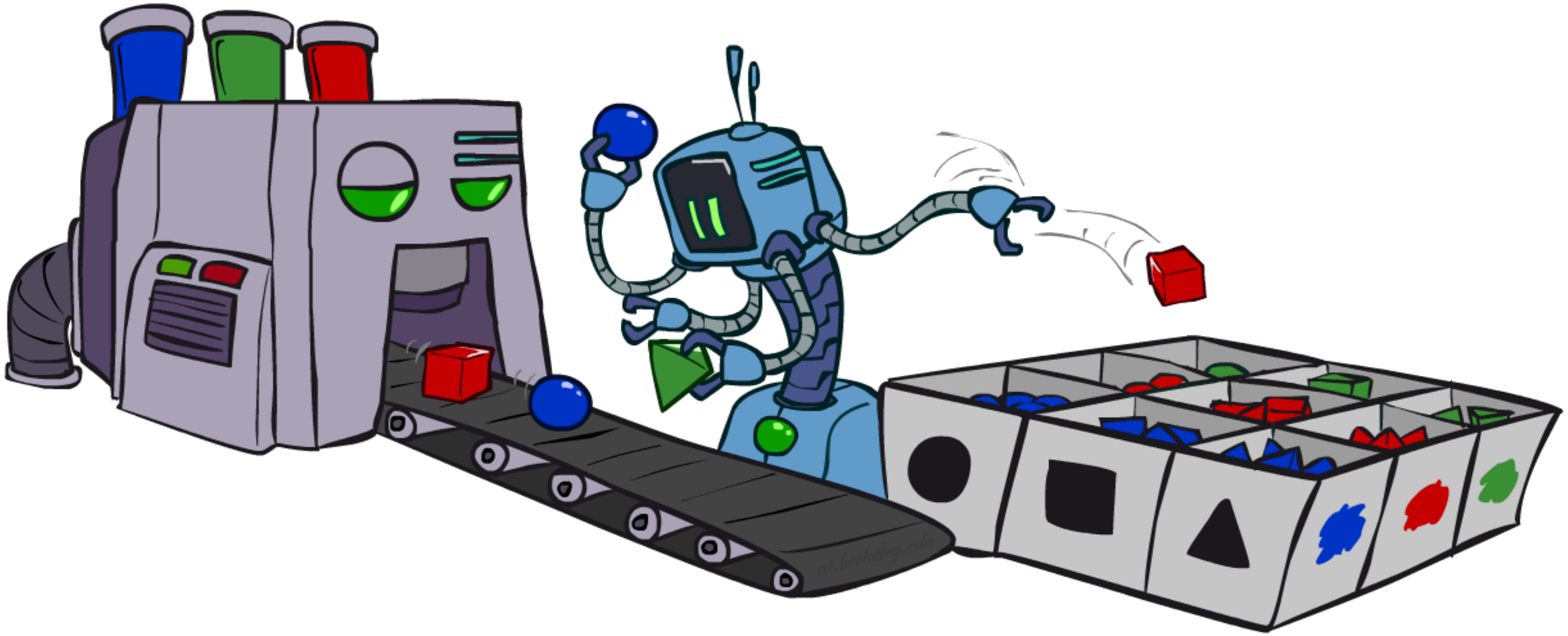
CS 188: Artificial Intelligence

Bayes Nets: Approximate Inference



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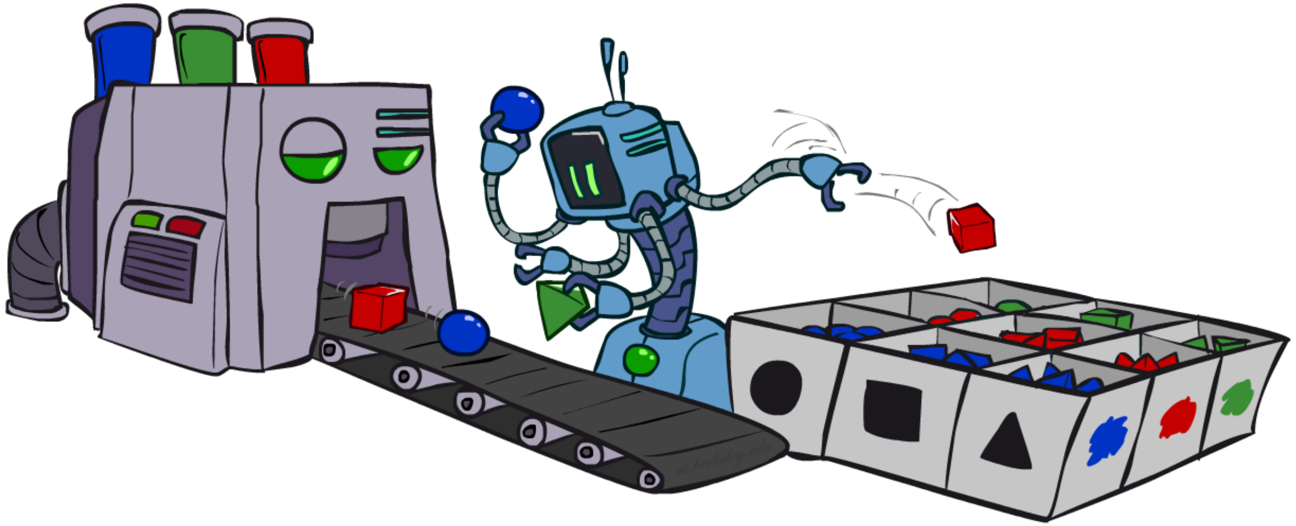
(Slides adapted from Stuart Russell and Dawn Song)

Sampling

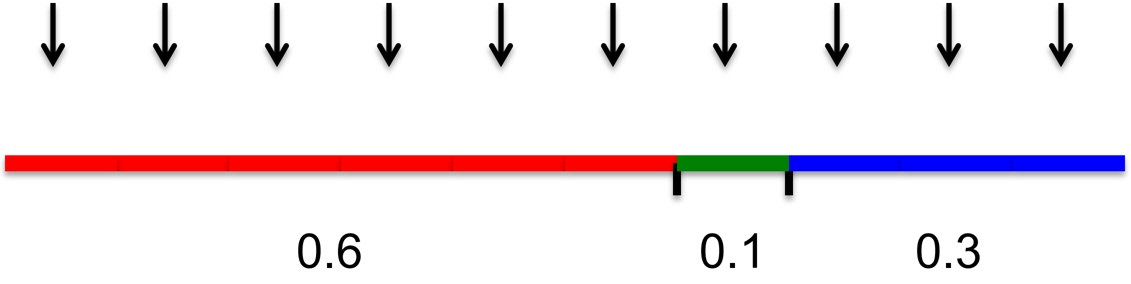
- Basic idea
  - Draw N samples from a ***sampling distribution*** S
  - Compute an approximate posterior probability
  - Show this converges to the true probability P

- Why sample?
  - Often very fast to get a decent approximate answer
  - The algorithms are very simple and general (easy to apply to fancy models)
  - They require very little memory ($O(n)$)
  - They can be applied to large models, whereas exact algorithms blow up



