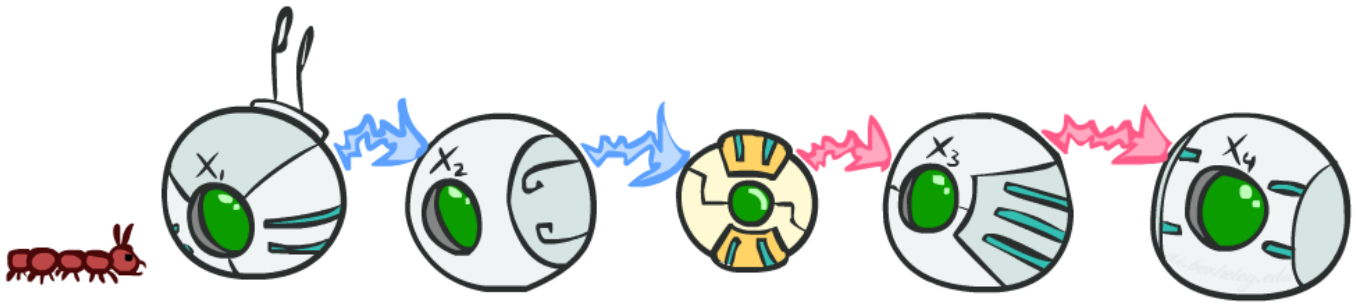
CS 188: Artificial Intelligence

Markov Models



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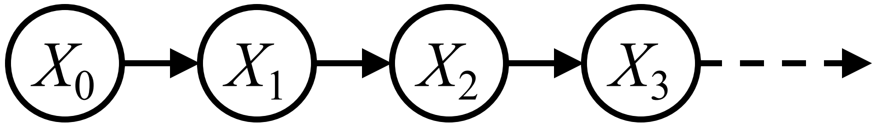
(Slides adapted from Pieter Abbeel, Dan Klein, Anca Dragan, Stuart Russell and Dawn Song)

Uncertainty and Time

- Often, we want to reason about a ***sequence*** of observations where the state of the underlying system is ***changing***
  - Speech recognition
  - Robot localization
  - User attention
  - Medical monitoring
  - Global climate
- Need to introduce time into our models

Markov Models (aka Markov chain/process)

- Value of X at a given time is called the ***state*** (usually discrete, finite)



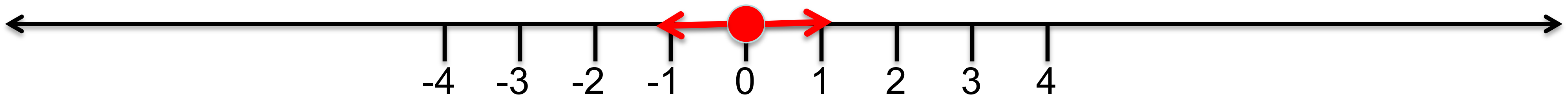
$P(X_0)$ $P(X_t \mid X_{t-1})$

- The ***transition model*** $P(X_t \mid X_{t-1})$ specifies how the state evolves over time
- ***Stationarity*** assumption: transition probabilities are the same at all times
- ***Markov*** assumption: “future is independent of the past given the present”
  - X_{t+1} is independent of $X_0, \ldots, X_{t-1}$ given X_t
  - This is a ***first-order*** Markov model (a kth-order model allows dependencies on k earlier steps)
- Current observation independent of all else given current state
- Joint distribution $P(X_0, \ldots, X_T) = P(X_0) \prod_t P(X_t \mid X_{t-1})$

Quiz: are Markov models a special case of Bayes nets?

- Yes and no!
- Yes:
  - Directed acyclic graph, joint = product of conditionals
- No:
  - Infinitely many variables (unless we truncate)
  - Repetition of transition model not part of standard Bayes net syntax

Example: Random walk in one dimension



- State: location on the unbounded integer line
- Initial probability: starts at 0
- Transition model: $P(X_t = k \mid X_{t-1} = k \pm 1) = 0.5$
- Applications: particle motion in crystals, stock prices, gambling, genetics, etc.
- Questions:
  - How far does it get as a function of t?
    - Expected distance is $O(\sqrt{t})$
  - Does it get back to 0 or can it go off for ever and not come back?
    - In 1D and 2D, returns w.p. 1; in 3D, returns w.p. 0.34053733

Example: n-gram models

- State: word at position *t* in text (can also build letter n-grams)
- Transition model (probabilities come from empirical frequencies):
  - Unigram (zero-order): $P(Word_t = i)$
    - "logical are as are confusion a may right tries agent goal the was . . ."
  - Bigram (first-order): $P(Word_t = i \mid Word_{t-1} = j)$
    - "systems are very similar computational approach would be represented . . ."
  - Trigram (second-order): $P(Word_t = i \mid Word_{t-1} = j, Word_{t-2} = k)$
    - "planning and scheduling are integrated the success of naive bayes model is . . ."
- Applications: text classification, language classification, speech recognition

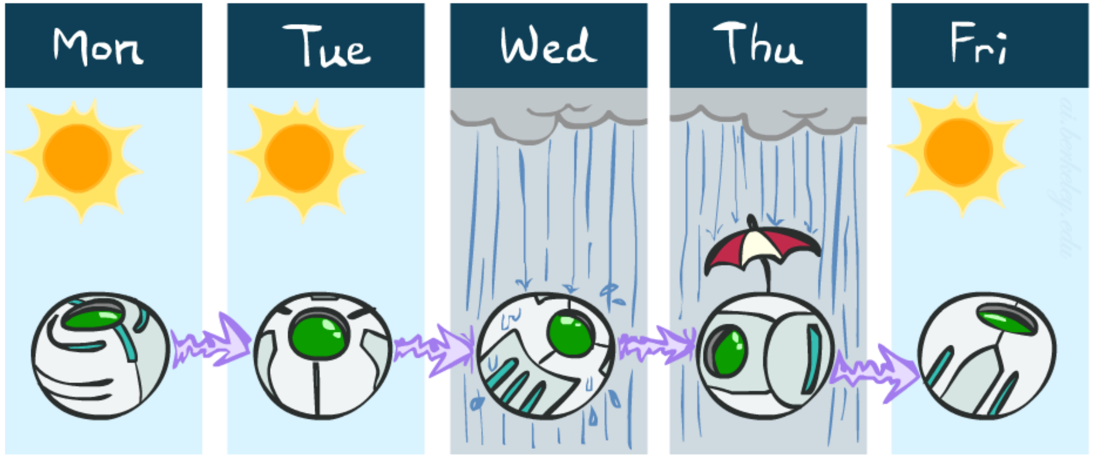
Example: Weather

- States {rain, sun}
- Initial distribution $P(X_0)$

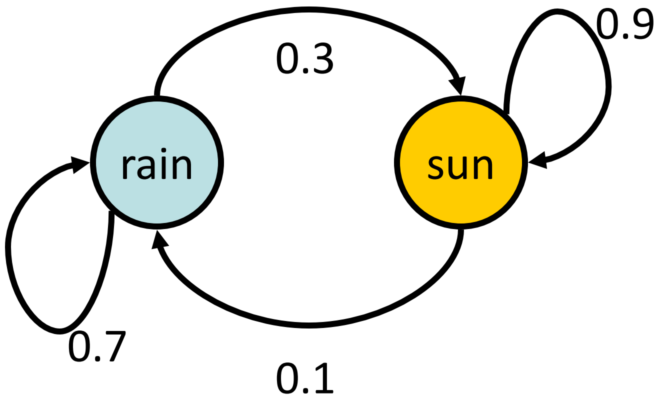
$P(X_0)$	
sun	rain
0.5	0.5

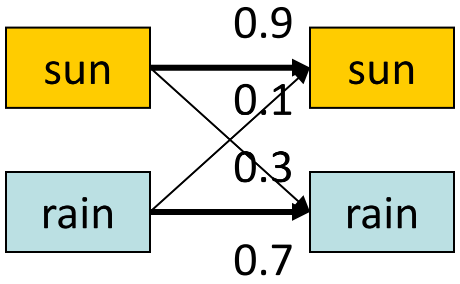
- Transition model $P(X_t \mid X_{t-1})$

