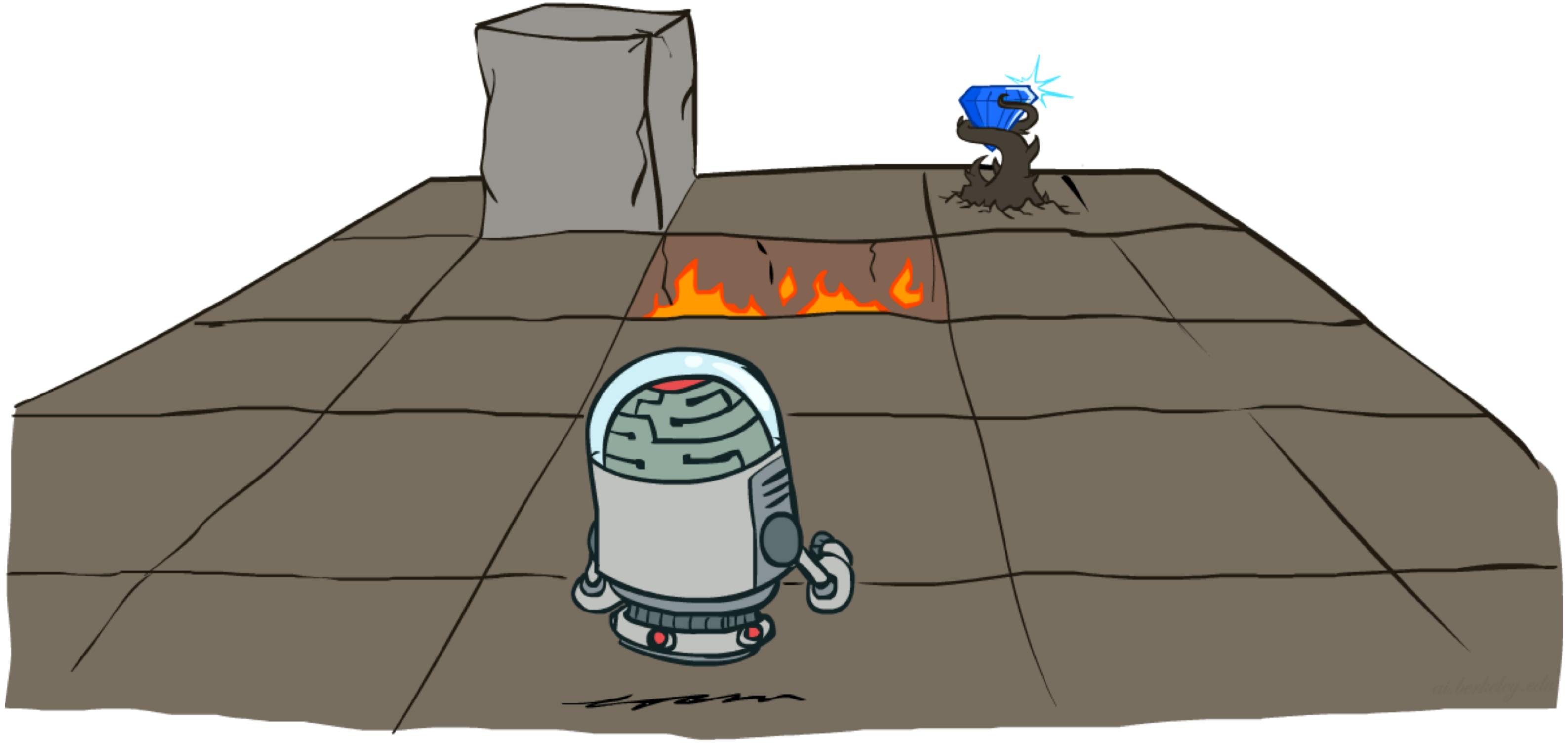
# CS 188: Artificial Intelligence

## Markov Decision Processes II



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[These slides adapted from Dan Klein and Pieter Abbeel]

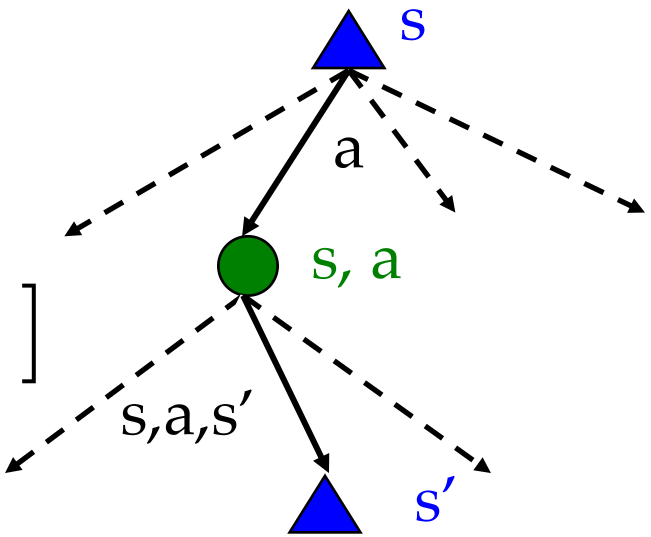
# Values of States

- Recursive definition of value:

$$V^*(s) = \max_a Q^*(s,a)$$

$$Q^*(s,a) = \sum_{s'} T(s,a,s')\left[R(s,a,s') + \gamma V^*(s')\right]$$

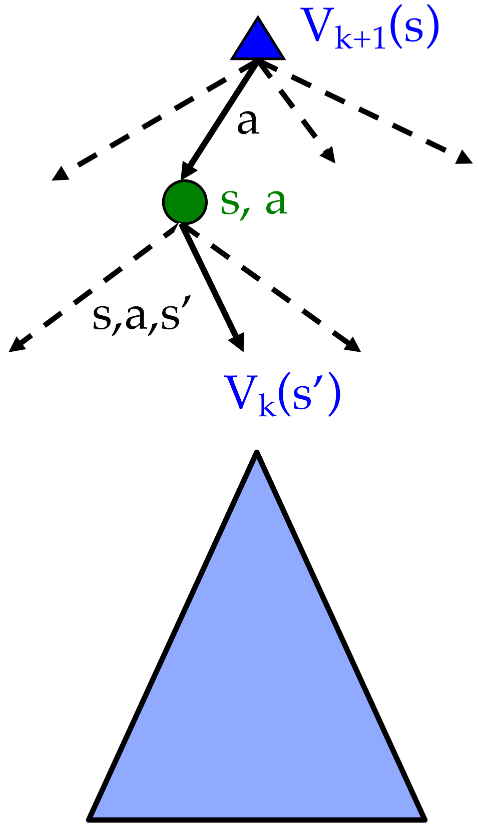
$$V^*(s) = \max_a \sum_{s'} T(s,a,s')\left[R(s,a,s') + \gamma V^*(s')\right]$$



# Value Iteration

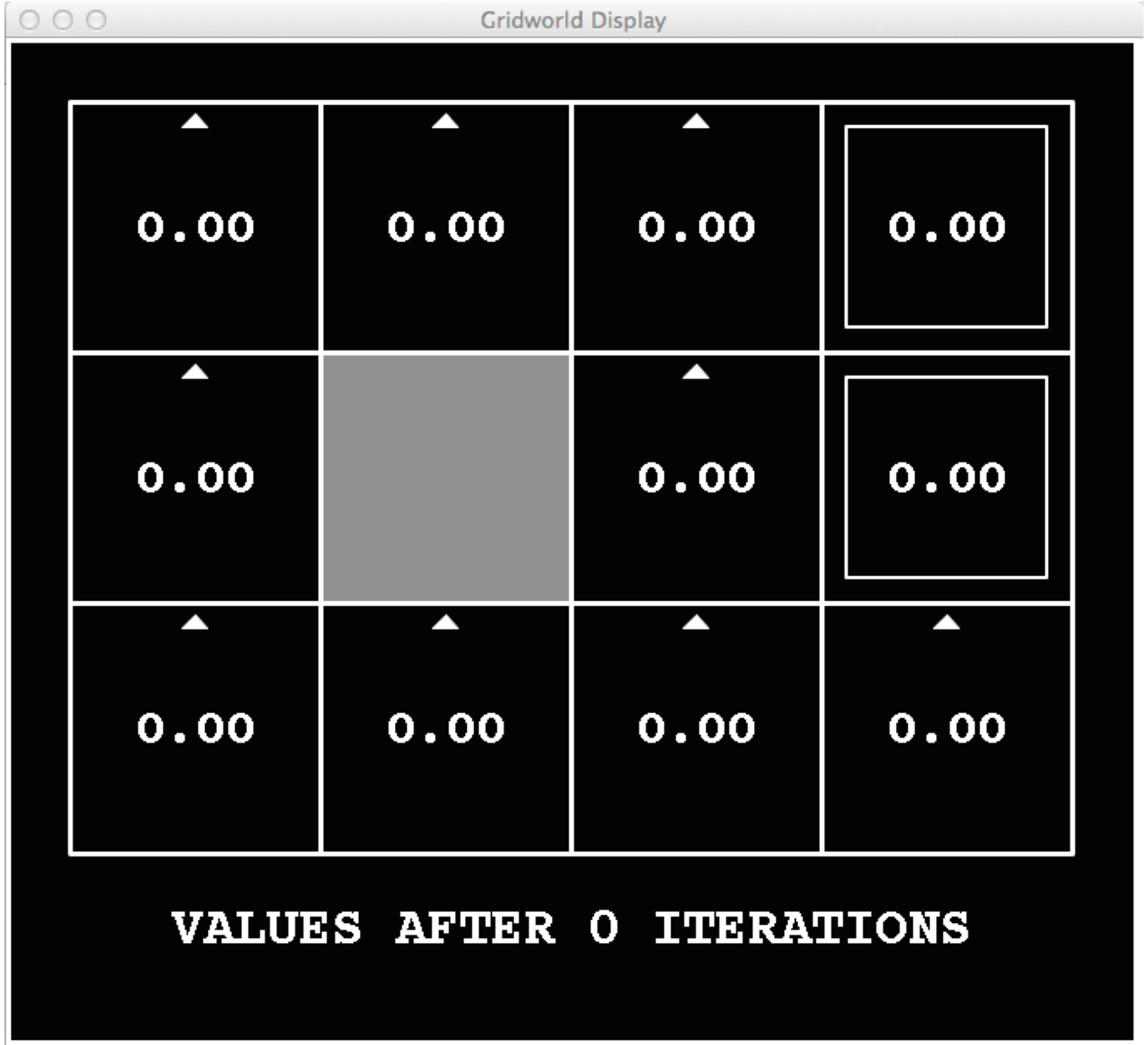
- Start with $V_0(s) = 0$: no time steps left means an expected reward sum of zero
- Given vector of $V_k(s)$ values, do one ply of expectimax from each state:

$$V_{k+1}(s) \leftarrow \max_a \sum_{s'} T(s,a,s') \left[ R(s,a,s') + \gamma V_k(s') \right]$$