Sampling basics: discrete (*categorical*) distribution

- To simulate a biased d-sided coin:
  - Step 1: Get sample u from uniform distribution over $[0, 1)$
    - E.g. random() in python
  - Step 2: Convert this sample u into an outcome for the given distribution by associating each outcome x with a $P(x)$-sized sub-interval of [0,1)

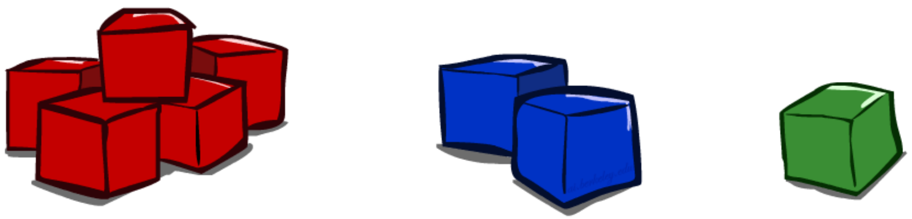


- Example

C	*P*(*C*)
red	0.6
green	0.1
blue	0.3

$0.0 \le u < 0.6, \rightarrow C=red$
$0.6 \le u < 0.7, \rightarrow C=green$
$0.7 \le u < 1.0, \rightarrow C=blue$

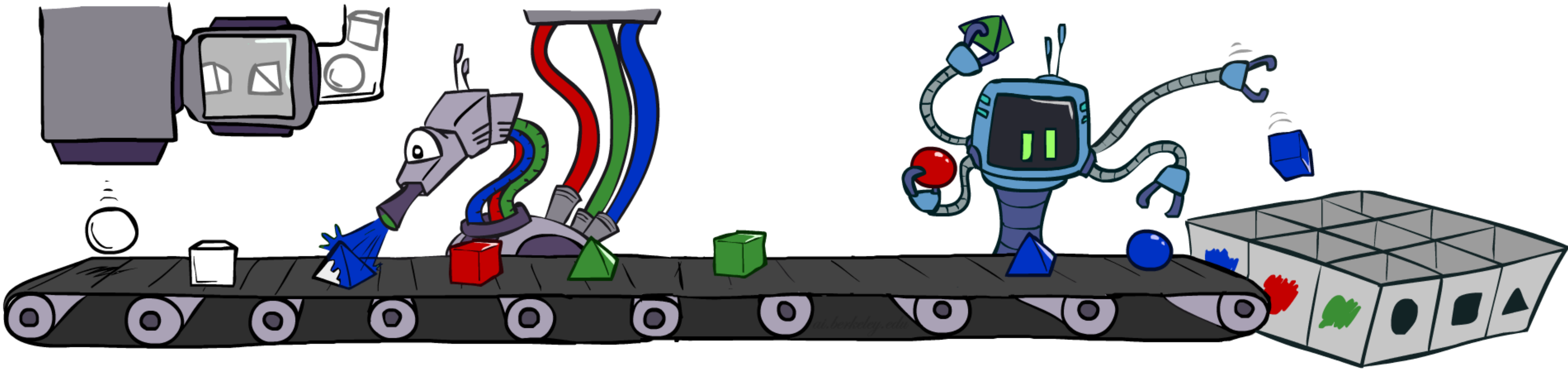
- If random() returns $u = 0.83$, then the sample is $C = blue$
- E.g, after sampling 8 times:



Sampling in Bayes Nets

- Prior Sampling
- Rejection Sampling
- Likelihood Weighting
- Gibbs Sampling

Prior Sampling



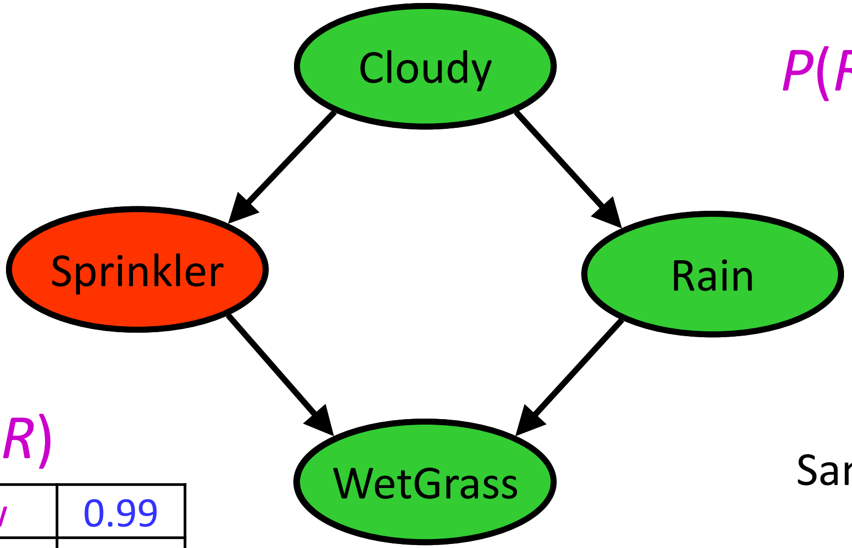
Prior Sampling

$P(C)$

c	0.5
¬c	0.5

$P(S \mid C)$

c	s	0.1
	¬s	0.9
¬c	s	0.5
	¬s	0.5



$P(R \mid C)$

c	r	0.8
	¬r	0.2
¬c	r	0.2
	¬r	0.8

$P(W \mid S,R)$

s	r	w	0.99
		¬w	0.01
	¬r	w	0.90
		¬w	0.10
¬s	r	w	0.90
		¬w	0.10
	¬r	w	0.01
		¬w	0.99

Samples:

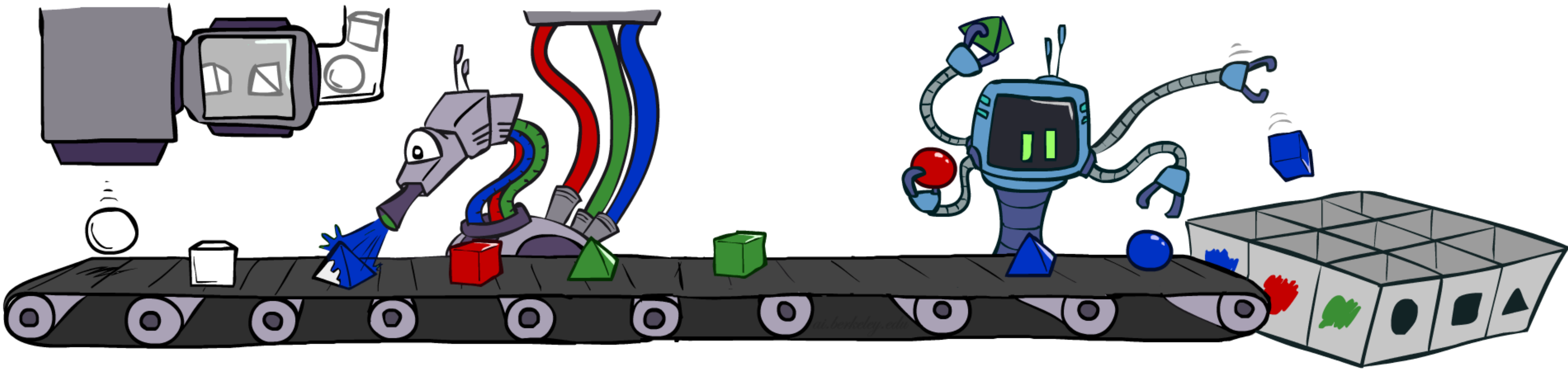
c, ¬s, r, w

¬c, s, ¬r, w

…

Prior Sampling

- For i=1, 2, …, n (in topological order)
  - Sample X_i from $P(X_i \mid parents(X_i))$
- Return $(x_1, x_2, \ldots, x_n)$