X_{t-1}	$P(X_t \mid X_{t-1})$	
	sun	rain
sun	0.9	0.1
rain	0.3	0.7



Two new ways of representing the same CPT

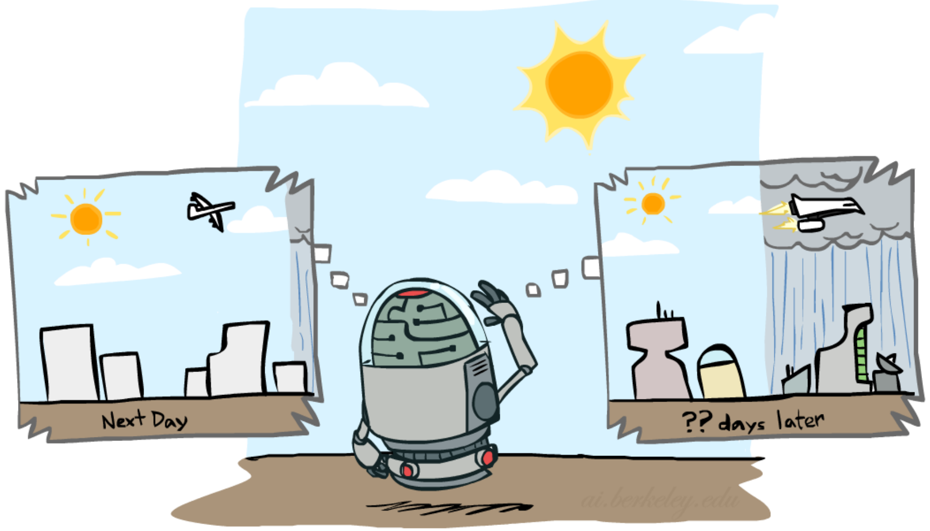




Weather prediction

- Time 0: <0.5,0.5>

X_{t-1}	$P(X_t \mid X_{t-1})$	
	sun	rain
sun	0.9	0.1
rain	0.3	0.7

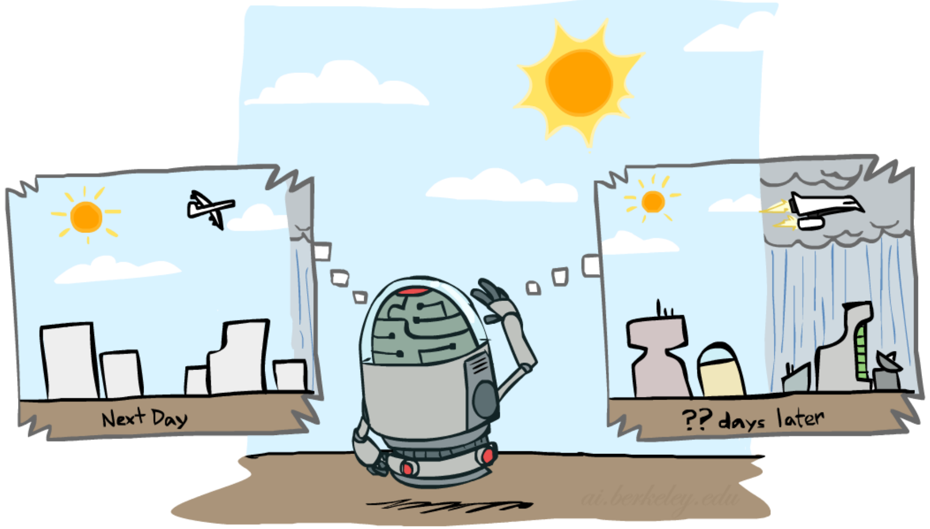


- What is the weather like at time 1?
  - $P(X_1) = \sum_{x_0} P(X_1, X_0 = x_0)$
  - $= \sum_{x_0} P(X_0 = x_0)\, P(X_1 \mid X_0 = x_0)$
  - $= 0.5\langle 0.9, 0.1\rangle + 0.5\langle 0.3, 0.7\rangle = \langle 0.6, 0.4\rangle$

Weather prediction, contd.

- Time 1: <0.6,0.4>

X_{t-1}	$P(X_t \| X_{t-1})$	
	sun	rain
sun	0.9	0.1
rain	0.3	0.7

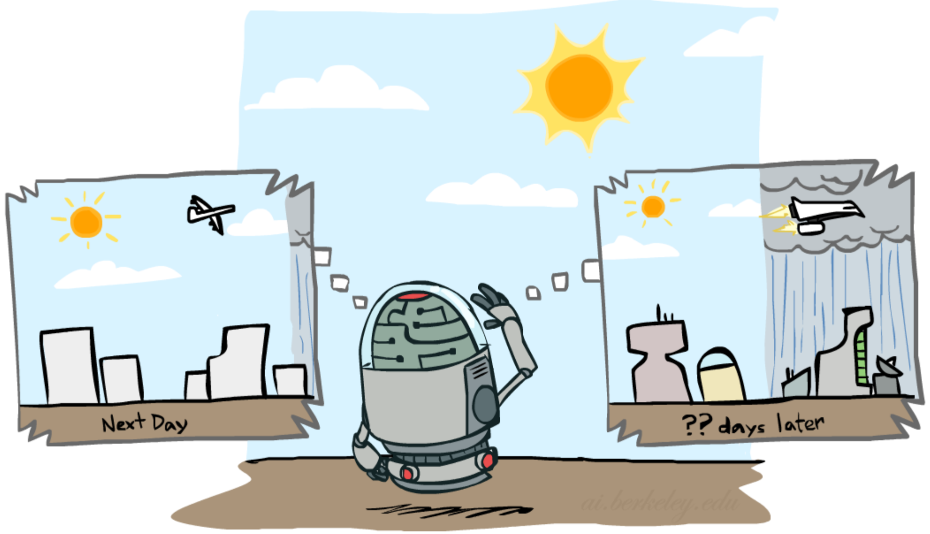


- What is the weather like at time 2?
  - $P(X_2) = \sum_{x_1} P(X_2, X_1 = x_1)$
  - $= \sum_{x_1} P(X_1 = x_1)\, P(X_2 \mid X_1 = x_1)$
  - $= 0.6\langle 0.9, 0.1\rangle + 0.4\langle 0.3, 0.7\rangle = \langle 0.66, 0.34\rangle$

Weather prediction, contd.

- Time 2: <0.66,0.34>

X_{t-1}	$P(X_t \mid X_{t-1})$	
	sun	rain
sun	0.9	0.1
rain	0.3	0.7



- What is the weather like at time 3?
  - $P(X_3) = \sum_{x_2} P(X_3, X_2{=}x_2)$
  - $= \sum_{x_2} P(X_2{=}x_2)\, P(X_3 \mid X_2{=}x_2)$
  - $= 0.66\langle 0.9, 0.1\rangle + 0.34\langle 0.3, 0.7\rangle = \langle 0.696, 0.304\rangle$

Mini-Forward algorithm

Probability from previous iteration

Transition model

- What is the state at time t?
  - $P(X_t) = \sum_{x_{t-1}} P(X_t, X_{t-1}=x_{t-1})$
  - $= \sum_{x_{t-1}} P(X_{t-1}=x_{t-1}) \, P(X_t \mid X_{t-1}=x_{t-1})$
- Iterate this update starting at $t=0$
  - This is called a ***recursive*** update: $P_t = g(P_{t-1}) = g(g(g(g(\ldots P_0))))$

And the same thing in linear algebra

- What is the weather like at time 2?
  - $P(X_2) = 0.6\langle 0.9,0.1\rangle + 0.4\langle 0.3,0.7\rangle = \langle 0.66,0.34\rangle$
- In matrix-vector form:
  - $P(X_2) = \begin{pmatrix} 0.9 & 0.3 \\ 0.1 & 0.7 \end{pmatrix} \begin{pmatrix} 0.6 \\ 0.4 \end{pmatrix} = \begin{pmatrix} 0.66 \\ 0.34 \end{pmatrix}$

X_{t-1}	$P(X_t \mid X_{t-1})$	
	sun	rain
sun	0.9	0.1
rain	0.3	0.7

- I.e., multiply by T^T, transpose of transition matrix

Stationary Distributions

- The limiting distribution is called the ***stationary distribution*** P_∞ of the chain
- It satisfies $P_\infty = P_{\infty+1} = T^T P_\infty$
- Solving for P_∞ in the example:

$$\begin{pmatrix} 0.9 & 0.3 \\ 0.1 & 0.7 \end{pmatrix} \begin{pmatrix} p \\ 1-p \end{pmatrix} = \begin{pmatrix} p \\ 1-p \end{pmatrix}$$

$0.9p + 0.3(1-p) = p$

$p = 0.75$

Stationary distribution is <0.75,0.25> ***regardless of starting distribution***

Example Run of Mini-Forward Algorithm

- From initial observation of sun

$$\underbrace{\left\langle \begin{matrix} 1.0 \\ 0.0 \end{matrix} \right\rangle}_{P(X_1)} \quad \underbrace{\left\langle \begin{matrix} 0.9 \\ 0.1 \end{matrix} \right\rangle}_{P(X_2)} \quad \underbrace{\left\langle \begin{matrix} 0.84 \\ 0.16 \end{matrix} \right\rangle}_{P(X_3)} \quad \underbrace{\left\langle \begin{matrix} 0.804 \\ 0.196 \end{matrix} \right\rangle}_{P(X_4)} \Rightarrow \underbrace{\left\langle \begin{matrix} 0.75 \\ 0.25 \end{matrix} \right\rangle}_{P(X_\infty)}$$

- From initial observation of rain

$$\underbrace{\left\langle \begin{matrix} 0.0 \\ 1.0 \end{matrix} \right\rangle}_{P(X_1)} \quad \underbrace{\left\langle \begin{matrix} 0.3 \\ 0.7 \end{matrix} \right\rangle}_{P(X_2)} \quad \underbrace{\left\langle \begin{matrix} 0.48 \\ 0.52 \end{matrix} \right\rangle}_{P(X_3)} \quad \underbrace{\left\langle \begin{matrix} 0.588 \\ 0.412 \end{matrix} \right\rangle}_{P(X_4)} \Rightarrow \underbrace{\left\langle \begin{matrix} 0.75 \\ 0.25 \end{matrix} \right\rangle}_{P(X_\infty)}$$

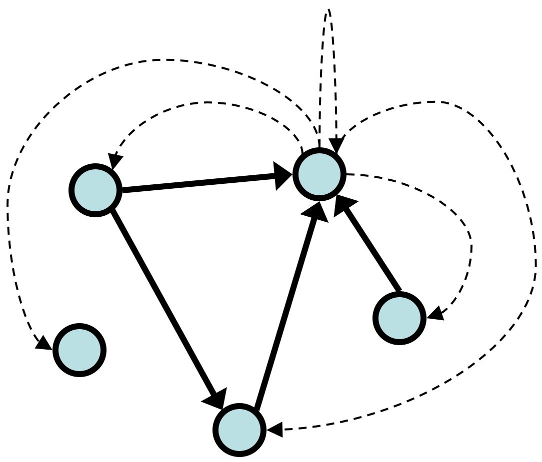
- From yet another initial distribution $P(X_1)$:

$$\underbrace{\left\langle \begin{matrix} p \\ 1-p \end{matrix} \right\rangle}_{P(X_1)} \quad \cdots \quad \Rightarrow \underbrace{\left\langle \begin{matrix} 0.75 \\ 0.25 \end{matrix} \right\rangle}_{P(X_\infty)}$$