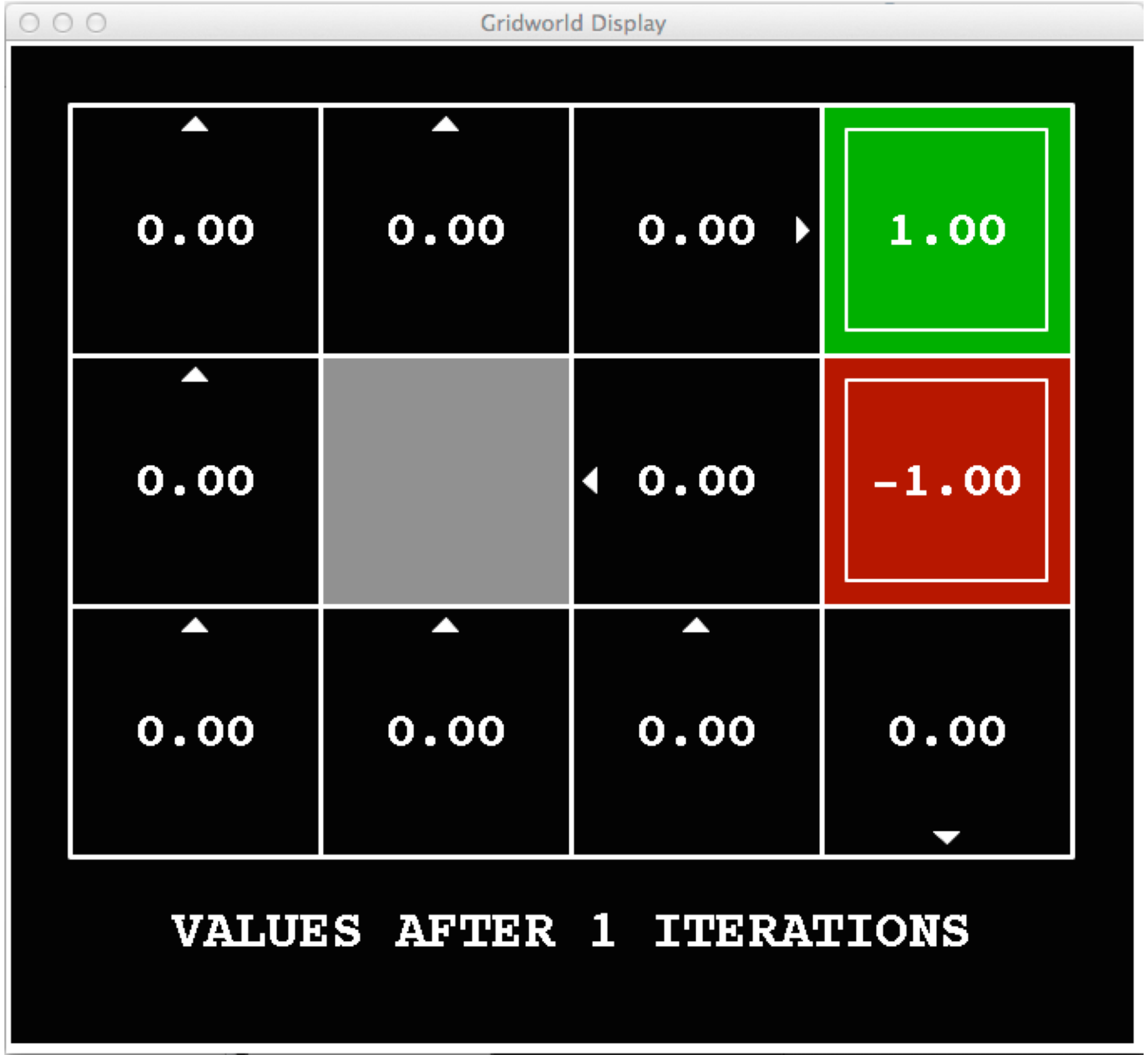

- Repeat until convergence, which yields V*
- Complexity of each iteration: $O(S^2A)$
- Theorem: will converge to unique optimal values
  - Basic idea: approximations get refined towards optimal values
  - Policy may converge long before values do