Prior Sampling

- This process generates samples with probability:

  $S_{PS}(x_1,\ldots,x_n) = \prod_i P(x_i \mid parents(X_i)) = P(x_1,\ldots,x_n)$

  …i.e. the BN's joint probability

- Let the number of samples of an event be $N_{PS}(x_1,\ldots,x_n)$
- Estimate from N samples is $Q_N(x_1,\ldots,x_n) = N_{PS}(x_1,\ldots,x_n)/N$
- Then $\lim_{N\to\infty} Q_N(x_1,\ldots,x_n) = \lim_{N\to\infty} N_{PS}(x_1,\ldots,x_n)/N$

  $= S_{PS}(x_1,\ldots,x_n)$

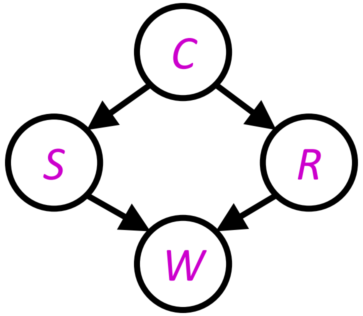
  $= P(x_1,\ldots,x_n)$

- I.e., the sampling procedure is ***consistent***

Example

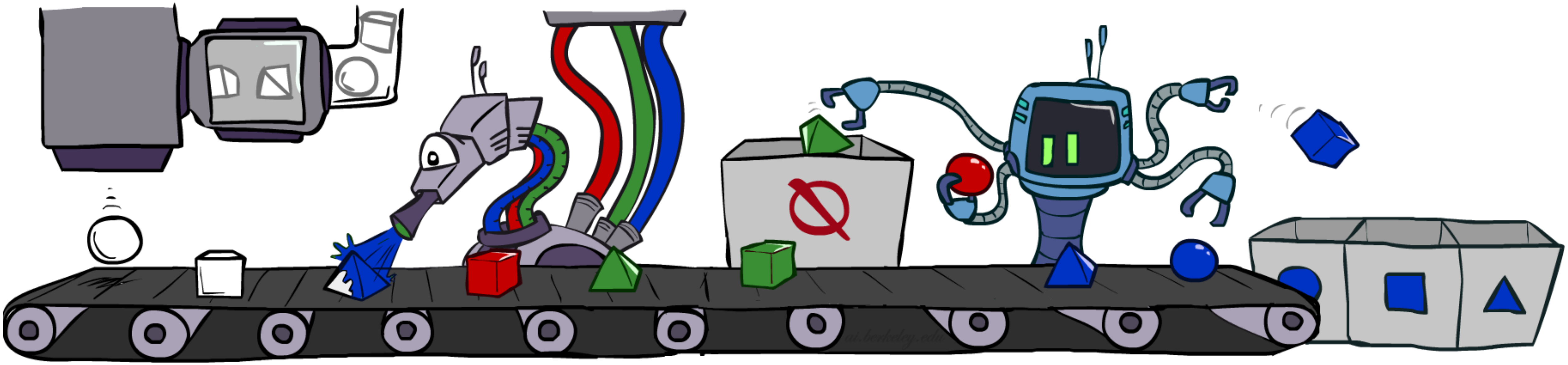
- We'll get a bunch of samples from the BN:

  c, ¬s, r, w
  c, s, r, w
  ¬c, s, r, ¬w
  c, ¬s, r, w
  ¬c, ¬s, ¬r, w



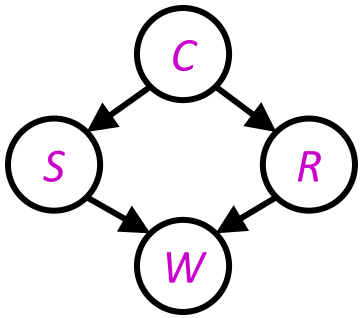
- If we want to know *P*(*W*)
  - We have counts <w:4, ¬w:1>
  - Normalize to get *P*(*W*) = <w:0.8, ¬w:0.2>
  - This will get closer to the true distribution with more samples
  - Can estimate anything else, too
    - P(C | ¬ w)?

Rejection Sampling



Rejection Sampling

- A simple application of prior sampling for estimating conditional probabilities
  - Let's say we want $P(C \mid r, w) = \alpha P(C, r, w)$
  - For these counts, samples with $\neg r$ or $\neg w$ ***are not relevant***
  - So count the *C* outcomes for samples with *r, w* and reject all other samples
- This is called ***rejection sampling***
  - It is also consistent for conditional probabilities (i.e., correct in the limit)