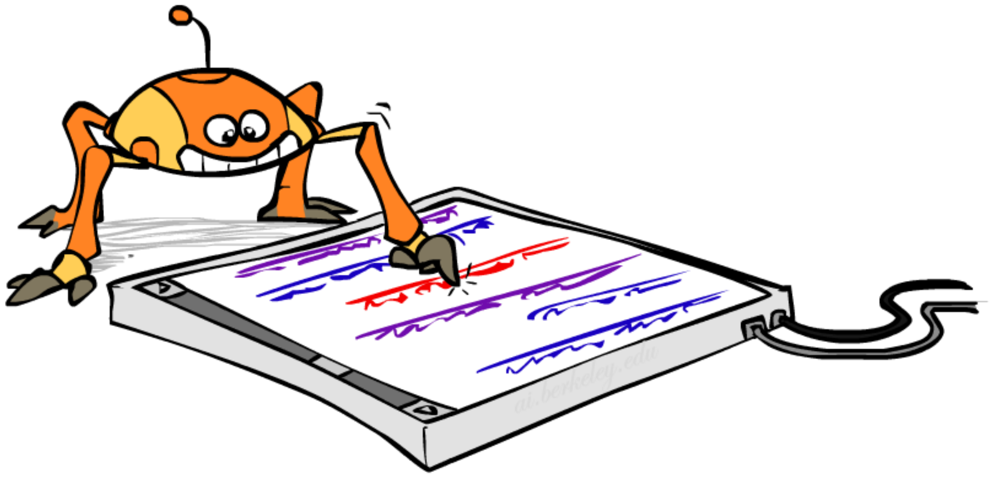
[Demo: L13D1,2,3]

Application of Stationary Distribution: Web Link Analysis

- PageRank over a web graph
  - Each web page is a possible value of a state
  - Initial distribution: uniform over pages
  - Transitions:
    - With prob. c, uniform jump to a random page
    - With prob. 1-c, follow a random outlink



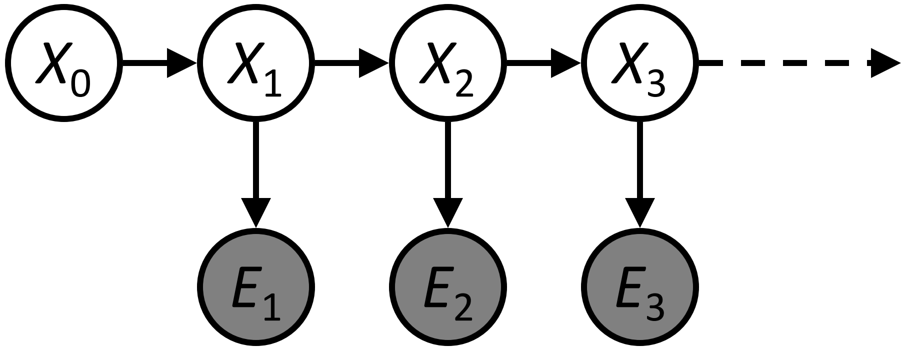
- Stationary distribution
  - Will spend more time on highly reachable pages
  - Google 1.0 returned the set of pages containing all your keywords in decreasing rank, now all search engines use link analysis along with many other factors (rank actually getting less important over time)



Hidden Markov Models

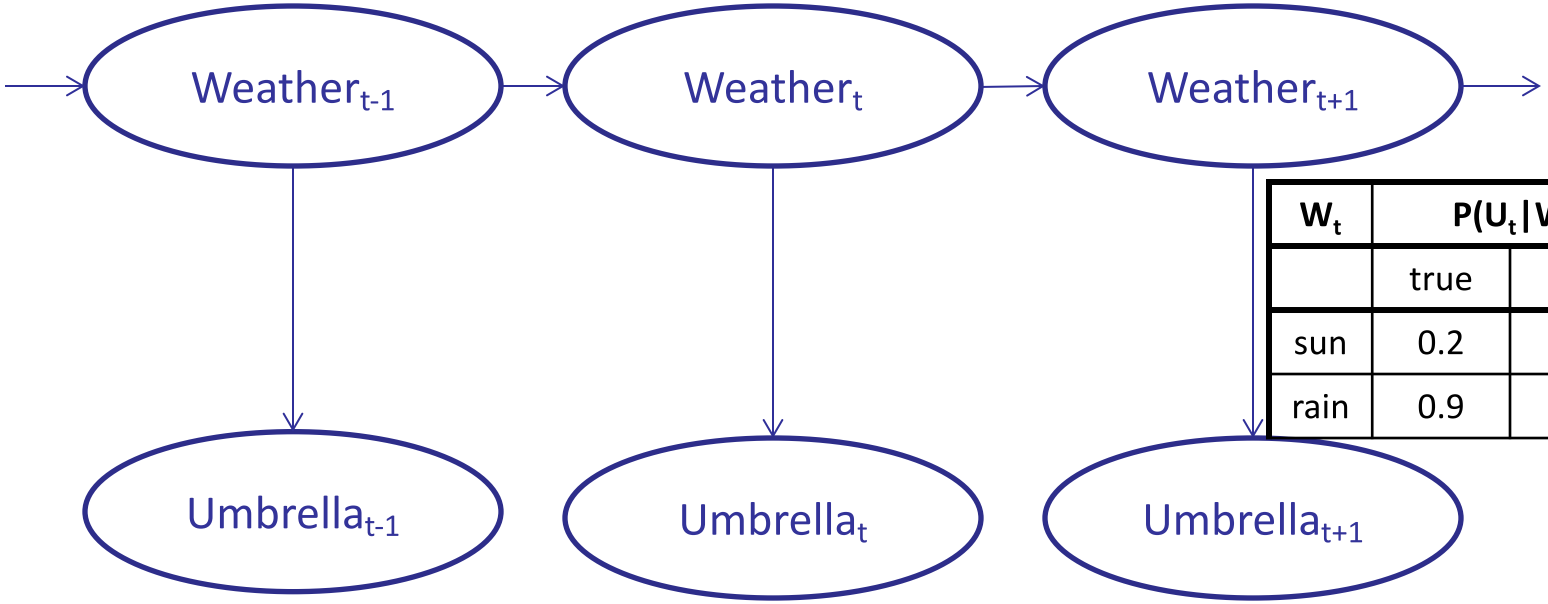
Hidden Markov Models

- Usually the true state is not observed directly
- Hidden Markov models (HMMs)
  - Underlying Markov chain over states X
  - You observe evidence E at each time step
  - X_t is a single discrete variable; E_t may be continuous and may consist of several variables



Example: Weather HMM

W_{t-1}	$P(W_t \mid W_{t-1})$	
	sun	rain
sun	0.9	0.1
rain	0.3	0.7

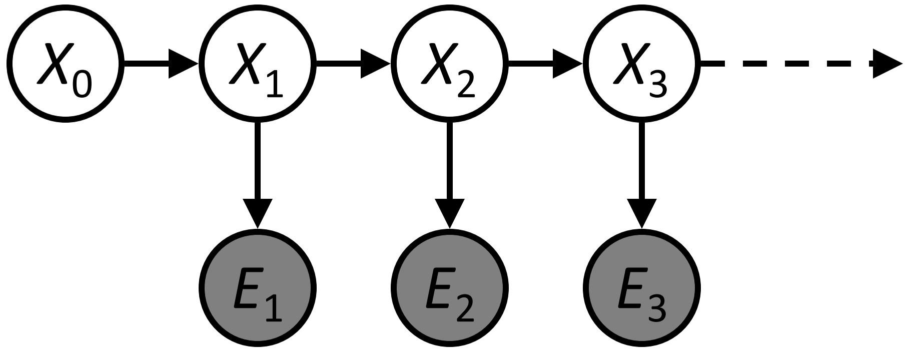


W_t	$P(U_t \mid W_t)$	
	true	false
sun	0.2	0.8
rain	0.9	0.1

- An HMM is defined by:
  - Initial distribution: $P(X_0)$
  - Transition model: $P(X_t \mid X_{t-1})$
  - Sensor model: $P(E_t \mid X_t)$

HMM as probability model

- Joint distribution for Markov model: $P(X_0,\ldots, X_T) = P(X_0) \prod_{t=1:T} P(X_t \mid X_{t-1})$
- Joint distribution for hidden Markov model:
  $P(X_0,X_1,\ldots, X_T,E_T) = P(X_0) \prod_{t=1:T} P(X_t \mid X_{t-1})\, P(E_t \mid X_t)$
- Future states are independent of the past given the present
- Current evidence is independent of everything else given the current state
- Are evidence variables independent of each other?