# k=0




Noise = 0.2
Discount = 0.9
Living reward = 0

# k=1




Noise = 0.2
Discount = 0.9
Living reward = 0

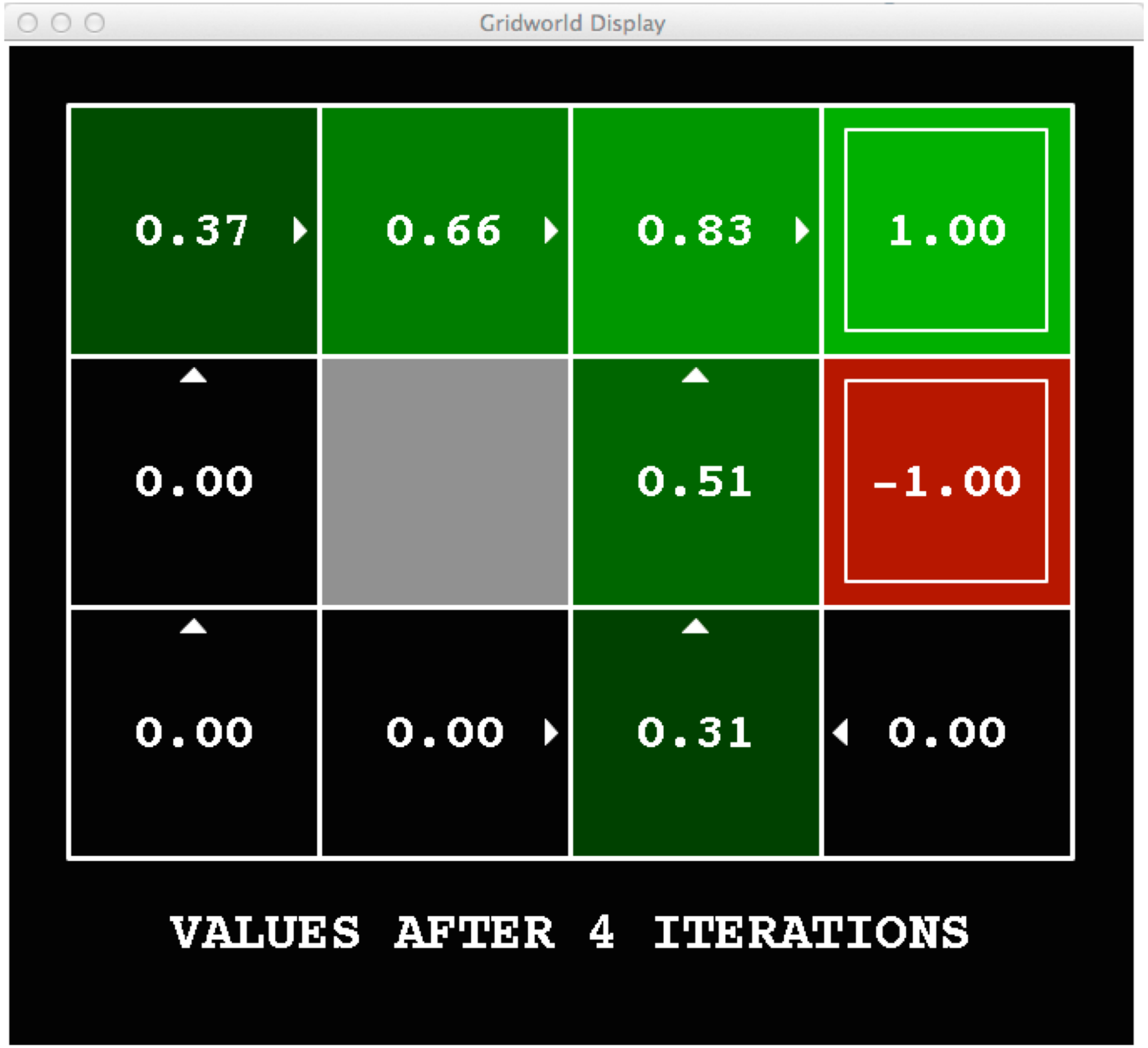
# k=2


Noise = 0.2
Discount = 0.9
Living reward = 0

# k=3

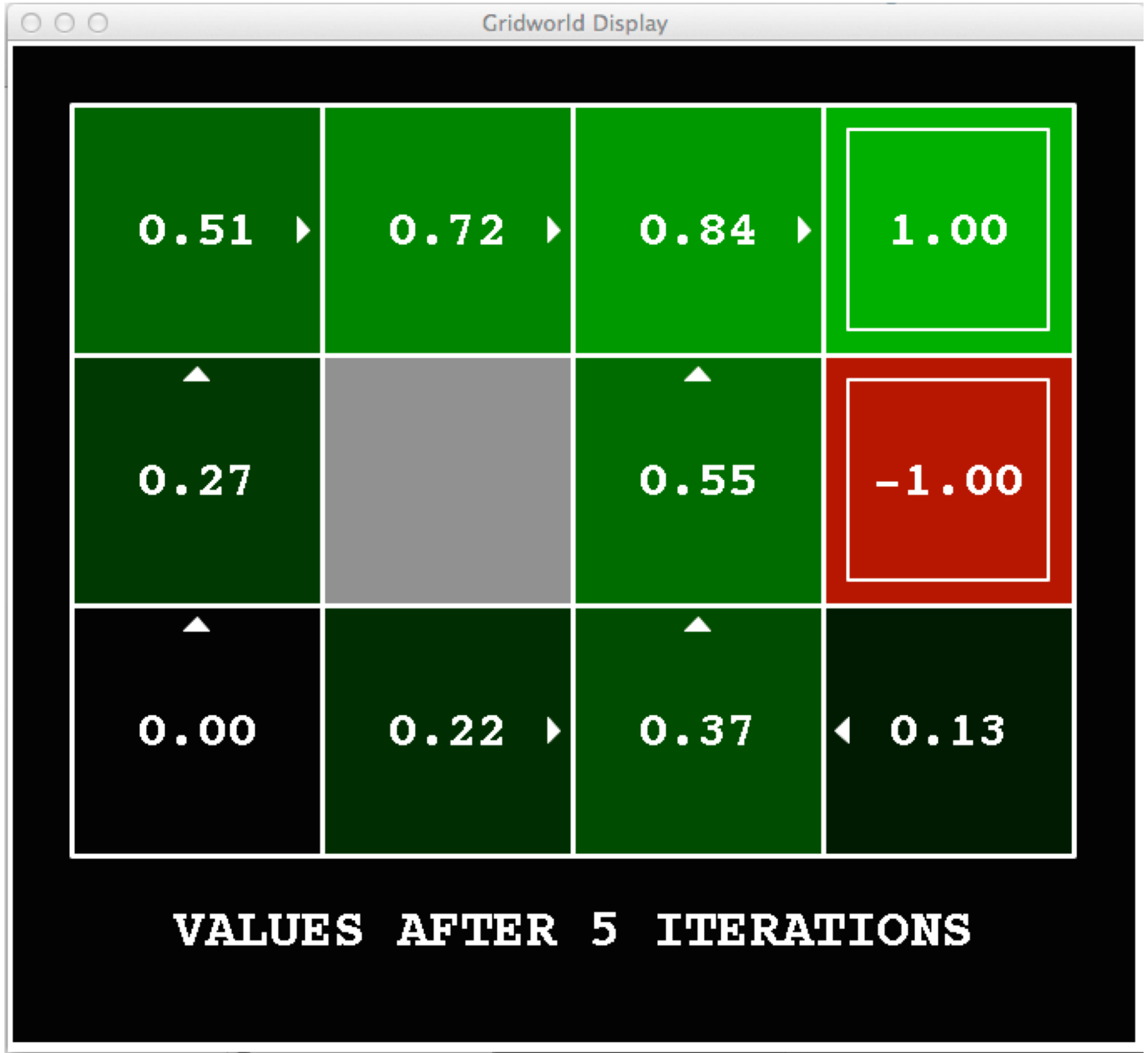

Noise = 0.2
Discount = 0.9
Living reward = 0

# k=4



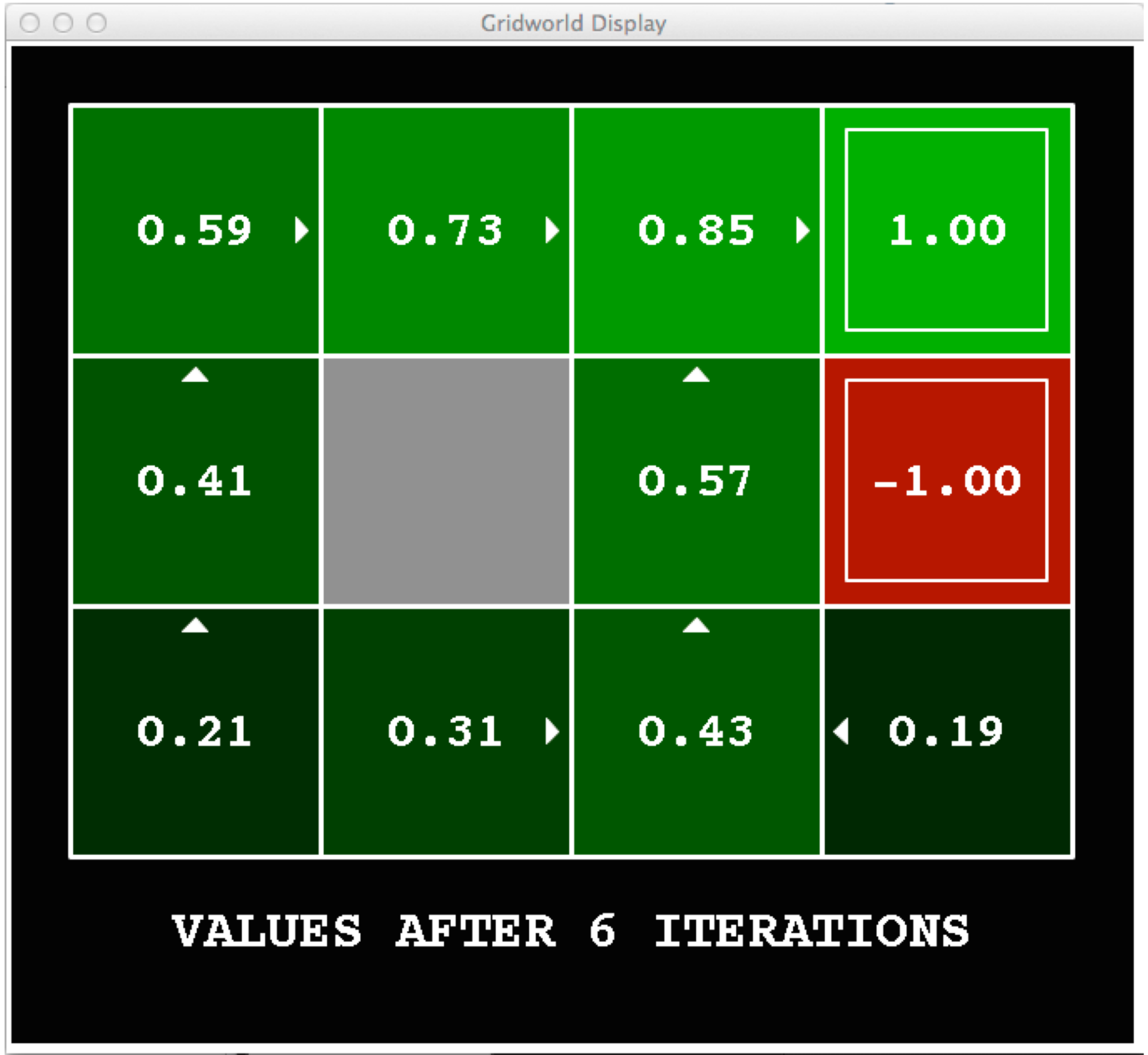

Noise = 0.2
Discount = 0.9
Living reward = 0

# k=5



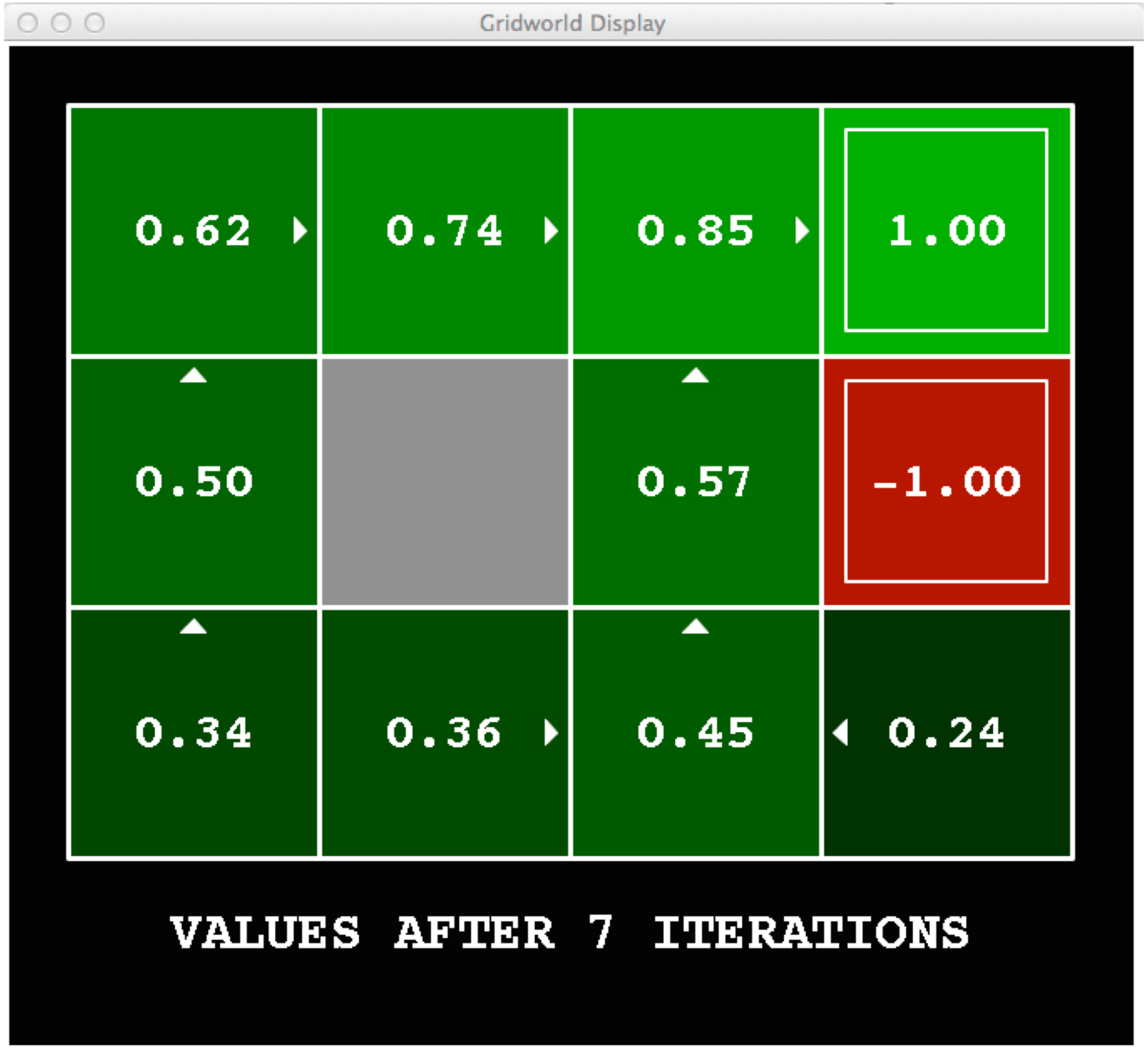

Noise = 0.2
Discount = 0.9
Living reward = 0

