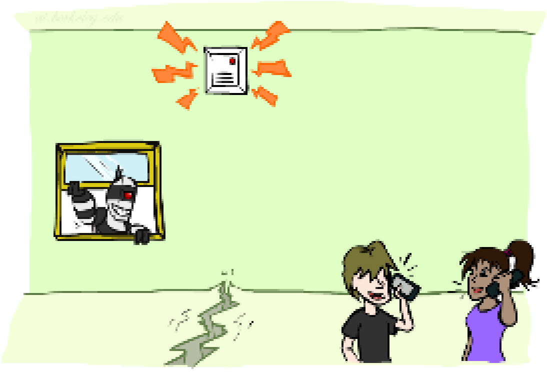
$$= \frac{1}{4} * \frac{1}{4} = \frac{1}{16}$$

Example: Alarm Network

B	P(B)
+b	0.001
-b	0.999



E	P(E)
+e	0.002
-e	0.998



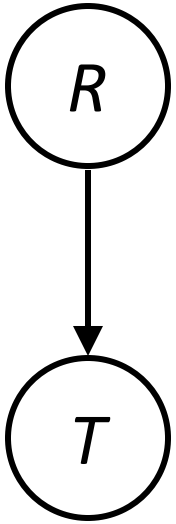
A	J	P(J|A)
+a	+j	0.9
+a	-j	0.1
-a	+j	0.05
-a	-j	0.95

A	M	P(M|A)
+a	+m	0.7
+a	-m	0.3
-a	+m	0.01
-a	-m	0.99

B	E	A	P(A|B,E)
+b	+e	+a	0.95
+b	+e	-a	0.05
+b	-e	+a	0.94
+b	-e	-a	0.06
-b	+e	+a	0.29
-b	+e	-a	0.71
-b	-e	+a	0.001
-b	-e	-a	0.999

Example: Traffic

- Causal direction



$P(R)$

+r	1/4
-r	3/4

$P(T|R)$

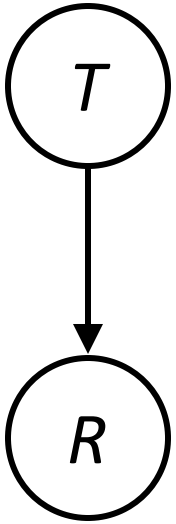
+r	+t	3/4
	-t	1/4
-r	+t	1/2
	-t	1/2

$P(T,R)$

+r	+t	3/16
+r	-t	1/16
-r	+t	6/16
-r	-t	6/16

Example: Reverse Traffic

- Reverse causality?



$P(T)$

+t	9/16
-t	7/16

$P(R|T)$

+t	+r	1/3
	-r	2/3

-t	+r	1/7
	-r	6/7

$P(R,T)$

+r	+t	3/16
+r	-t	1/16
-r	+t	6/16
-r	-t	6/16

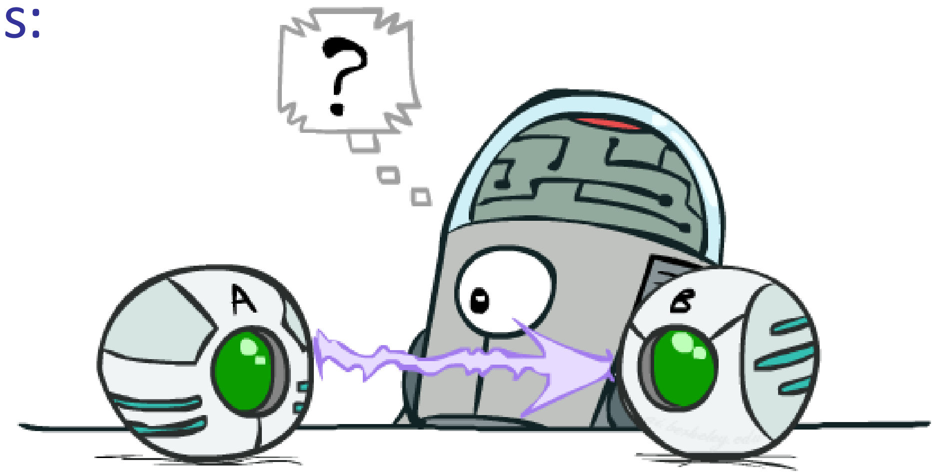
$P(+r,-t) =$

$P(-t)P(+r|-t)$

$$= \frac{7}{16} * \frac{1}{7} = \frac{1}{16}$$

Causality?

- When Bayes' nets reflect the true causal patterns:
  - Often simpler (nodes have fewer parents)
  - Often easier to think about
  - Often easier to elicit from experts



- BNs need not actually be causal
  - Sometimes no causal net exists over the domain (especially if variables are missing)
  - E.g. consider the variables *Traffic* and *Drips*
  - End up with arrows that reflect correlation, not causation

- What do the arrows really mean?
  - Topology may happen to encode causal structure
  - Topology really encodes conditional independence

$$P(x_i|x_1 \ldots x_{i-1}) = P(x_i|\text{parents}(X_i))$$

Bayes' Nets

- So far: how a Bayes' net encodes a joint distribution
- Next: how to answer queries about that distribution
  - Today:
    - First assembled BNs using an intuitive notion of conditional independence as causality
    - Then saw that key property is conditional independence
  - Main goal: answer queries about conditional independence and influence
- After that: how to answer numerical queries (inference)