Useful notation:

$X_{a:b} = X_a , X_{a+1}, \ldots, X_b$

Real HMM Examples

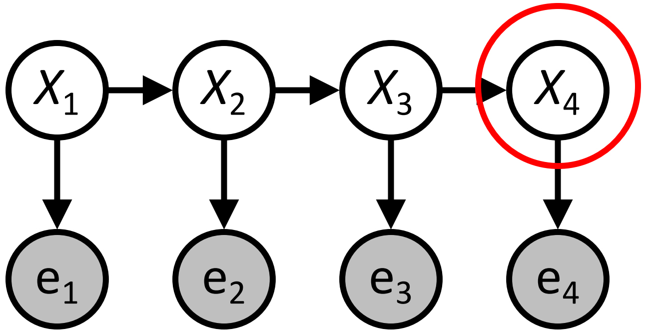
- Speech recognition HMMs:
  - Observations are acoustic signals (continuous valued)
  - States are specific positions in specific words (so, tens of thousands)
- Machine translation HMMs:
  - Observations are words (tens of thousands)
  - States are translation options
- Robot tracking:
  - Observations are range readings (continuous)
  - States are positions on a map (continuous)
- Molecular biology:
  - Observations are nucleotides ACGT
  - States are coding/non-coding/start/stop/splice-site etc.

Inference tasks

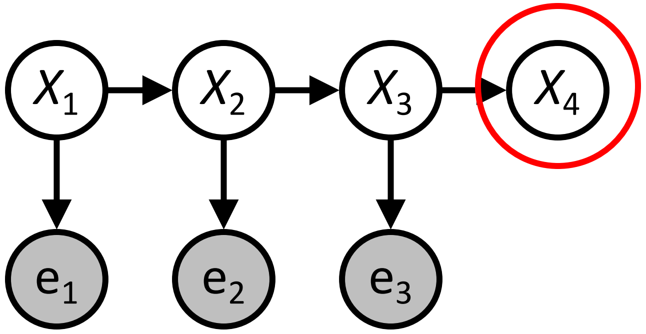
- ***Filtering***: $P(X_t|e_{1:t})$
  - ***belief state***—input to the decision process of a rational agent
- ***Prediction***: $P(X_{t+k}|e_{1:t})$ for $k > 0$
  - evaluation of possible action sequences; like filtering without the evidence
- ***Smoothing***: $P(X_k|e_{1:t})$ for $0 \leq k < t$
  - better estimate of past states, essential for learning
- ***Most likely explanation***: $\arg\max_{x_{1:t}} P(x_{1:t} \mid e_{1:t})$
  - speech recognition, decoding with a noisy channel

Inference tasks

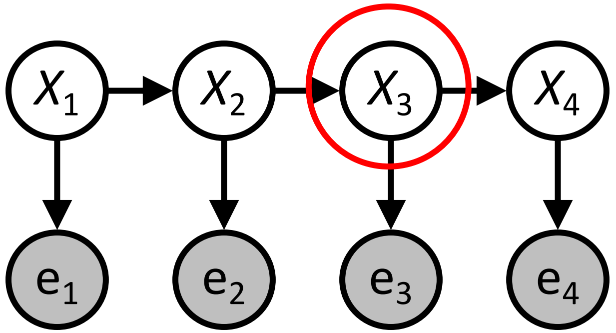
Filtering: $P(X_t | e_{1:t})$



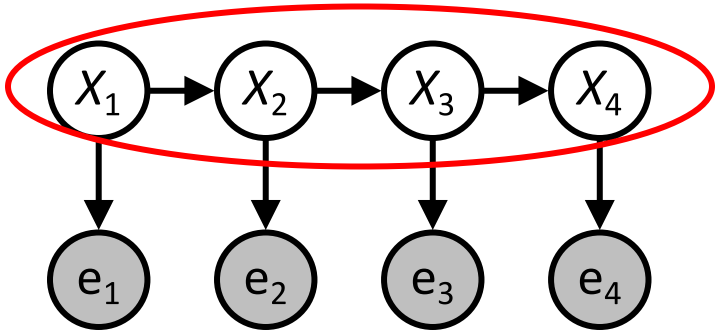
Prediction: $P(X_{t+k} | e_{1:t})$



Smoothing: $P(X_k | e_{1:t})$, $k<t$



Explanation: $P(X_{1:t} | e_{1:t})$



Filtering / Monitoring

- Filtering, or monitoring, or state estimation, is the task of maintaining the distribution $\boldsymbol{f}_{1:t} = P(X_t | e_{1:t})$ over time

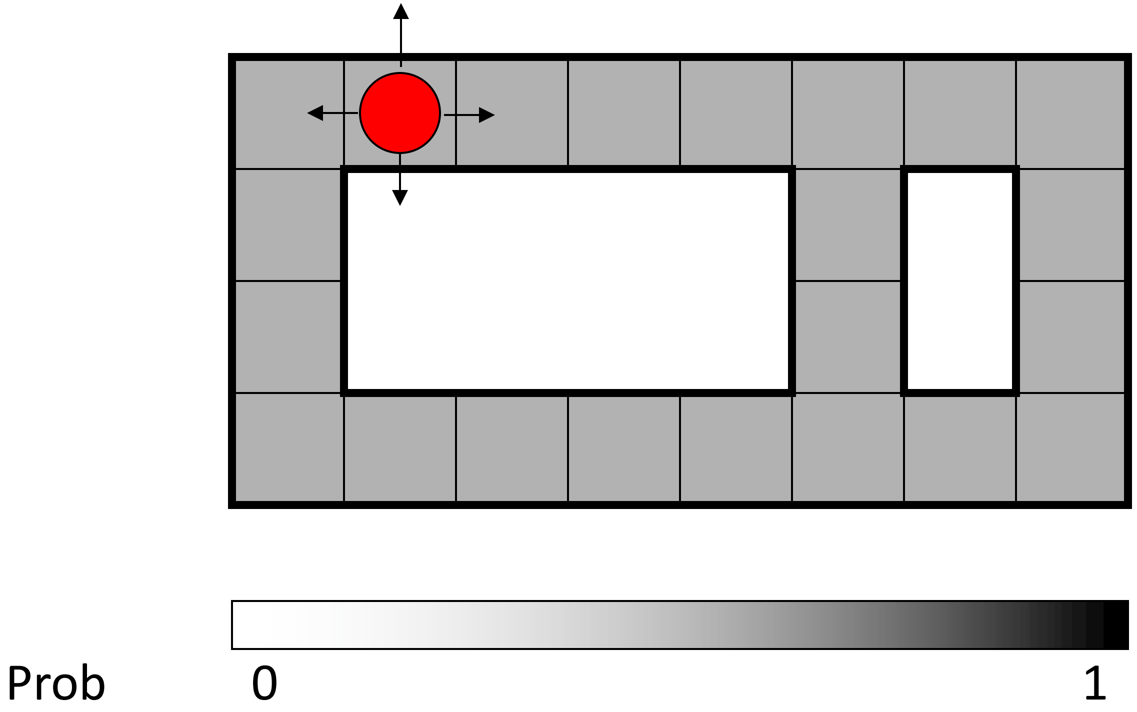
- We start with $\boldsymbol{f}_0$ in an initial setting, usually uniform

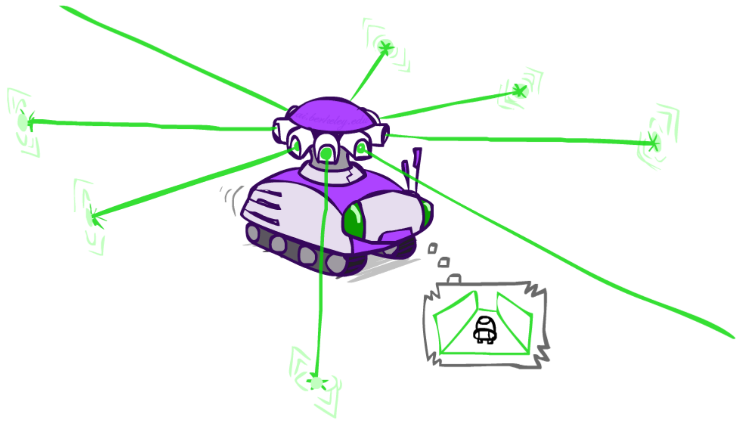
- Filtering is a fundamental task in engineering and science

- The Kalman filter (continuous variables, linear dynamics, Gaussian noise) was invented in 1960 and used for trajectory estimation in the Apollo program; core ideas used by Gauss for planetary observations