# k=6



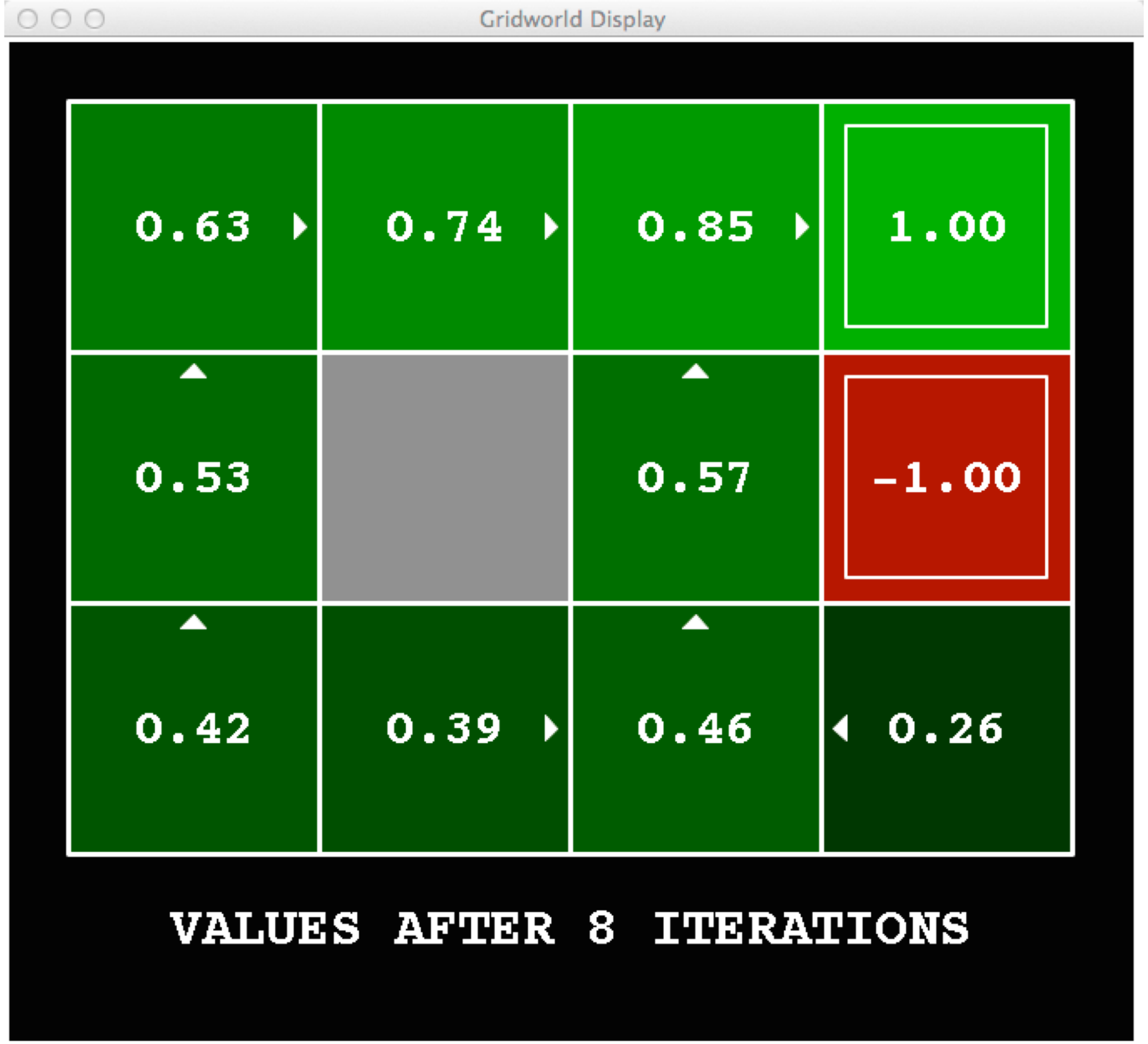

Noise = 0.2
Discount = 0.9
Living reward = 0

# k=7



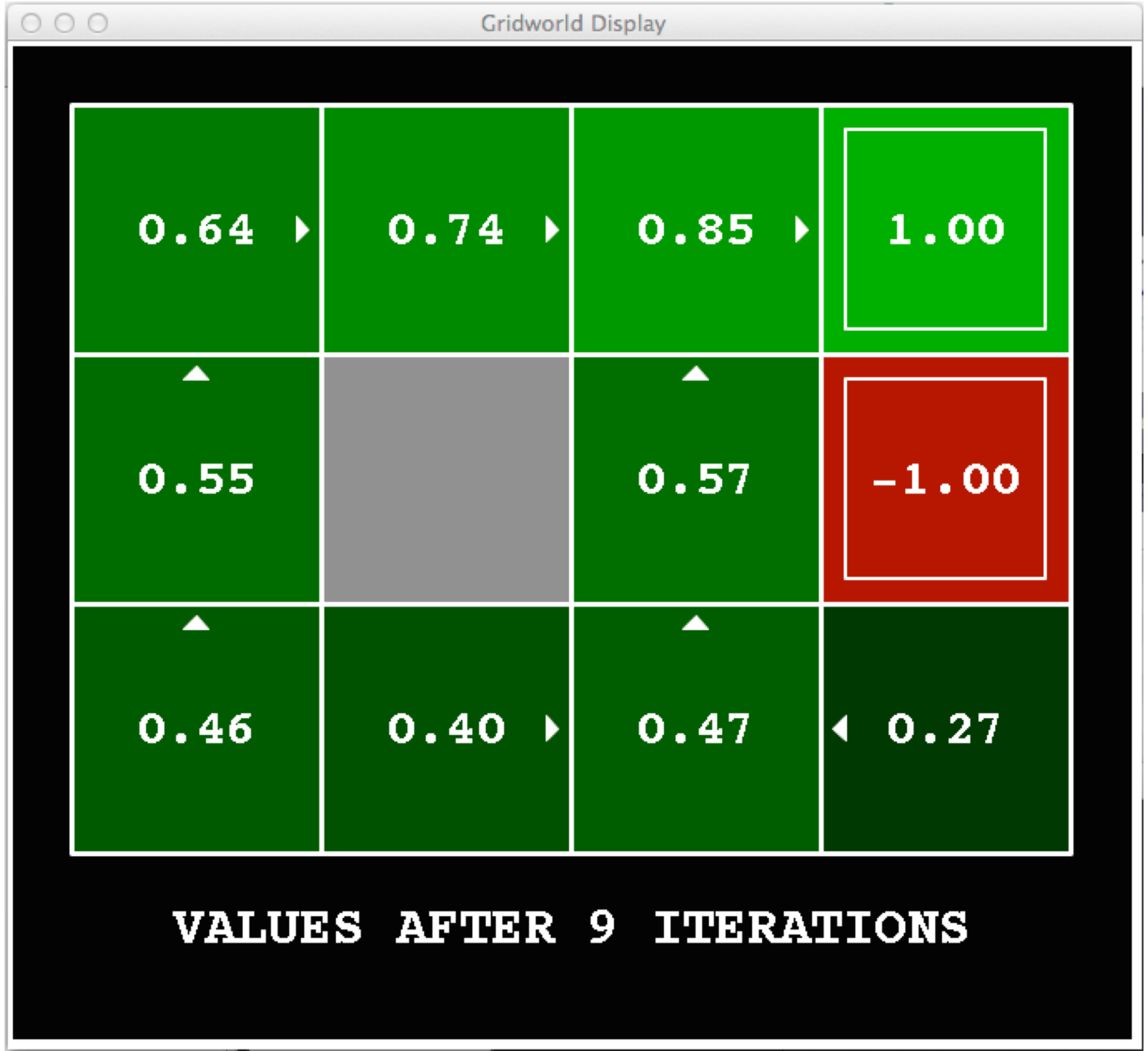

Noise = 0.2
Discount = 0.9
Living reward = 0

# k=8



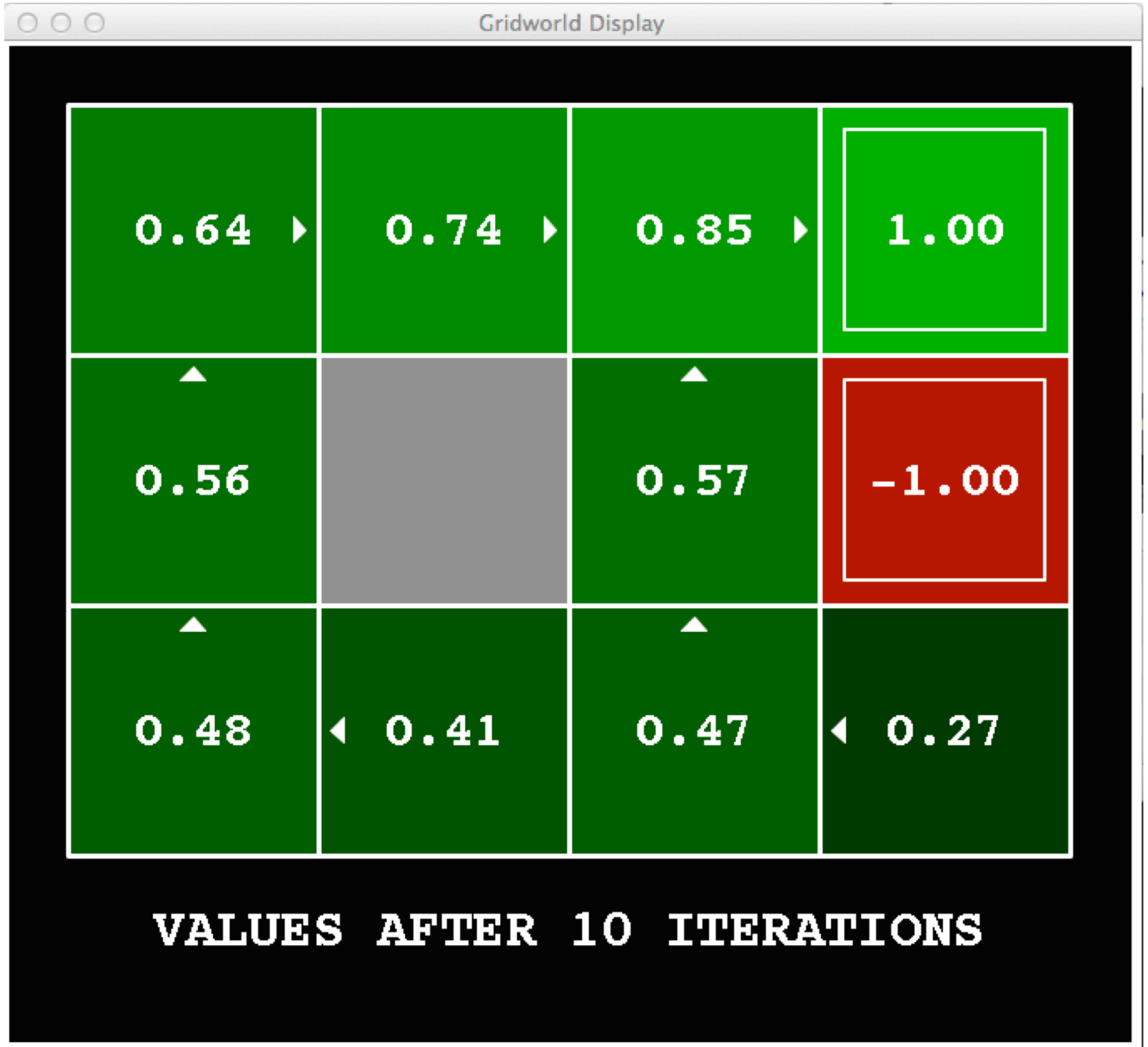

Noise = 0.2
Discount = 0.9
Living reward = 0

# k=9



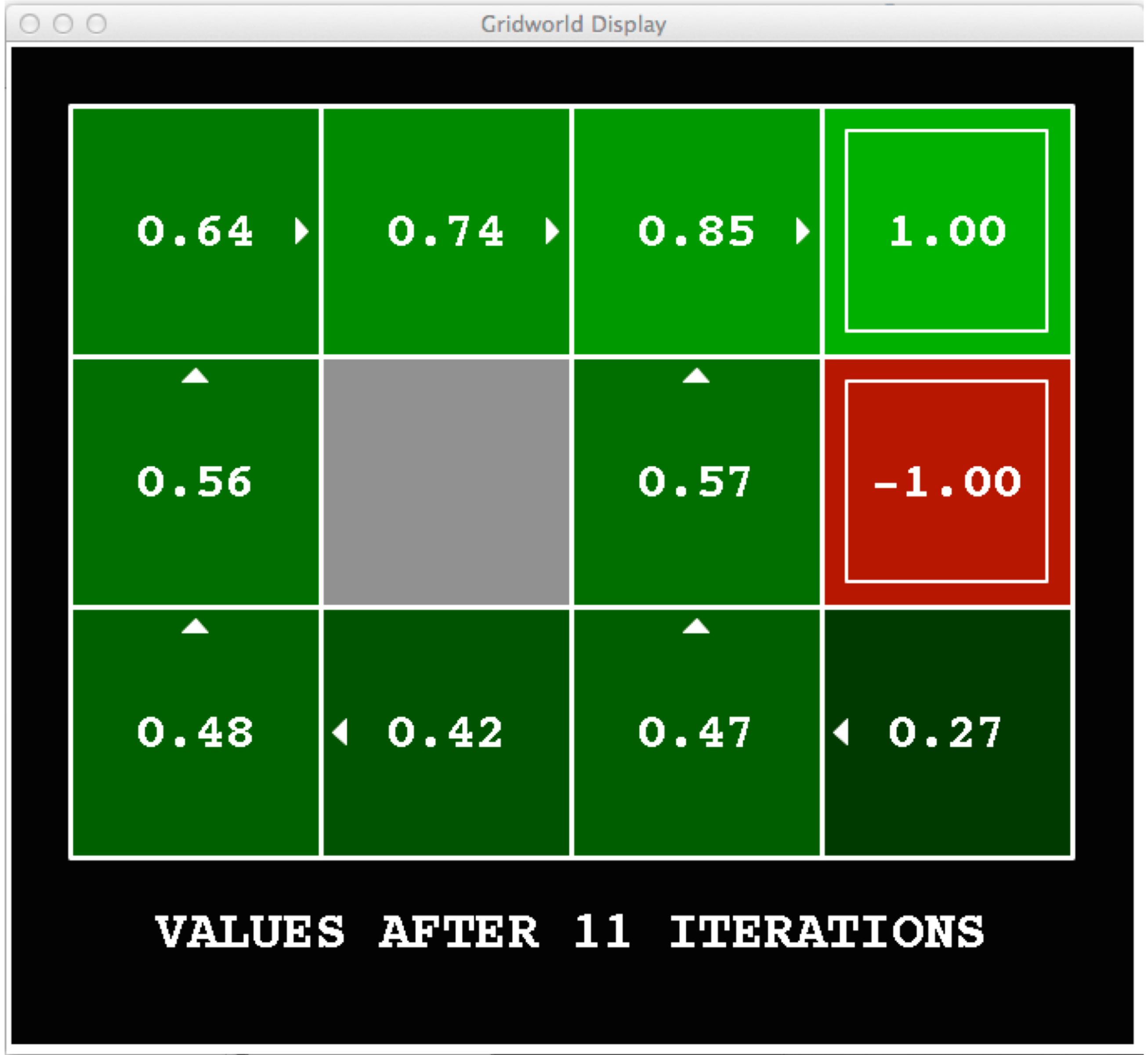

Noise = 0.2
Discount = 0.9
Living reward = 0

# k=10



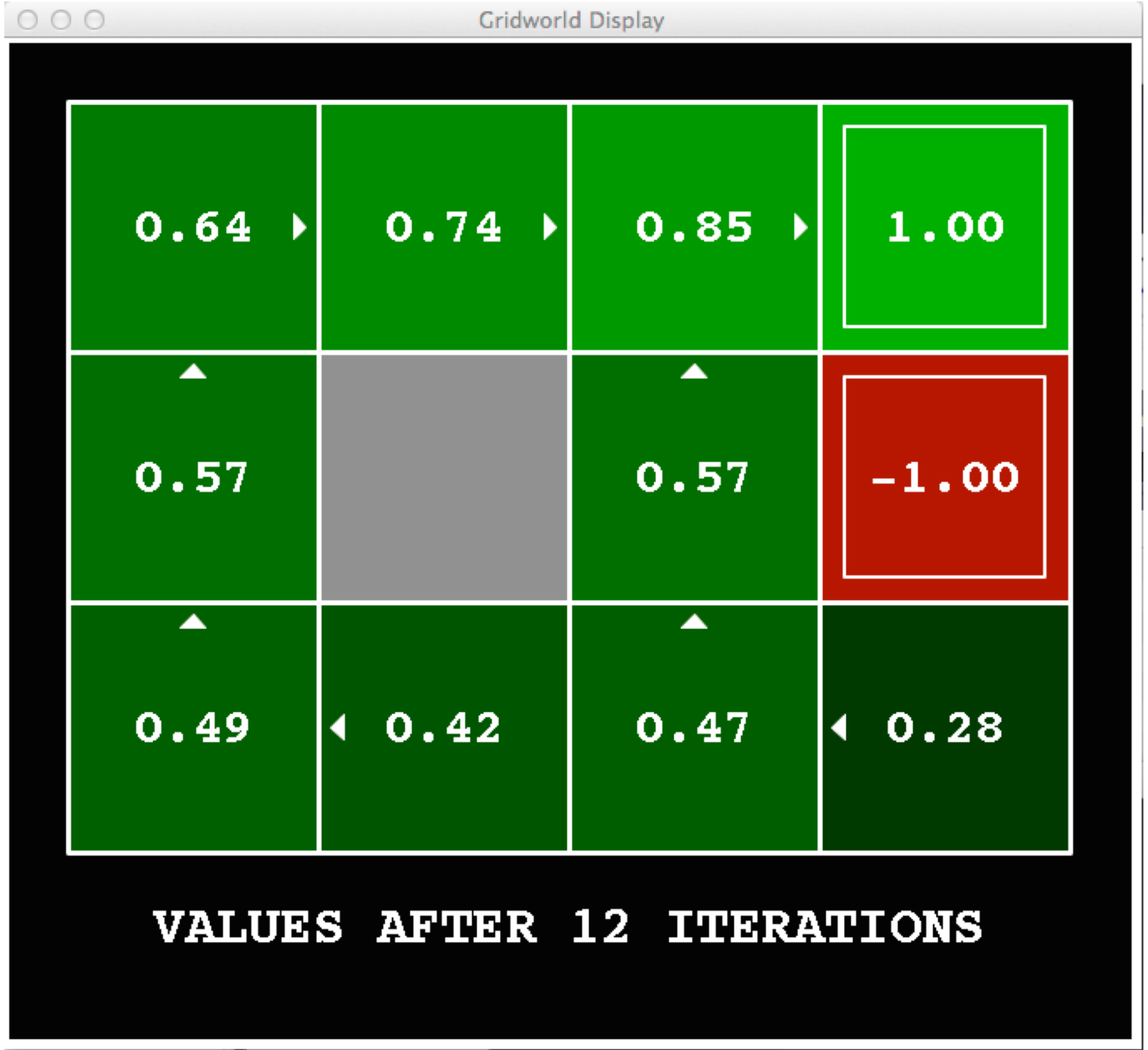

Noise = 0.2
Discount = 0.9
Living reward = 0