c, ¬s, r, w

~~c, s, ¬r~~

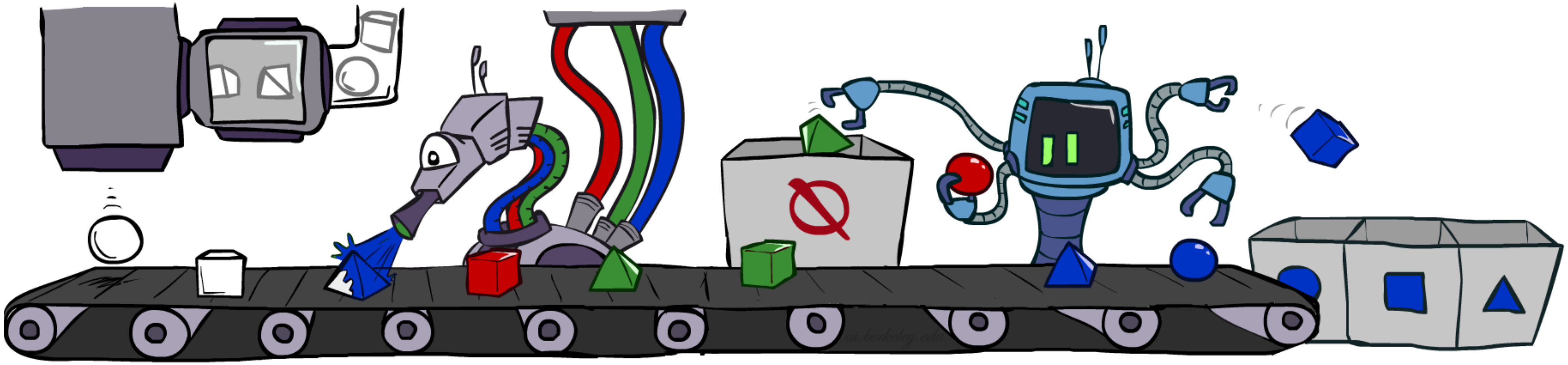
~~¬c, s, r, ¬w~~

~~c, ¬s, ¬r~~

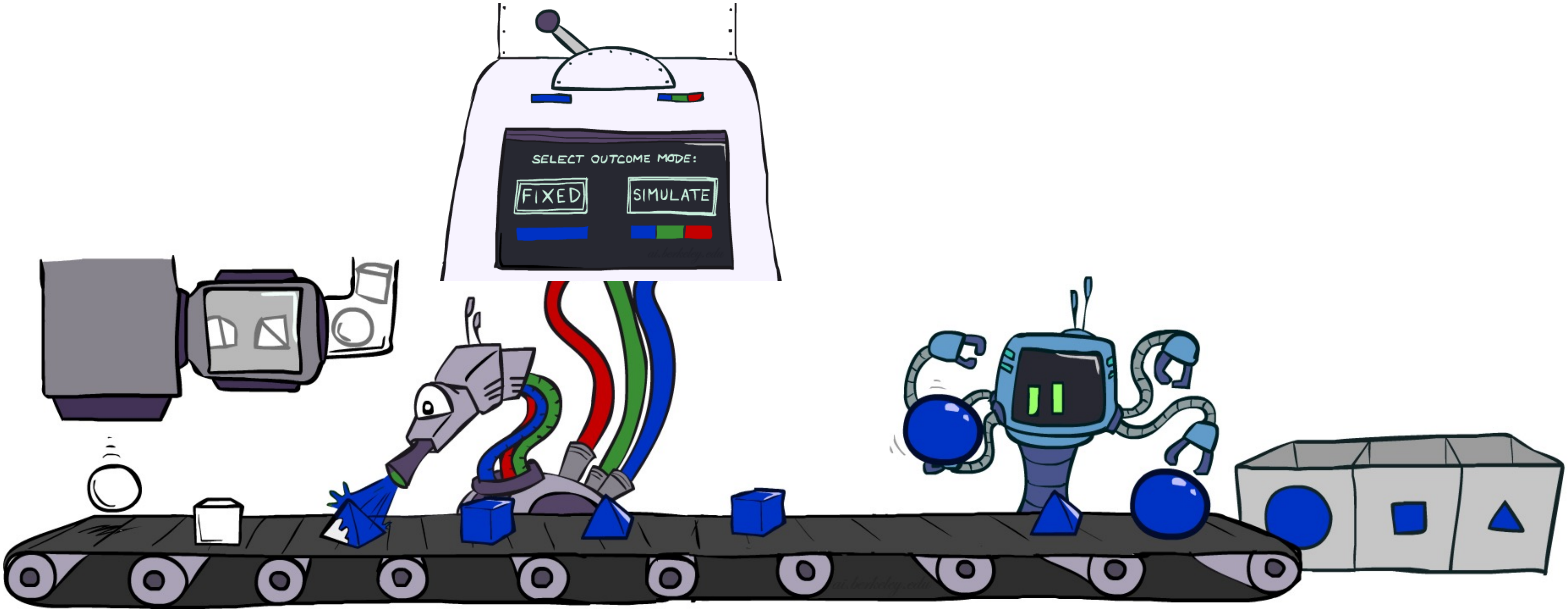
¬c, ¬s, r, w

Rejection Sampling

- Input: evidence $e_1,..,e_k$
- For i=1, 2, …, n
  - Sample X_i from $P(X_i \mid parents(X_i))$
  - If x_i not consistent with evidence
    - Reject: Return, and no sample is generated in this cycle
- Return $(x_1, x_2, …, x_n)$

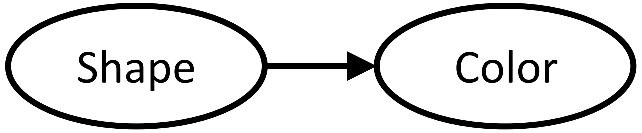


Likelihood Weighting

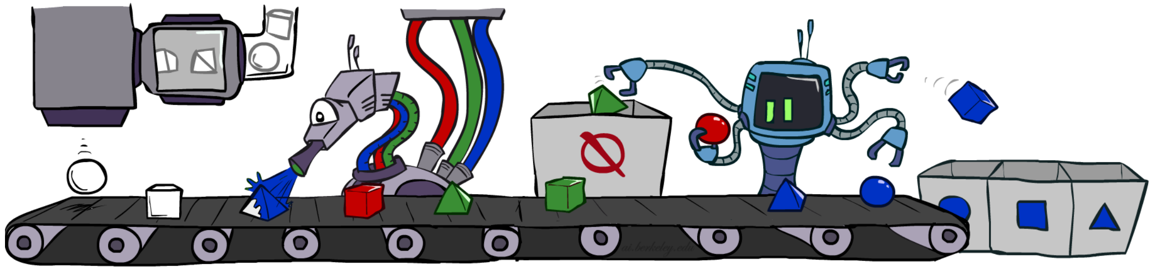


Likelihood Weighting

- Problem with rejection sampling:
  - If evidence is unlikely, rejects lots of samples
  - Evidence not exploited as you sample
  - Consider P(*Shape*|*Color=blue*)

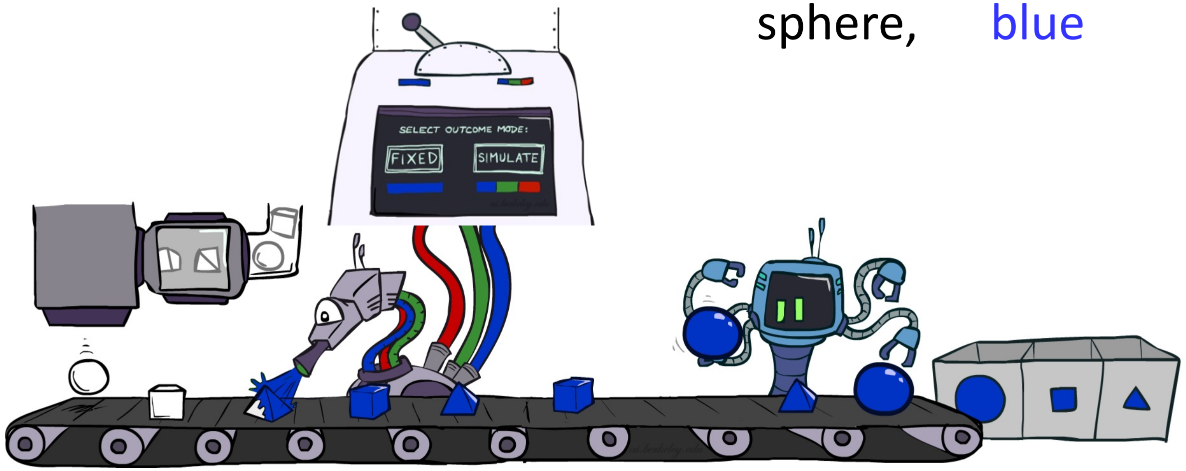


~~pyramid, green~~
~~pyramid, red~~
sphere, blue
~~cube, red~~
~~sphere, green~~



- Idea: fix evidence variables, sample the rest
  - Problem: sample distribution not consistent!
  - Solution: ***weight*** each sample by probability of evidence variables given parents

pyramid, blue
pyramid, blue
sphere, blue
cube, blue
sphere, blue



Likelihood Weighting

P(*C*)