Example: Robot Localization

Example from
Michael Pfeiffer



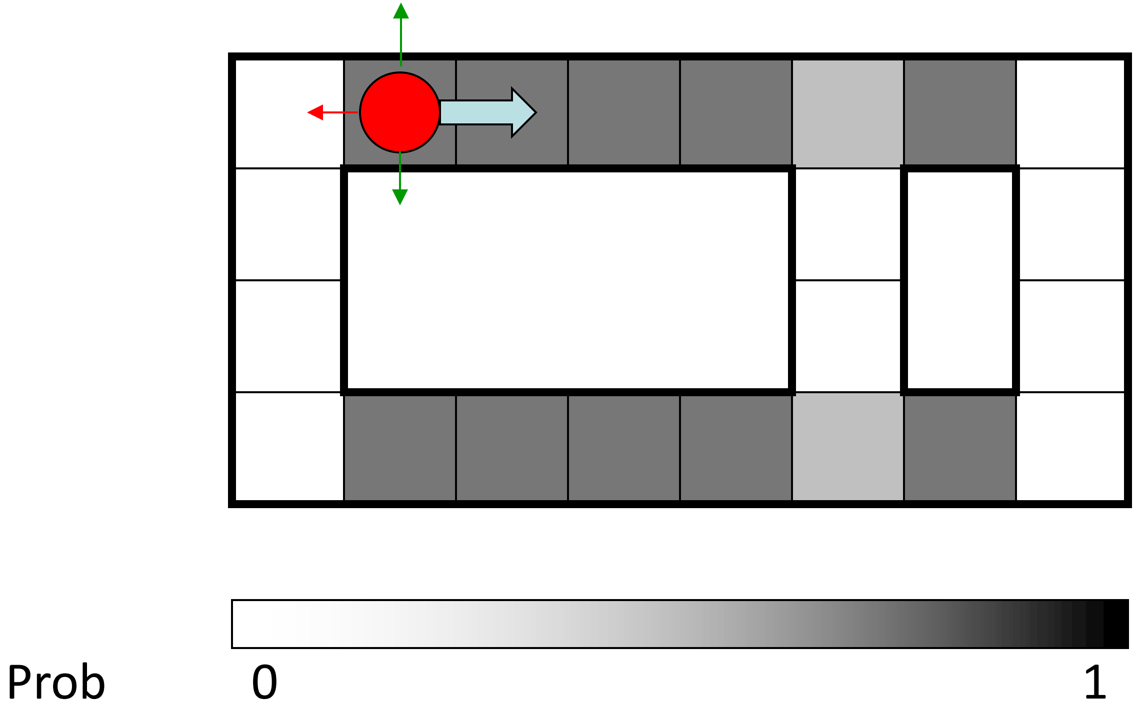


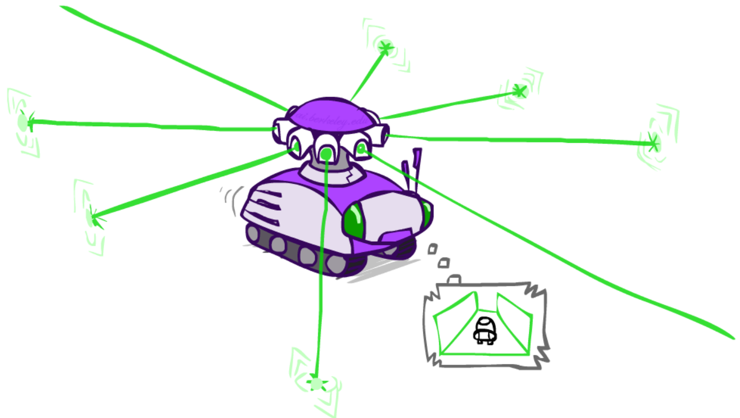
t=0

Sensor model: four bits for wall/no-wall in each direction, never more than 1 mistake

Transition model: action may fail with small prob.

Example: Robot Localization



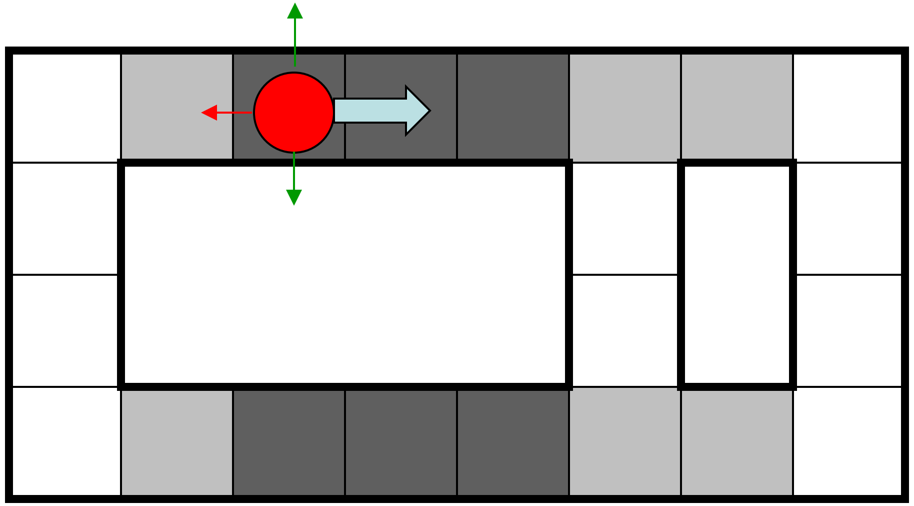


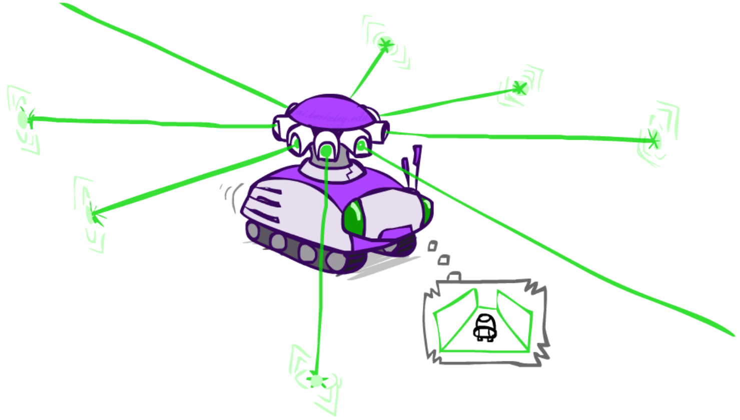
t=1

Lighter grey: was ***possible*** to get the reading,

but ***less likely*** (required 1 mistake)