# k=11



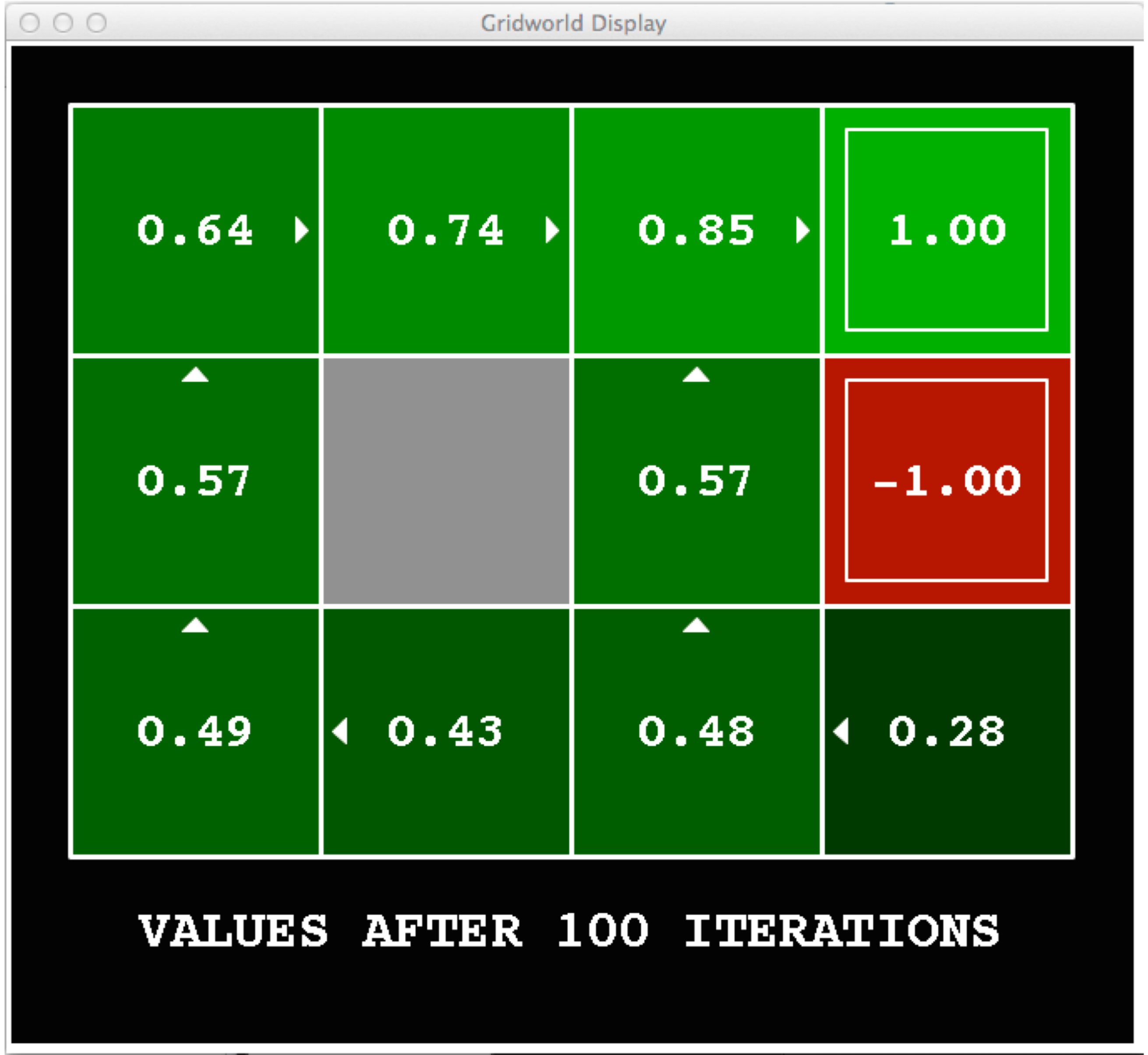

Noise = 0.2
Discount = 0.9
Living reward = 0

# k=12



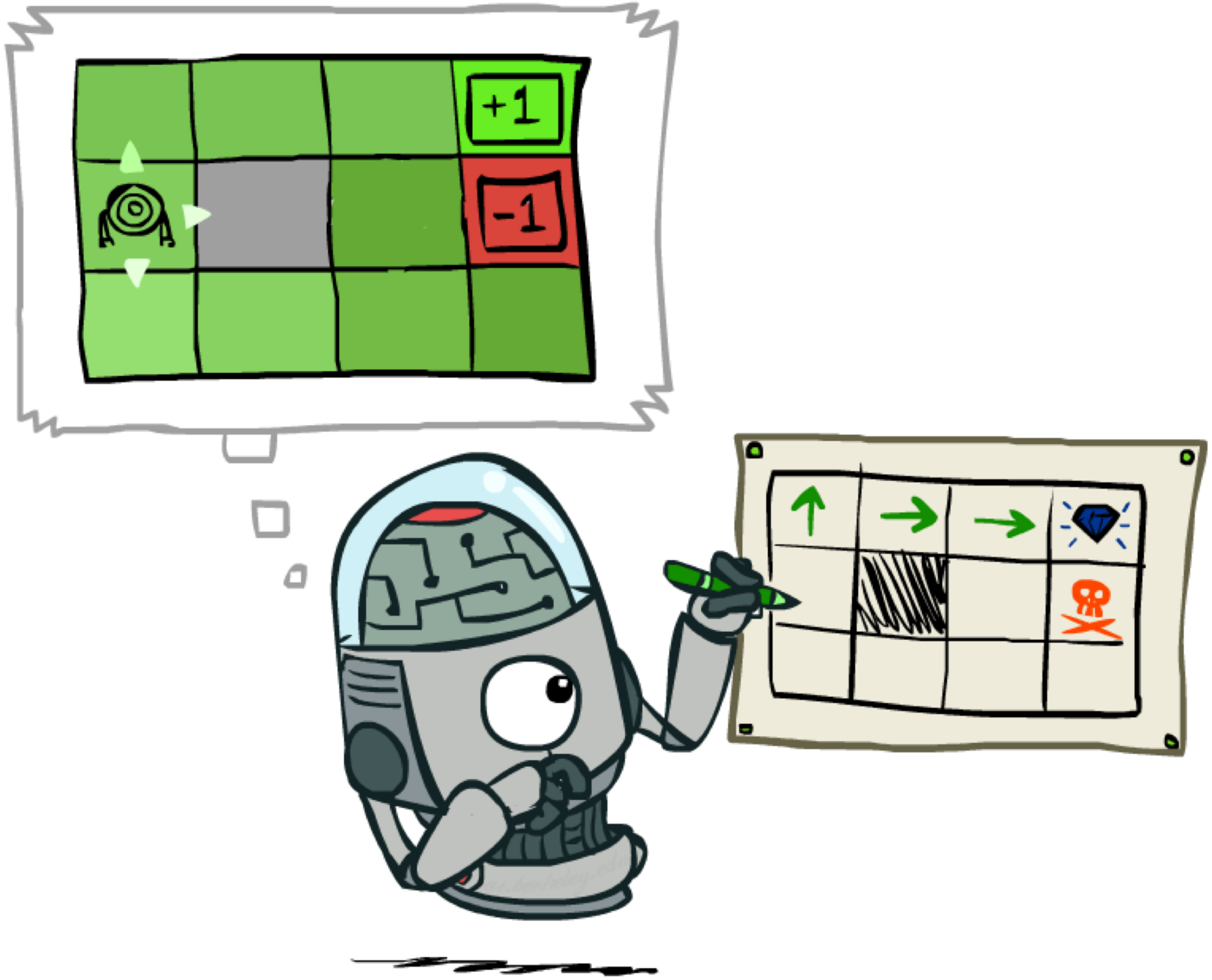

Noise = 0.2
Discount = 0.9
Living reward = 0

# k=100




Noise = 0.2
Discount = 0.9
Living reward = 0

# Policy Extraction

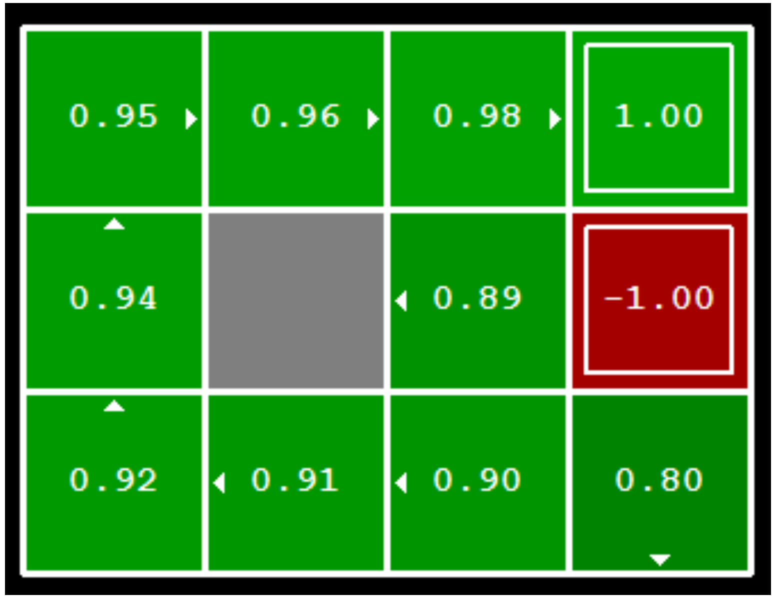


# Computing Actions from Values

- Let's imagine we have the optimal values V*(s)

- How should we act?
  - It's not obvious!

- We need to do a mini-expectimax (one step)