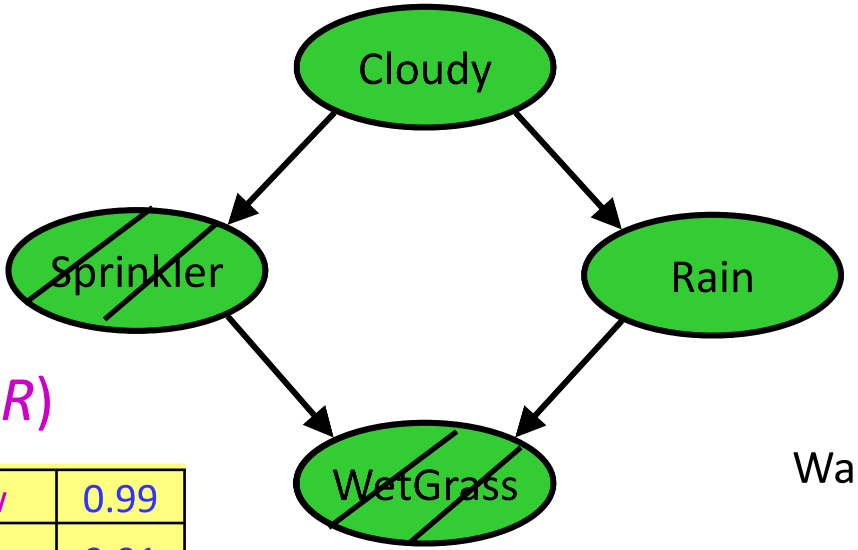
c	0.5
¬c	0.5

P(*S* | *C*)

c	s	0.1
	¬s	0.9
¬c	s	0.5
	¬s	0.5

P(*R* | *C*)

c	r	0.8
	¬r	0.2
¬c	r	0.2
	¬r	0.8



P(*W* | *S*,*R*)

s	r	w	0.99
		¬w	0.01
	¬r	w	0.90
		¬w	0.10
¬s	r	w	0.90
		¬w	0.10
	¬r	w	0.01
		¬w	0.99

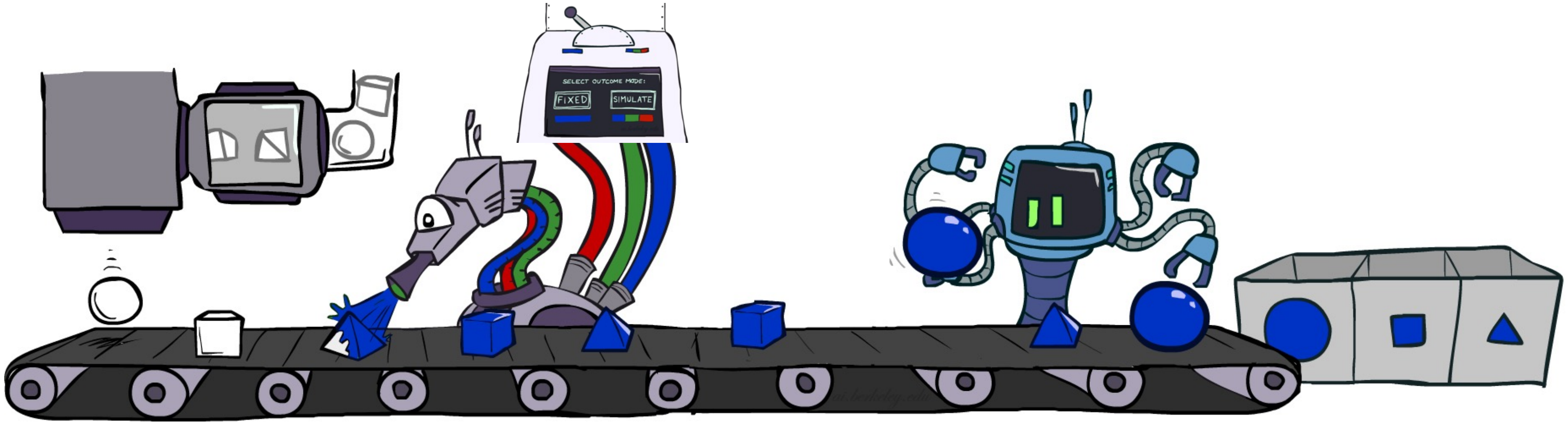
Want to estimate P (C | s, w)

Samples:

c*, *s*, *r*, *w ***w* = 1.0 x 0.1 x 0.99**

Likelihood Weighting

- Input: evidence $e_1,..,e_k$
- $w = 1.0$
- for i=1, 2, …, n
  - if X_i is an evidence variable
    - x_i = observed value$_i$ for X_i
    - Set $w = w * P(x_i \mid parents(X_i))$
  - else
    - Sample x_i from $P(X_i \mid parents(X_i))$
- return $(x_1, x_2, \ldots, x_n)$, w