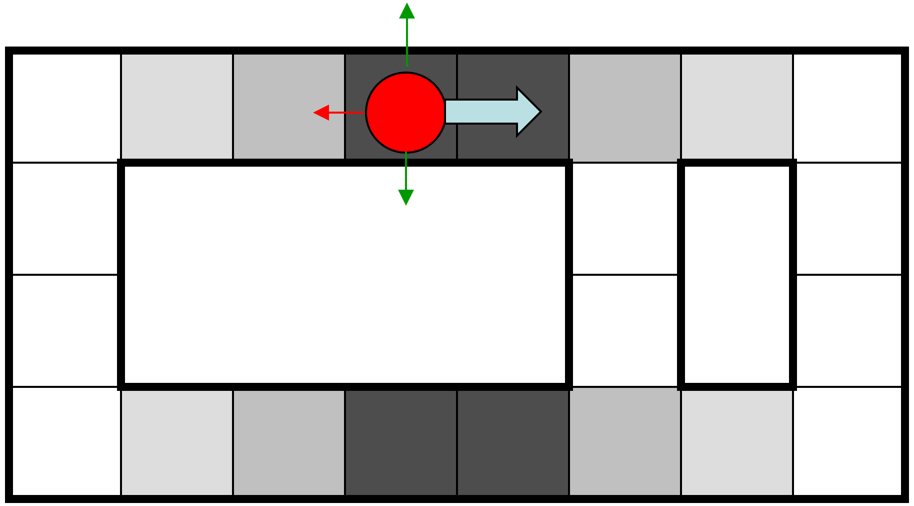
Example: Robot Localization

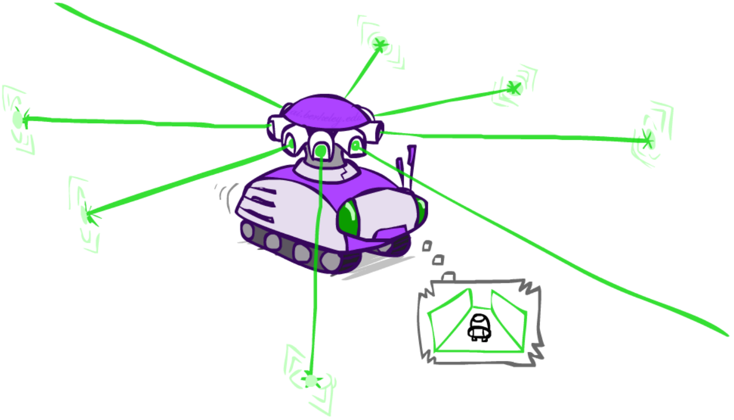


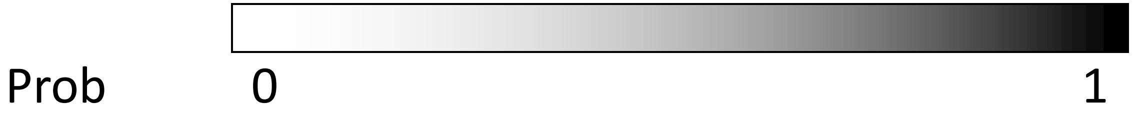


t=2

Example: Robot Localization

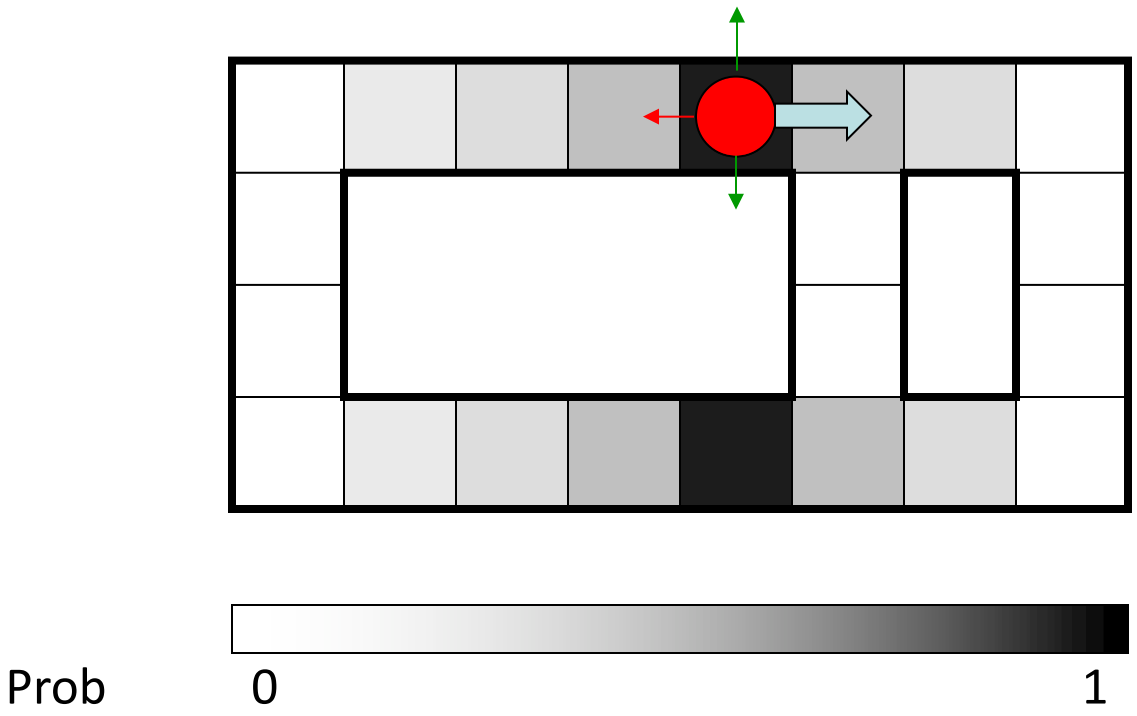


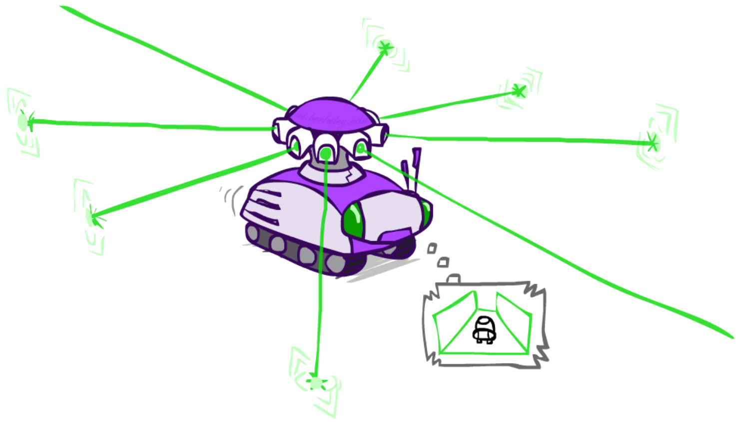




t=3

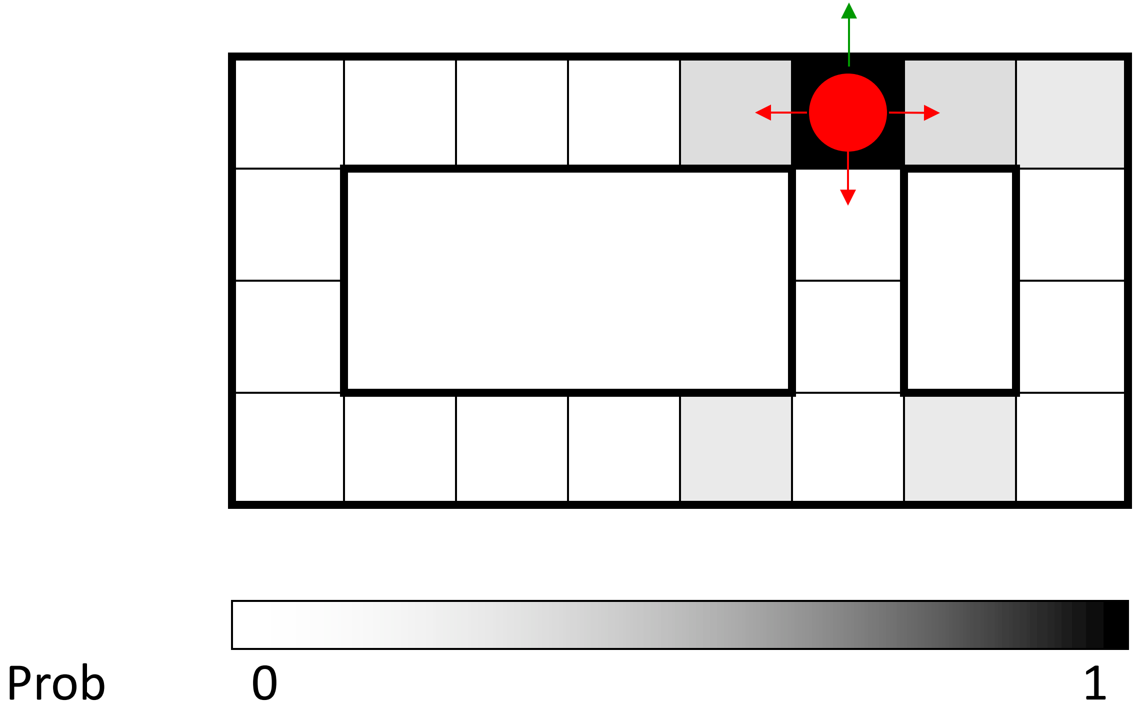
Example: Robot Localization

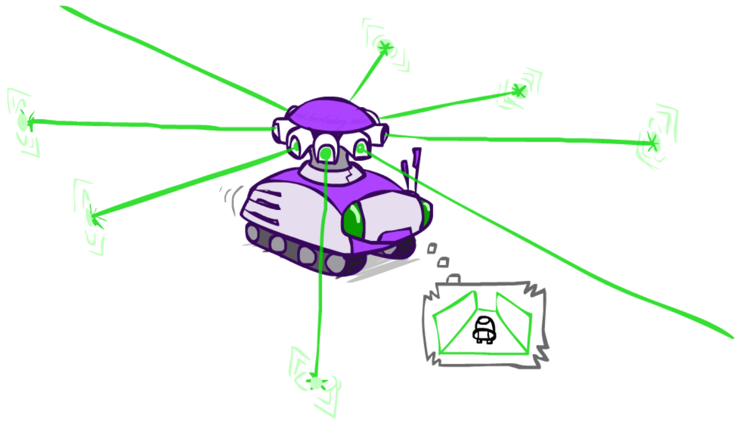




t=4

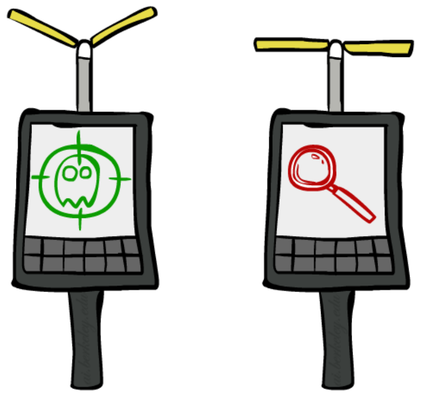
Example: Robot Localization

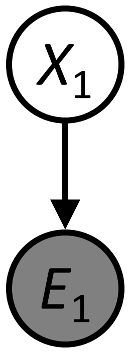


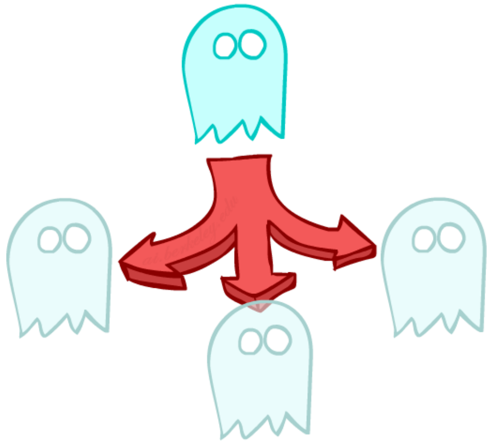


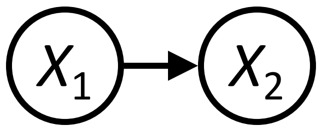
t=5

Inference: Base Cases









$$P(X_1|e_1)$$

$$P(X_1|e_1) = \frac{P(X_1, e_1)}{\sum_{x_1} P(x_1, e_1)}$$

$$P(X_1|e_1) = \frac{P(e_1|X_1)P(X_1)}{\sum_{x_1} P(e_1|x_1)P(x_1)}$$

$$P(X_2)$$

$$P(X_2) = \sum_{x_1} P(x_1, X_2)$$

$$P(X_2) = \sum_{x_1} P(X_2|x_1)P(x_1)$$

Filtering algorithm

- Aim: devise a ***recursive filtering*** algorithm of the form
  - $P(X_{t+1} | e_{1:t+1}) = g(e_{t+1}, P(X_t | e_{1:t}))$

- $P(X_{t+1} | e_{1:t+1}) =$

Filtering algorithm

- Aim: devise a ***recursive filtering*** algorithm of the form
  - $P(X_{t+1} | e_{1:t+1}) = g(e_{t+1}, P(X_t | e_{1:t}))$

- $P(X_{t+1} | e_{1:t+1}) = P(X_{t+1} | e_{1:t}, e_{t+1})$
- $= \alpha\, P(e_{t+1} | X_{t+1}, e_{1:t})\, P(X_{t+1} | e_{1:t})$
- $= \alpha\, P(e_{t+1} | X_{t+1})\, P(X_{t+1} | e_{1:t})$
- $= \alpha\, P(e_{t+1} | X_{t+1}) \sum_{x_t} P(x_t | e_{1:t})\, P(X_{t+1} | x_t, e_{1:t})$
- $= \alpha\, P(e_{t+1} | X_{t+1}) \sum_{x_t} P(x_t | e_{1:t})\, P(X_{t+1} | x_t)$