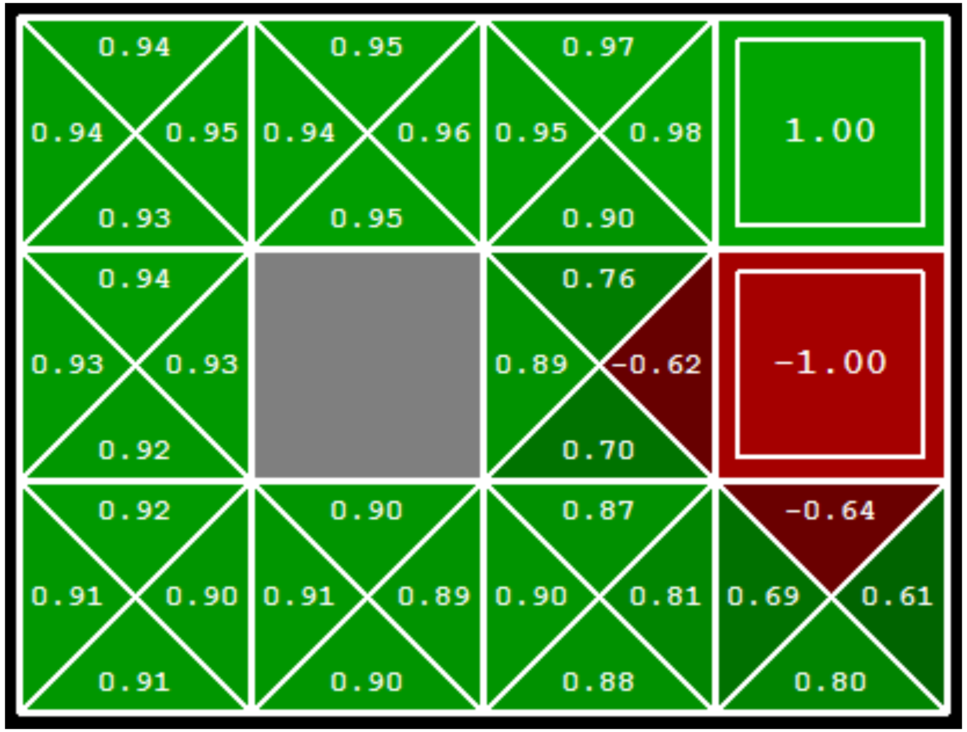



$$\pi^*(s) = \arg\max_a \sum_{s'} T(s,a,s')[R(s,a,s') + \gamma V^*(s')]$$

- This is called policy extraction, since it gets the policy implied by the values

# Computing Actions from Q-Values

- Let's imagine we have the optimal q-values:




- How should we act?
  - Completely trivial to decide!

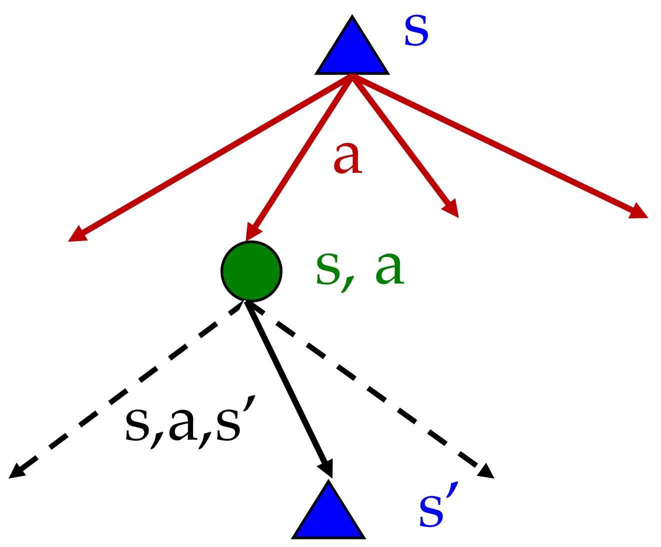
$$\pi^*(s) = \arg\max_a Q^*(s,a)$$

- Important lesson: actions are easier to select from q-values than values!

