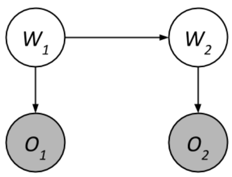
CS 188
Spring 2026

Regular Discussion 8 Solutions

1 HMMs

Consider the following Hidden Markov Model. O_1 and O_2 are supposed to be shaded.



W_1	$P(W_1)$
0	0.3
1	0.7

W_t	W_{t+1}	$P(W_{t+1}\|W_t)$
0	0	0.4
0	1	0.6
1	0	0.8
1	1	0.2

W_t	O_t	$P(O_t\|W_t)$
0	a	0.9
0	b	0.1
1	a	0.5
1	b	0.5

Suppose that we observe $O_1 = a$ and $O_2 = b$.
Using the forward algorithm, compute the probability distribution $P(W_2|O_1 = a, O_2 = b)$ one step at a time.

(a) Compute $P(W_1, O_1 = a)$.

$P(W_1, O_1 = a) = P(W_1)P(O_1 = a|W_1)$
$P(W_1 = 0, O_1 = a) = (0.3)(0.9) = 0.27$
$P(W_1 = 1, O_1 = a) = (0.7)(0.5) = 0.35$

(b) Using the previous calculation, compute $P(W_2, O_1 = a)$.

$P(W_2, O_1 = a) = \sum_{w_1} P(w_1, O_1 = a)P(W_2|w_1)$
$P(W_2 = 0, O_1 = a) = (0.27)(0.4) + (0.35)(0.8) = 0.388$
$P(W_2 = 1, O_1 = a) = (0.27)(0.6) + (0.35)(0.2) = 0.232$

(c) Using the previous calculation, compute $P(W_2, O_1 = a, O_2 = b)$.

$P(W_2, O_1 = a, O_2 = b) = P(W_2, O_1 = a)P(O_2 = b|W_2)$
$P(W_2 = 0, O_1 = a, O_2 = b) = (0.388)(0.1) = 0.0388$
$P(W_2 = 1, O_1 = a, O_2 = b) = (0.232)(0.5) = 0.116$

(d) Finally, compute $P(W_2|O_1 = a, O_2 = b)$.

Renormalizing the distribution above, we have
$P(W_2 = 0|O_1 = a, O_2 = b) = 0.0388/(0.0388 + 0.116) \approx 0.25$
$P(W_2 = 1|O_1 = a, O_2 = b) = 0.116/(0.0388 + 0.116) \approx 0.75$

Dynamic Bayesian Networks

The Decaying Sensor

We are tracking a robot moving through a 1D grid. Let L_t be the robot's true location at time t. The robot is equipped with a distance sensor that gives a reading S_t at time t. However, the sensor runs on a separate battery, B_t. As the battery drains, the sensor readings become less reliable.

The dynamics of this world are modeled by a DBN with the following properties:

- The robot's current location L_t depends only on its previous location L_{t-1}.
- The sensor's battery level B_t depends only on its previous battery level B_{t-1}.
- The sensor reading S_t depends on both the robot's current location L_t and the current battery level B_t.

1. Write down the full joint probability distribution for the first two time steps, $t = 1$ and $t = 2$, in terms of the initial state distributions, transition models, and sensor models.

   $P(L_1, B_1, S_1, L_2, B_2, S_2) = P(L_1)P(B_1)P(S_1 \mid L_1, B_1)P(L_2 \mid L_1)P(B_2 \mid B_1)P(S_2 \mid L_2, B_2)$

2. Are L_2 and B_2 guaranteed to be independent given no observations? Prove it algebraically by marginalizing the joint distribution.

   Yes. We can show this by summing out the unobserved variables (L_1, B_1) to find the marginal distribution $P(L_2, B_2)$. (We can ignore S_1 and S_2 here because summing over their domains evaluates to 1).

   $$P(L_2, B_2) = \sum_{L_1, B_1} P(L_1)P(B_1)P(L_2 \mid L_1)P(B_2 \mid B_1)$$

   Because the terms for L and B do not interact, we can push the summations inward:

   $$P(L_2, B_2) = \left(\sum_{L_1} P(L_2 \mid L_1)P(L_1)\right)\left(\sum_{B_1} P(B_2 \mid B_1)P(B_1)\right)$$

   Which simplifies to:

   $$P(L_2, B_2) = P(L_2)P(B_2)$$

   Since the joint factors into the product of the marginals, L_2 and B_2 are unconditionally independent.

3. Are L_2 and B_2 guaranteed to be independent given the sensor reading S_2? Briefly explain using the concept of common effects (explaining away).

   No. The sensor reading S_2 is a common effect of both the true location L_2 and the battery level B_2. Observing a common effect couples its causes. This is known as "explaining away." For example, if we observe a strangely erratic sensor reading (S_2), and we then discover the battery is nearly dead (B_2), it "explains away" the bad reading. This changes our belief about where the robot actually is (L_2) compared to if we hadn't known the battery state.