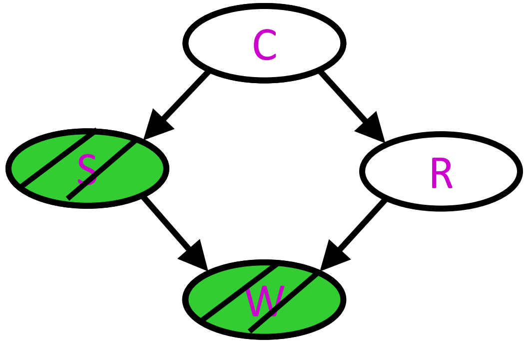
Likelihood Weighting

- Sampling distribution if **Z** sampled and **e** fixed evidence

  $$S_{WS}(\mathbf{z},\mathbf{e}) = \prod_j P(z_j \mid parents(Z_j))$$

- Now, samples have weights

  $$w(\mathbf{z},\mathbf{e}) = \prod_k P(e_k \mid parents(E_k))$$



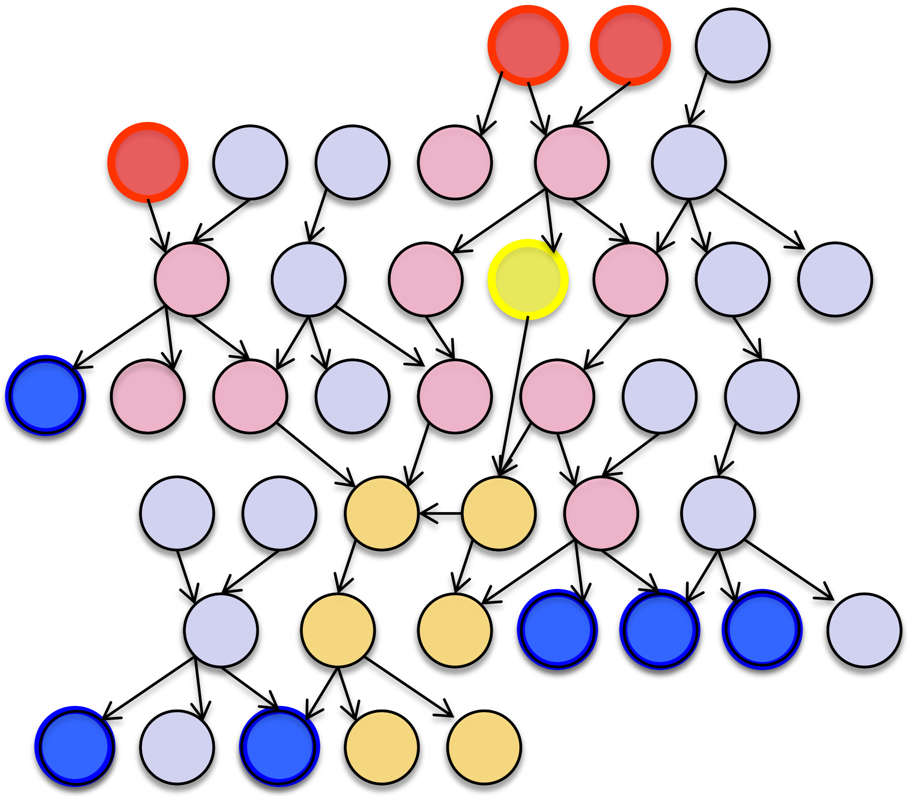
- Together, weighted sampling distribution is consistent

  $$S_{WS}(\mathbf{z},\mathbf{e}) \cdot w(\mathbf{z},\mathbf{e}) = \prod_j P(z_j \mid parents(Z_j)) \; \prod_k P(e_k \mid parents(E_k))$$
  $$= P(\mathbf{z},\mathbf{e})$$

- Likelihood weighting is an example of ***importance sampling***
  - Would like to estimate some quantity based on samples from P
  - P is hard to sample from, so use Q instead
  - Weight each sample x by $P(x)/Q(x)$

Likelihood Weighting

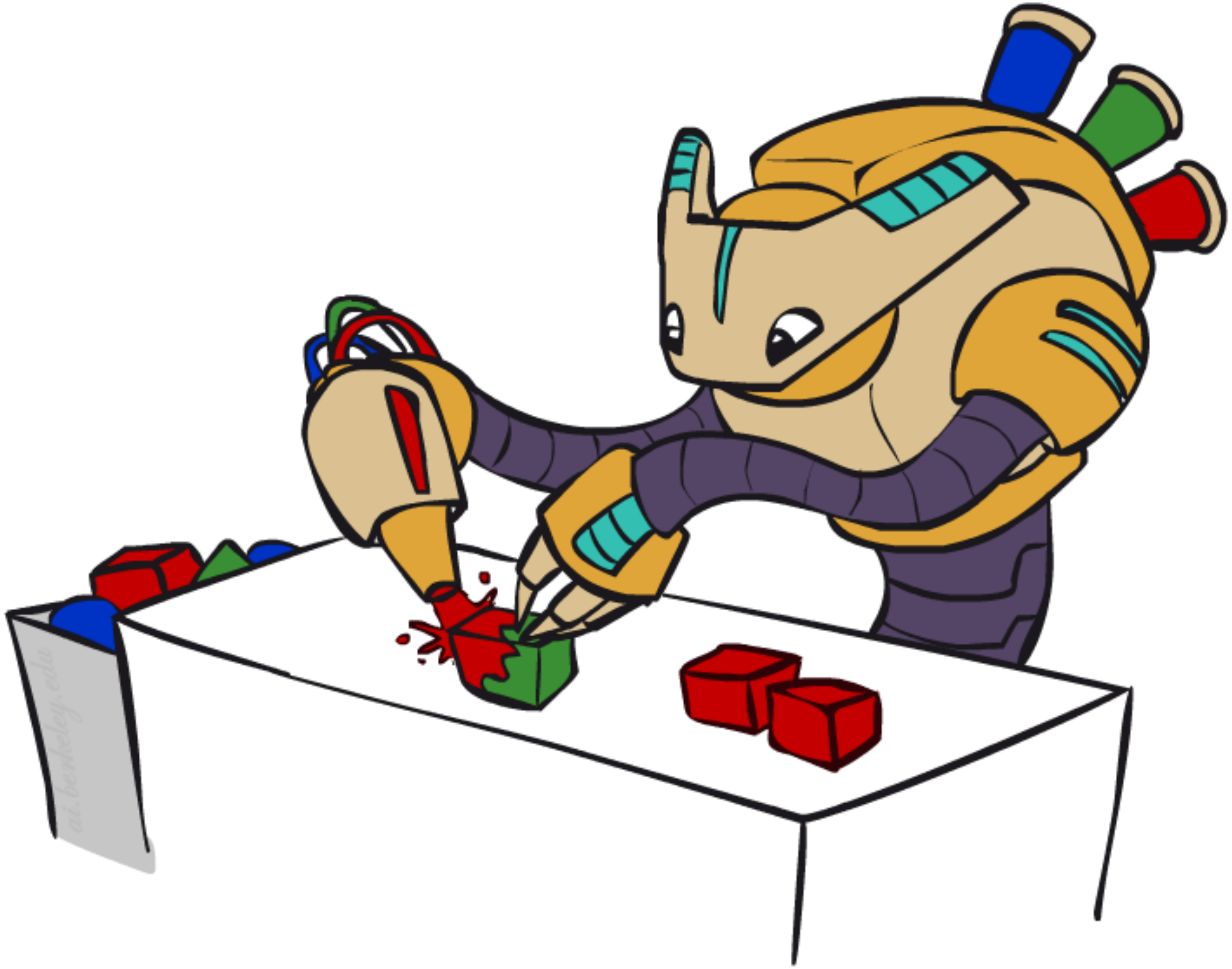
- Likelihood weighting is good
  - All samples are used
  - The values of ***downstream*** variables are influenced by ***upstream*** evidence



- Likelihood weighting still has weaknesses
  - The values of ***upstream*** variables are unaffected by ***downstream*** evidence
    - E.g., suppose evidence is a video of a traffic accident
  - With evidence in k leaf nodes, weights will be $O(2^{-k})$
  - With high probability, one lucky sample will have much larger weight than the others, dominating the result

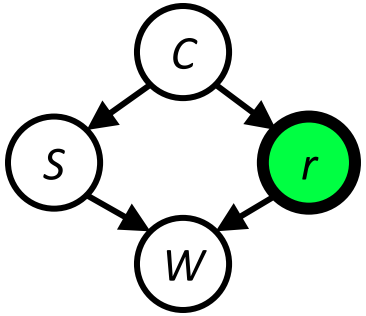
- We would like each variable to "see" ***all*** the evidence!