# Problems with Value Iteration

- Value iteration repeats the Bellman updates:

$$V_{k+1}(s) \leftarrow \max_a \sum_{s'} T(s,a,s') \left[ R(s,a,s') + \gamma V_k(s') \right]$$

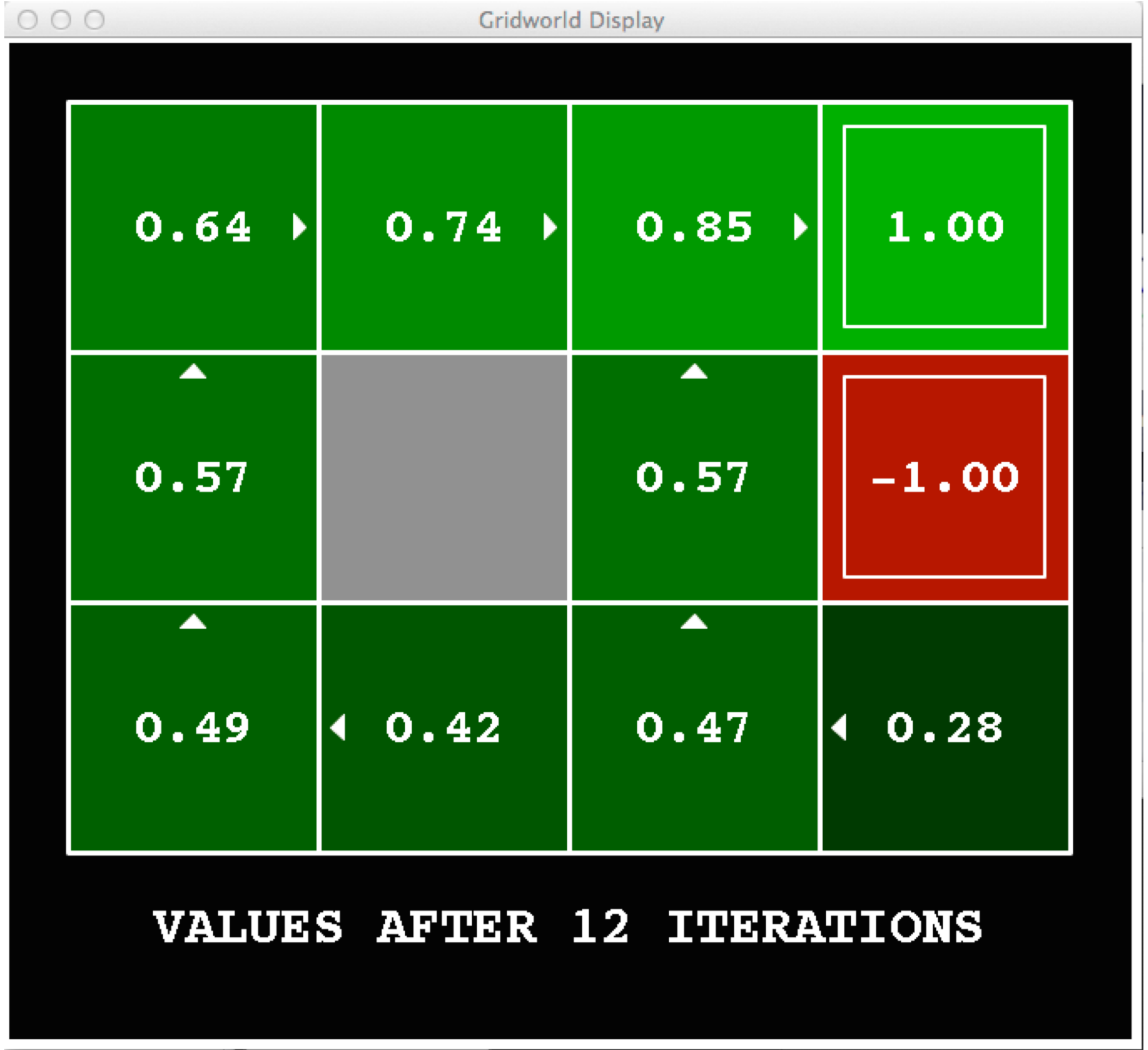



- Problem 1: It's slow – $O(S^2A)$ per iteration

- Problem 2: The "max" at each state rarely changes

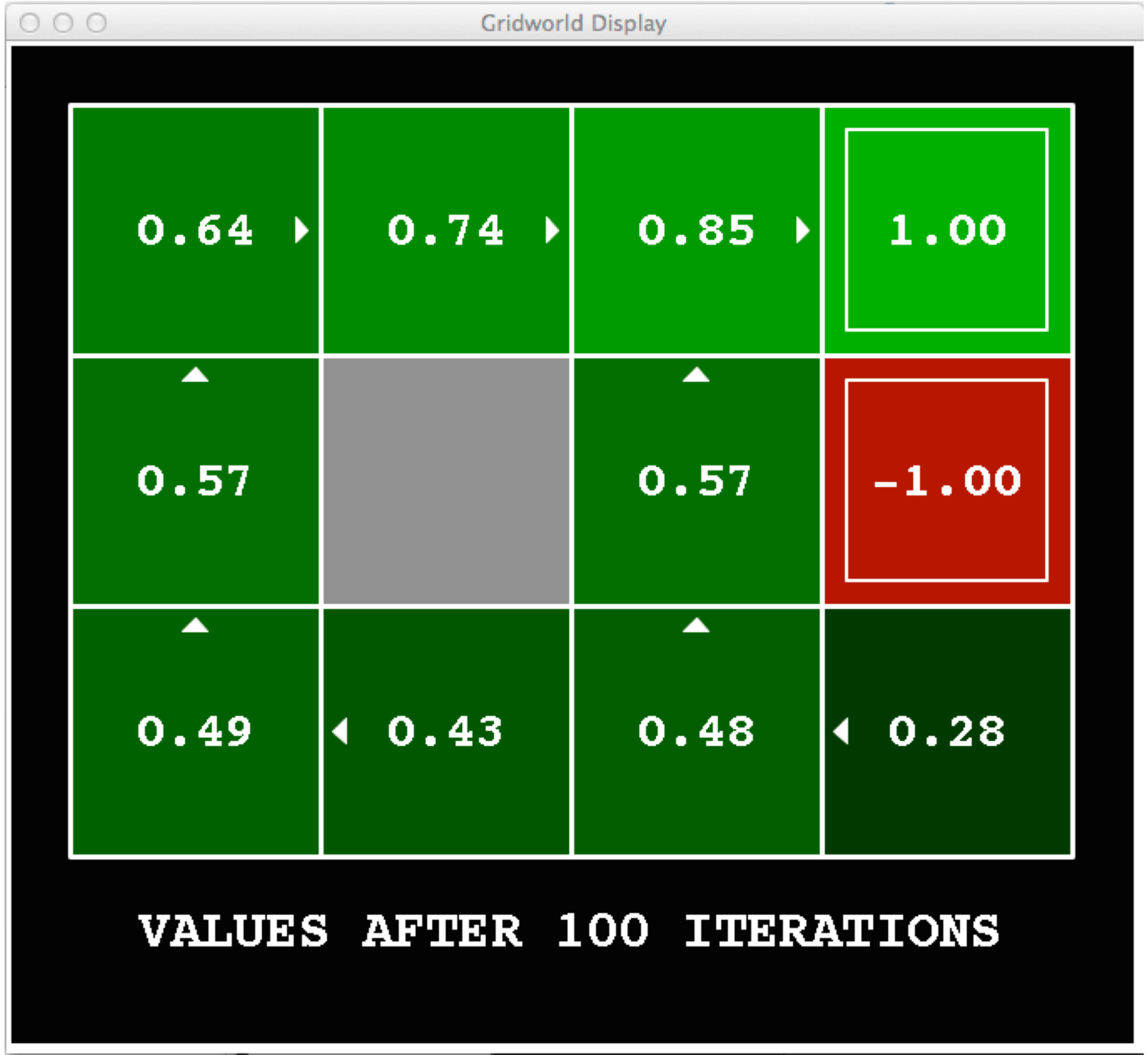
- Problem 3: The policy often converges long before the values