Apply Bayes' rule

Apply conditional independence

Condition on X_t

Apply conditional independence

Normalize

Update

Predict

Filtering algorithm

- $P(X_{t+1} | e_{1:t+1}) = \alpha\, P(e_{t+1} | X_{t+1}) \sum_{x_t} P(x_t \mid e_{1:t})\, P(X_{t+1} | x_t)$

Normalize — Update — Predict

- $\boldsymbol{f}_{1:t+1} = \text{FORWARD}(\boldsymbol{f}_{1:t}, e_{t+1})$
- Cost per time step: $O(|X|^2)$ where $|X|$ is the number of states
- Time and space costs are ***constant***, independent of t
- $O(|X|^2)$ is infeasible for models with many state variables
- We get to invent really cool approximate filtering algorithms

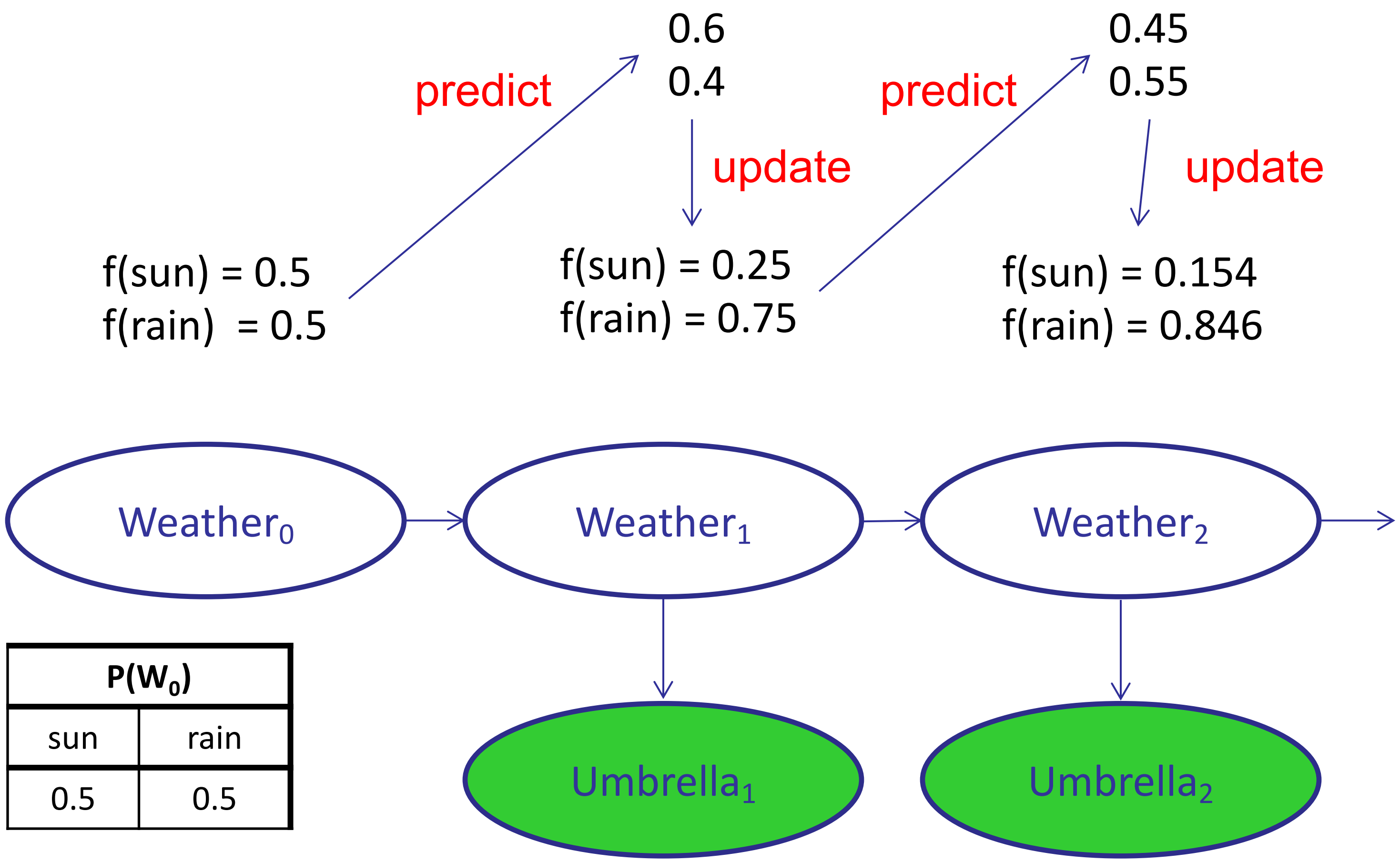
And the same thing in linear algebra

- Transition matrix T, observation matrix O_t
  - Observation matrix has state likelihoods for E_t along diagonal
  - E.g., for U_1 = true, $O_1 = \begin{pmatrix} 0.2 & 0 \\ 0 & 0.9 \end{pmatrix}$
- Filtering algorithm becomes
  - $\boldsymbol{f}_{1:t+1} = \alpha\, O_{t+1} T^{\mathrm{T}}\, \boldsymbol{f}_{1:t}$

X_{t-1}	$P(X_t \mid X_{t-1})$	
	sun	rain
sun	0.9	0.1
rain	0.3	0.7

W_t	$P(U_t \mid W_t)$	
	true	false
sun	0.2	0.8
rain	0.9	0.1

Example: Weather HMM

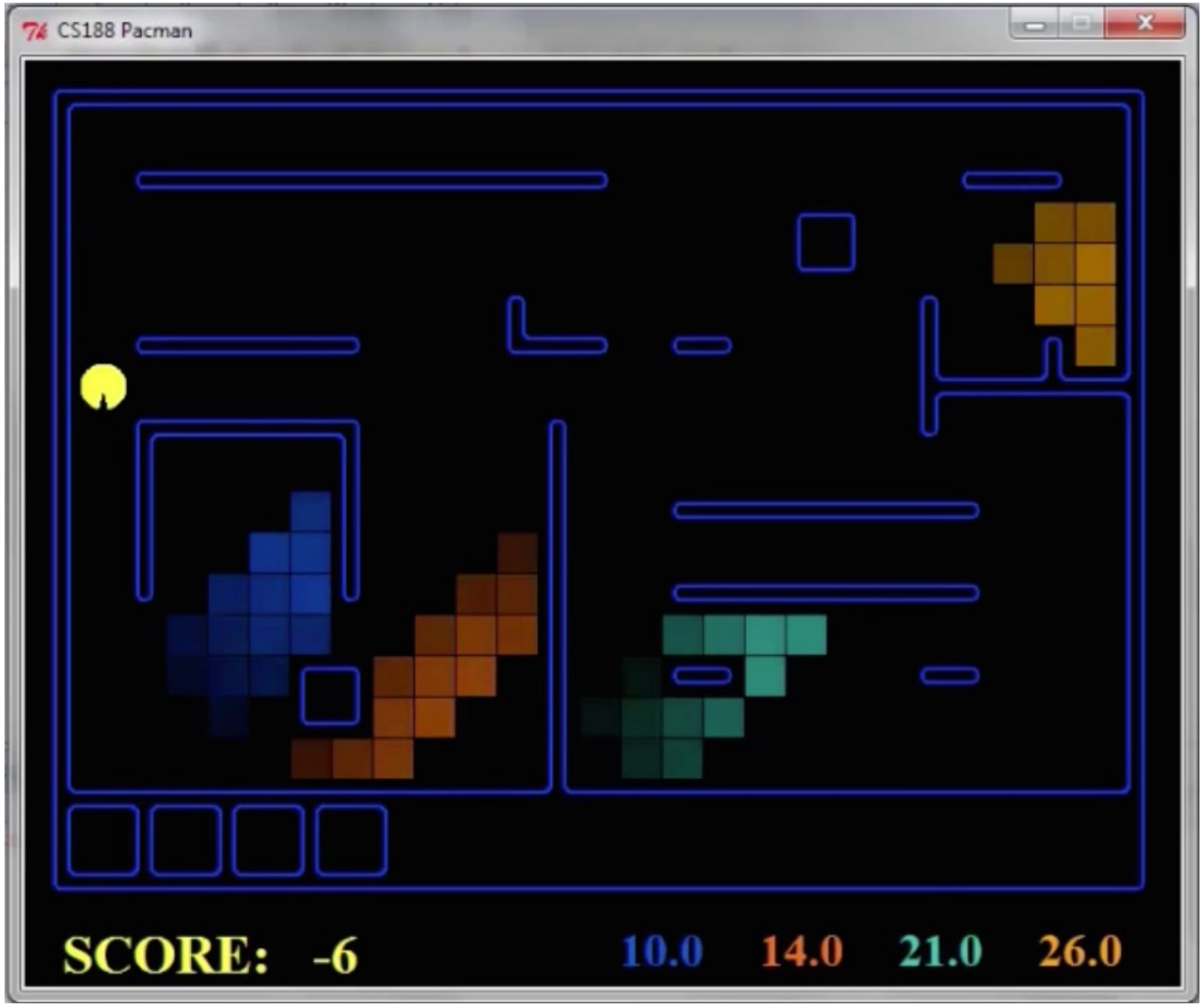


P(W_0)	
sun	rain
0.5	0.5

W_{t-1}	P(W_t|W_{t-1})	
	sun	rain
sun	0.9	0.1
rain	0.3	0.7

W_t	P(U_t|W_t)	
	true	false
sun	0.2	0.8
rain	0.9	0.1

Pacman – Hunting Invisible Ghosts with Sonar



[Demo: Pacman – Sonar – No Beliefs(L14D1)]