Gibbs Sampling

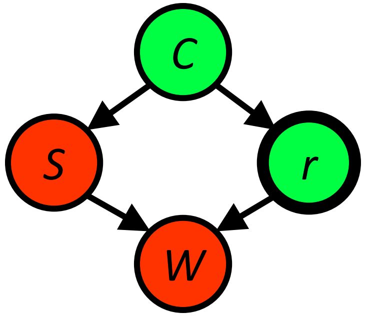


Gibbs Sampling Example: *P*(*S* | *r*)

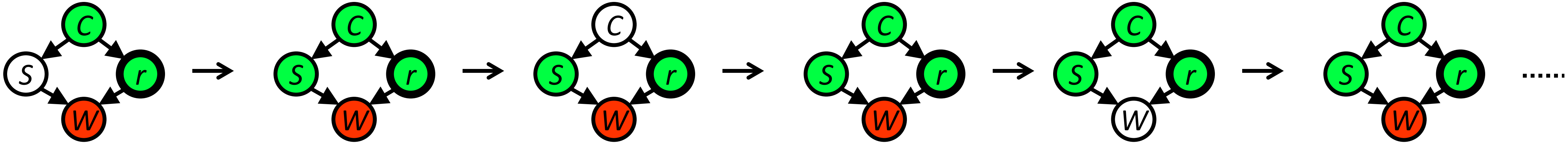
- Step 1: Fix evidence
  - *R* = true



- Step 2: Initialize other variables
  - Randomly



- Step 3: Repeat
  - Choose a non-evidence variable *X*
  - Resample *X* from *P*(*X* | all other variables)



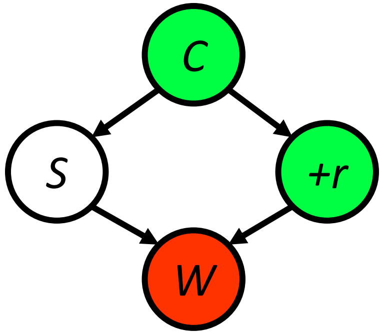
Sample from $P(S \mid c, r, \neg w)$

Sample from $P(C \mid s, r)$

Sample from $P(W \mid s, r)$

Resampling of One Variable

- Sample from P(S | +c, +r, -w)



$$P(S|+c,+r,-w)$$

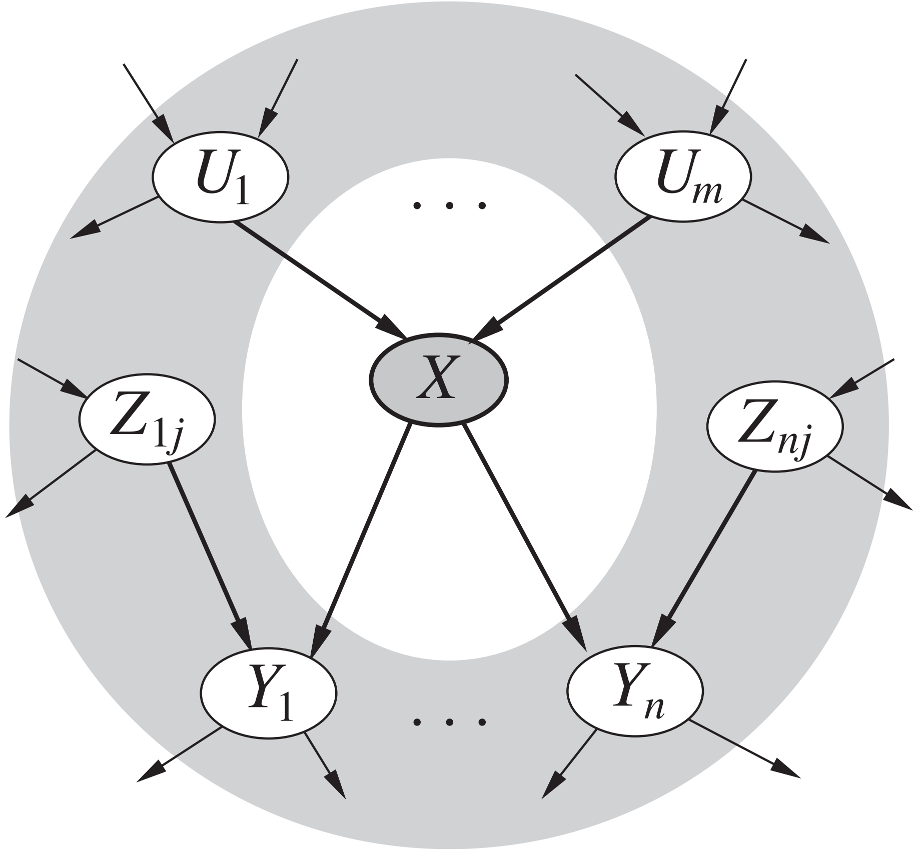
- Many things cancel out – only CPTs with S remain!
- More generally: only CPTs that have resampled variable need to be considered, and joined together

Resampling of One Variable

- Repeat many times
  - Sample a non-evidence variable X_i from

$$P(X_i \mid x_1,..,x_{i-1},x_{i+1},..,x_n) = P(X_i \mid markov\_blanket(X_i))$$

$$= \alpha\ P(X_i \mid parents\ (X_i))\ \prod_j P(y_j \mid parents(Y_j))$$



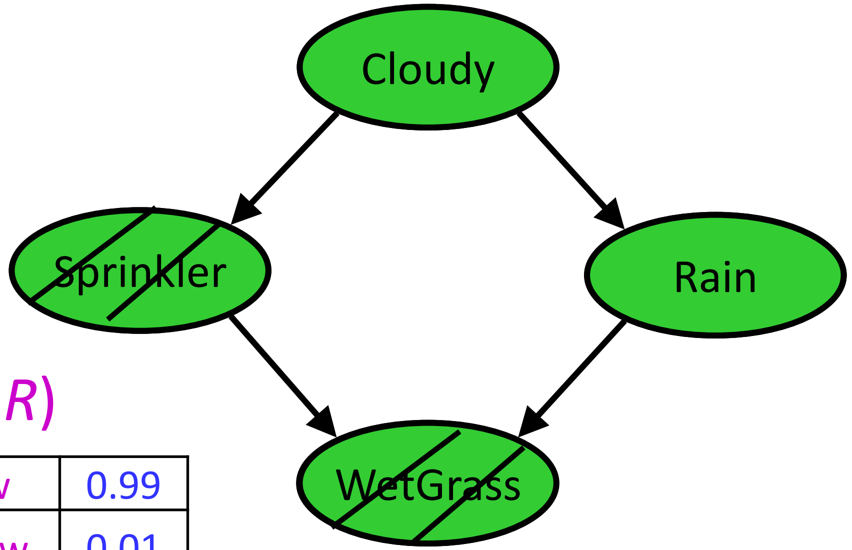
Gibbs Sampling

P(C)

c	0.5
¬c	0.5

P(S | C)

c	s	0.1
	¬s	0.9
¬c	s	0.5
	¬s	0.5



P(R | C)

c	r	0.8
	¬r	0.2
¬c	r	0.2
	¬r	0.8

P(W | S,R)

s	r	w	0.99
		¬w	0.01
	¬r	w	0.90
		¬w	0.10
¬s	r	w	0.90
		¬w	0.10
	¬r	w	0.01
		¬w	0.99