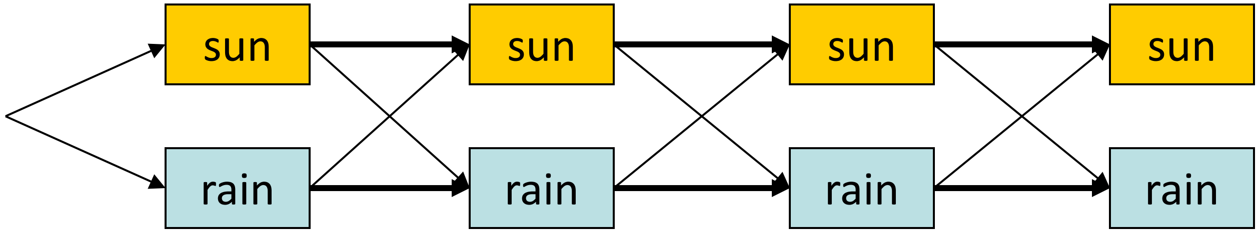
Most Likely Explanation

Inference tasks

- ***Filtering***: $P(X_t | e_{1:t})$
  - ***belief state***—input to the decision process of a rational agent
- ***Prediction***: $P(X_{t+k} | e_{1:t})$ for $k > 0$
  - evaluation of possible action sequences; like filtering without the evidence
- ***Smoothing***: $P(X_k | e_{1:t})$ for $0 \leq k < t$
  - better estimate of past states, essential for learning
- ***Most likely explanation***: $\arg\max_{x_{1:t}} P(x_{1:t} \mid e_{1:t})$
  - speech recognition, decoding with a noisy channel

Most likely explanation = most probable path

- ***State trellis***: graph of states and transitions over time

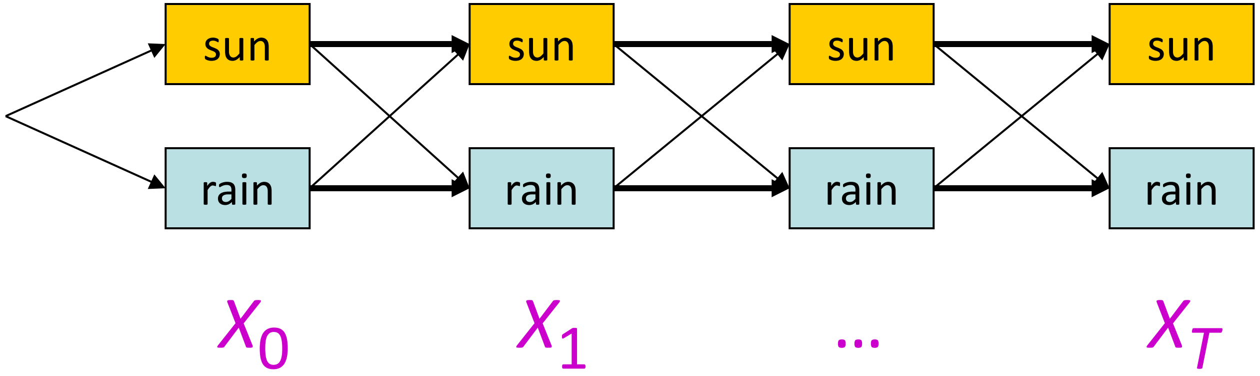


X_0 X_1 … X_T

- $\arg\max_{x_{1:t}} P(x_{1:t} \mid e_{1:t})$
  $= \arg\max_{x_{1:t}} \alpha\, P(x_{1:t}, e_{1:t})$
  $= \arg\max_{x_{1:t}} P(x_{1:t}, e_{1:t})$
  $= \arg\max_{x_{1:t}} P(x_0) \prod_t P(x_t \mid x_{t-1})\, P(e_t \mid x_t)$

- Each arc represents some transition $x_{t-1} \rightarrow x_t$
- Each arc has weight $P(x_t \mid x_{t-1})\, P(e_t \mid x_t)$ (arcs to initial states have weight $P(x_0)$)
- The ***product*** of weights on a path is proportional to that state sequence's probability
- Forward algorithm computes sums of paths, ***Viterbi algorithm*** computes best paths

Forward / Viterbi algorithms



Forward Algorithm (sum)

For each state at time t, keep track of the ***total probability of all paths*** to it

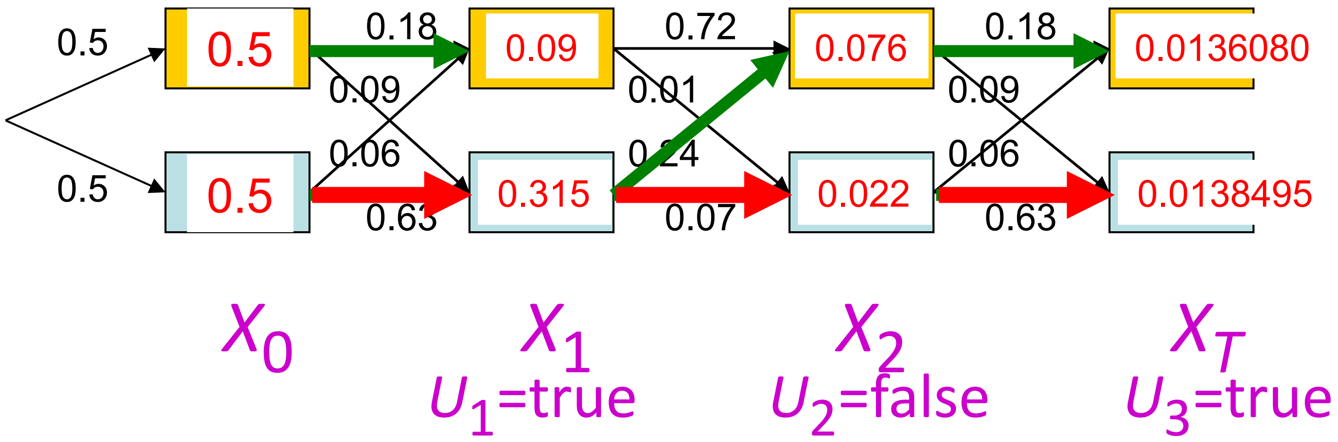
$$\boldsymbol{f}_{1:t+1} = \text{FORWARD}(\boldsymbol{f}_{1:t}, e_{t+1})$$
$$= \alpha \, P(e_{t+1}|X_{t+1}) \sum_{x_t} P(X_{t+1}|\, x_t) \, \boldsymbol{f}_{1:t}$$

Viterbi Algorithm (max)

For each state at time t, keep track of the ***maximum probability of any path*** to it

$$\boldsymbol{m}_{1:t+1} = \text{VITERBI}(\boldsymbol{m}_{1:t}, e_{t+1})$$
$$= P(e_{t+1}|X_{t+1}) \max_{x_t} P(X_{t+1}|\, x_t) \, \boldsymbol{m}_{1:t}$$

Viterbi algorithm contd.



W_{t-1}	$P(W_t \mid W_{t-1})$	
	sun	rain
sun	0.9	0.1
rain	0.3	0.7

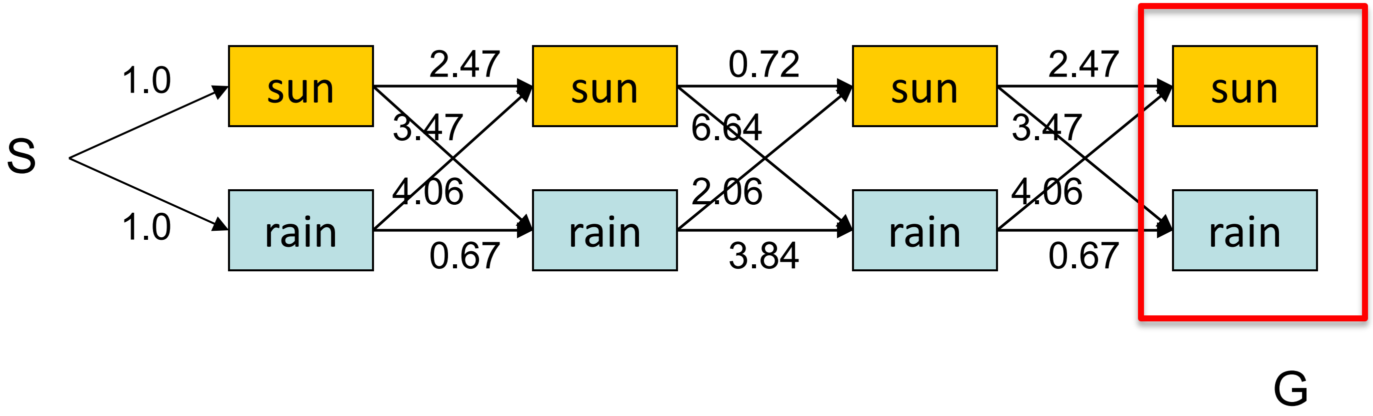
W_t	$P(U_t \mid W_t)$	
	true	false
sun	0.2	0.8
rain	0.9	0.1

Time complexity?
$O(|X|^2 T)$

Space complexity?
$O(|X| T)$

Number of paths?
$O(|X|^T)$

Viterbi in negative log space



W_{t-1}	$P(W_t \mid W_{t-1})$	
	sun	rain
sun	0.9	0.1
rain	0.3	0.7

W_t	$P(U_t \mid W_t)$	
	true	false
sun	0.2	0.8
rain	0.9	0.1

argmax of product of probabilities
= argmin of sum of negative log probabilities
= minimum-cost path

Viterbi is essentially breadth-first graph search
What about A*?