Want to estimate P (C | s, w)

(Arbitrarily) Pick R to resample

P (R | c, s, w) = α P(R | c) P (w | s,R)

(Arbitrarily) Pick C to resample

P (C | r, s, w) = α P(C) P(s | C) P (r | C)

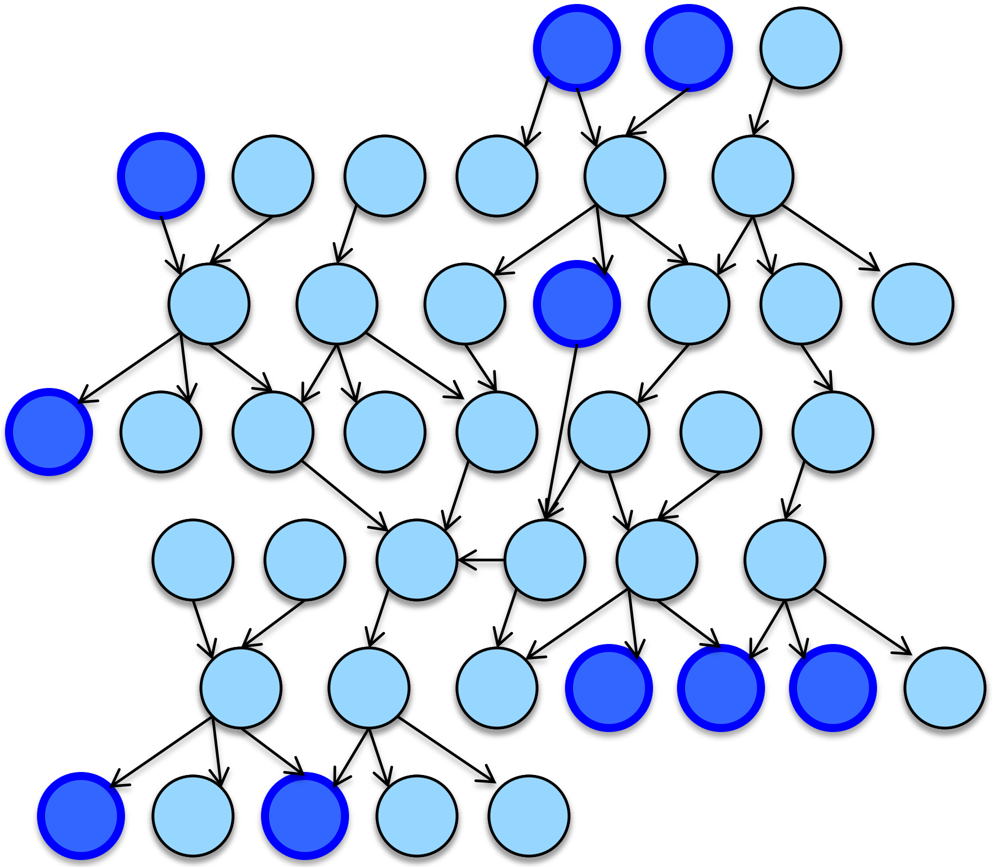
Samples:

c, s, r, w

c, s, r, w

¬ ***c, s, r, w***

Why would anyone do this?



Both upstream and downstream variables condition on evidence!

In contrast: likelihood weighting only conditions on upstream evidence, and hence weights obtained in likelihood weighting can sometimes be very small.

More details on Gibbs sampling

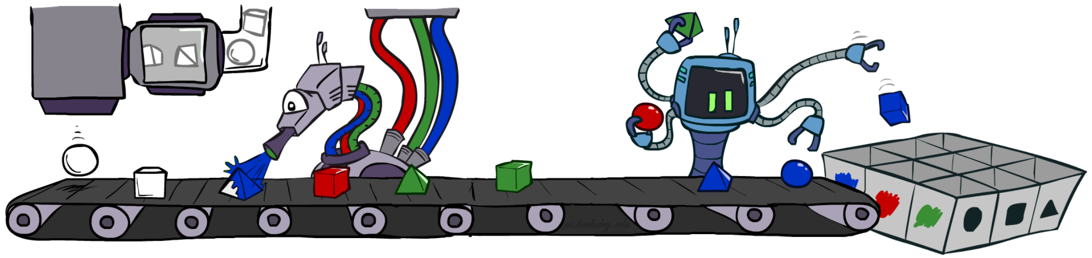
- Gibbs sampling belongs to a family of sampling methods called Markov chain Monte Carlo (MCMC)

- Theorem: In the limit of repeating this infinitely many times, the resulting samples come from the correct distribution.*

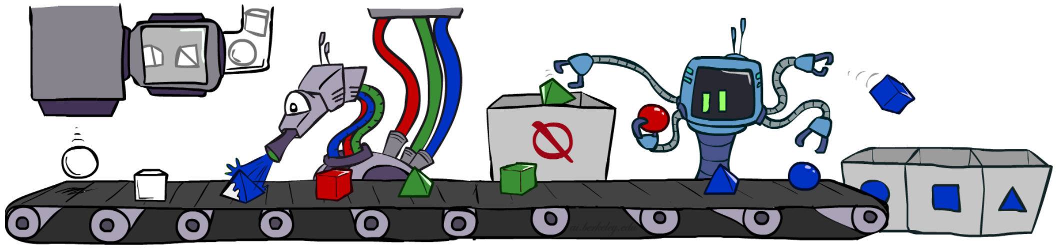
*: Provided all Gibbs distributions are bounded away from 0 and 1 and variable selection is fair

Bayes Net Sampling Summary

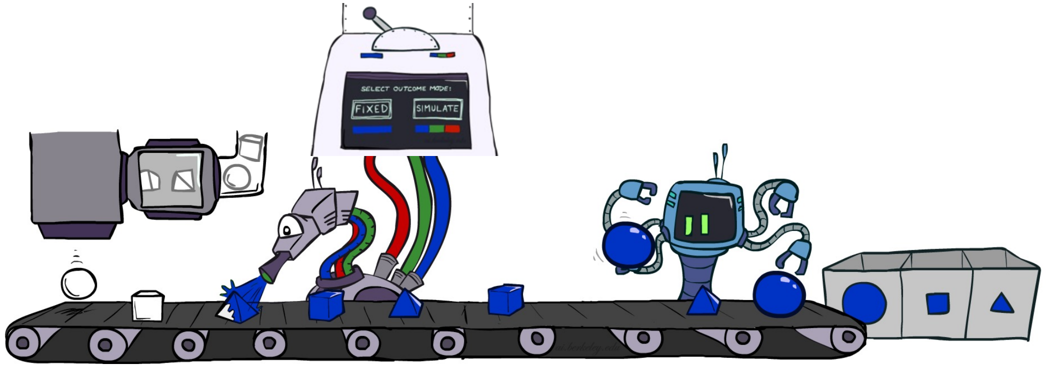
- Prior Sampling



- Rejection Sampling



- Likelihood Weighting



- Gibbs Sampling