# k=12



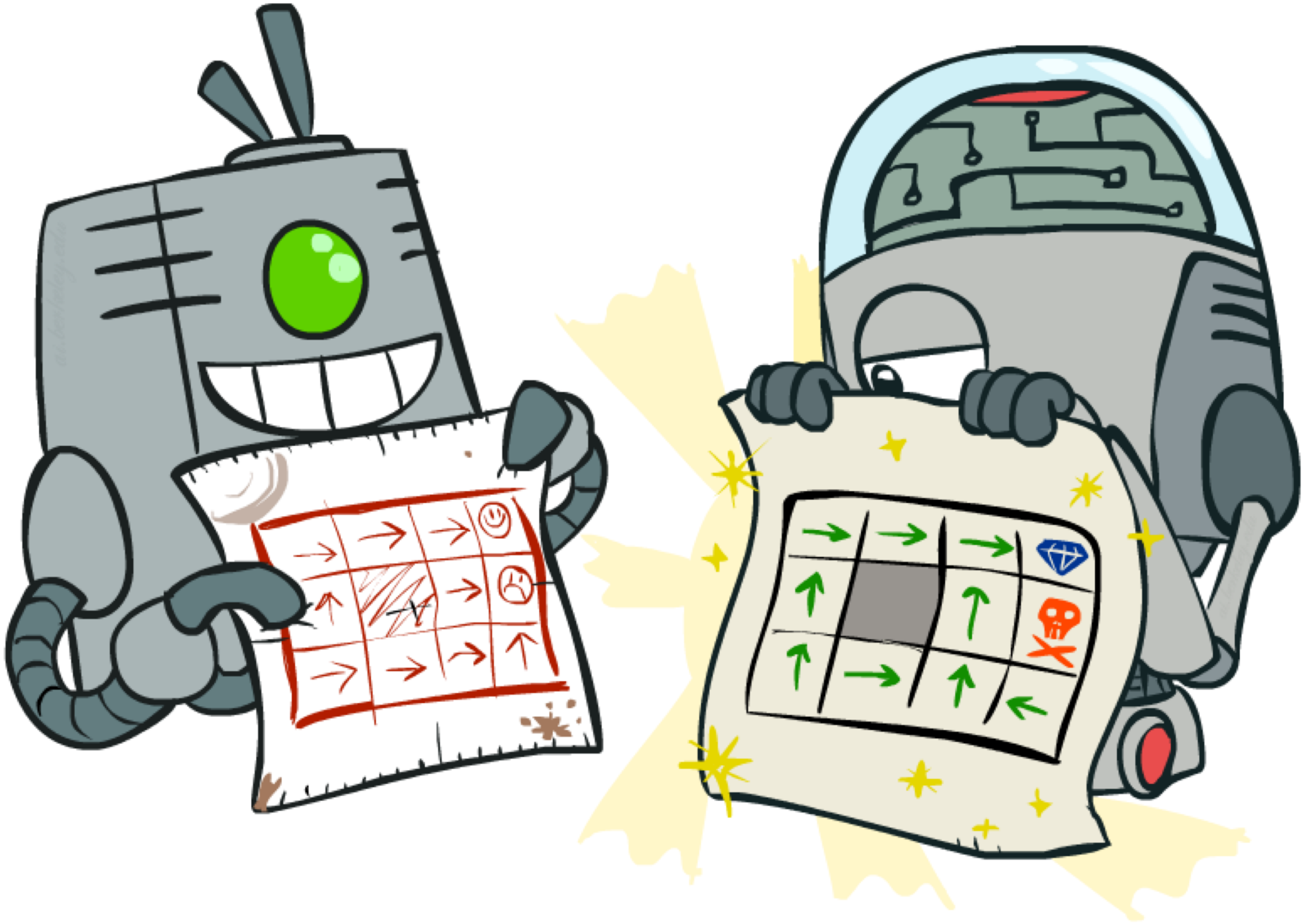

Noise = 0.2
Discount = 0.9
Living reward = 0

# k=100




Noise = 0.2
Discount = 0.9
Living reward = 0

# Policy Methods

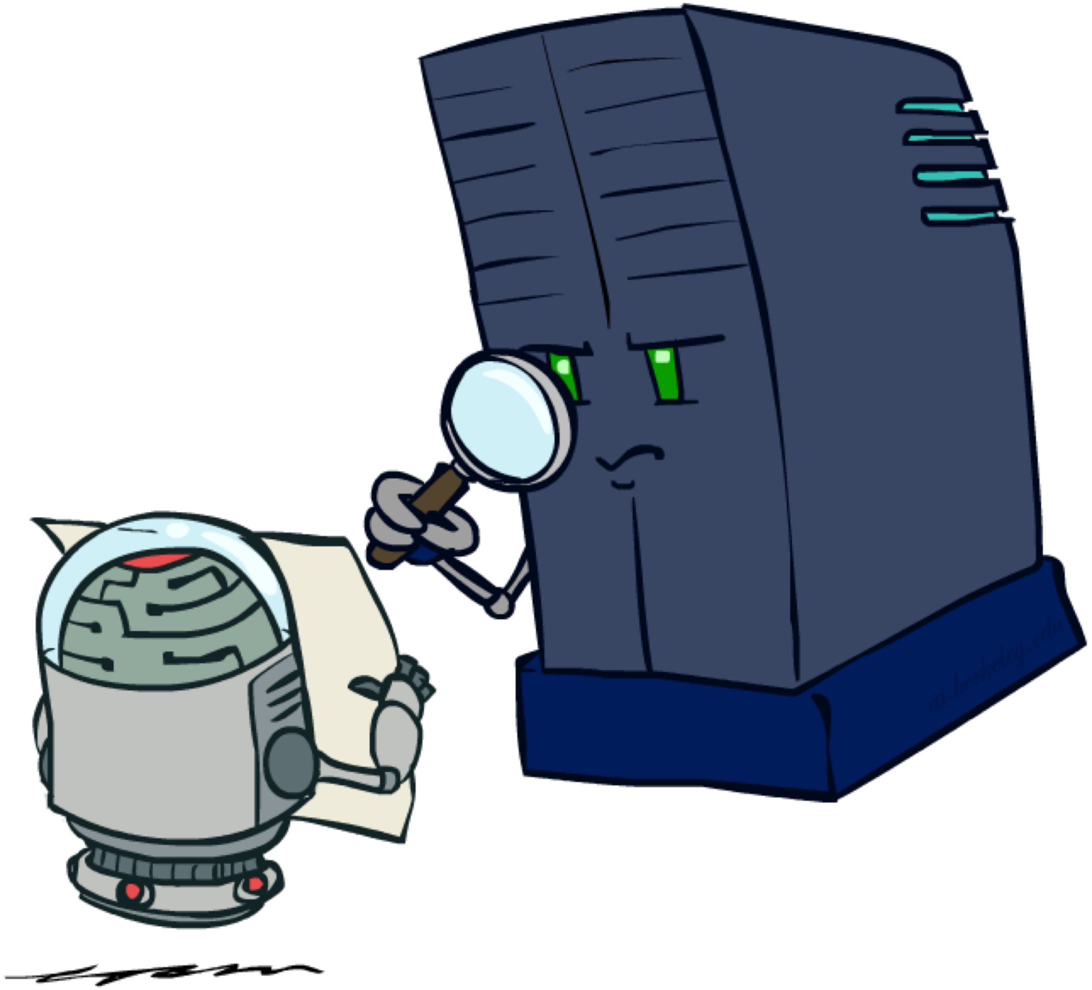


# Policy Iteration

- Alternative approach for optimal values:
  - Step 1: Policy Evaluation: calculate utilities for some fixed policy (not optimal utilities!) until convergence
  - Step 2: Policy Improvement: update policy using one-step look-ahead with resulting converged (but not optimal!) utilities as future values
  - Repeat steps until policy converges

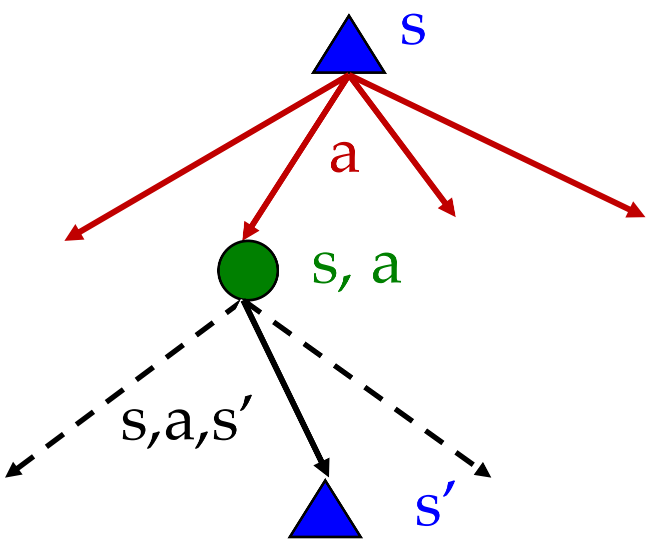
- This is Policy Iteration
  - It's still optimal!
  - Can converge (much) faster under some conditions

# Policy Evaluation

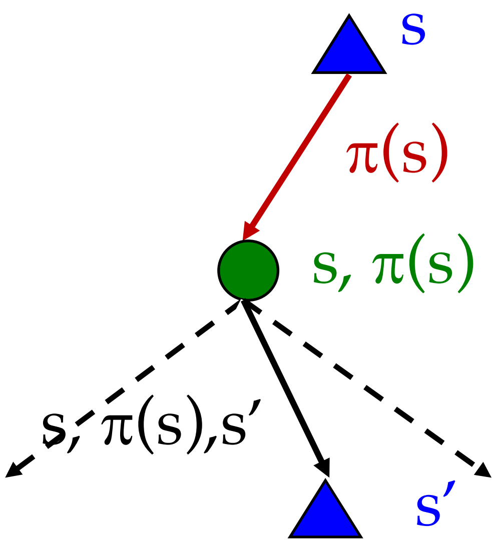


# Fixed Policies

Do the optimal action




Do what $\pi$ says to do




- Expectimax trees max over all actions to compute the optimal values
- If we fixed some policy $\pi(s)$, then the tree would be simpler – only one action per state
  - … though the tree's value would depend on which policy we fixed

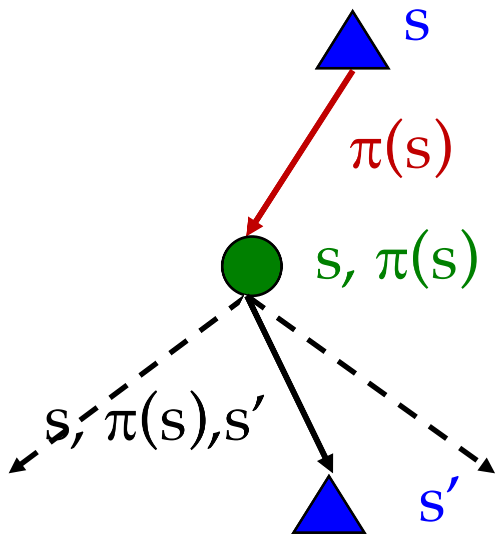
# Utilities for a Fixed Policy

- Another basic operation: compute the utility of a state s under a fixed (generally non-optimal) policy

- Define the utility of a state s, under a fixed policy $\pi$:

  $V^{\pi}(s)$ = expected total discounted rewards starting in s and following $\pi$

- Recursive relation (one-step look-ahead / Bellman equation):




$$V^{\pi}(s) = \sum_{s'} T(s, \pi(s), s')[R(s, \pi(s), s') + \gamma V^{\pi}(s')]$$

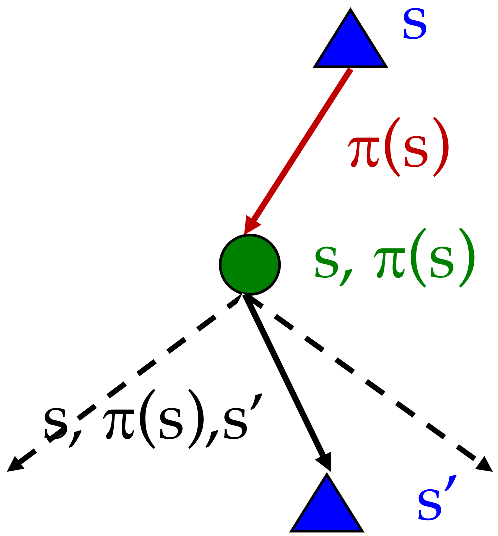
# Policy Evaluation

- How do we calculate the V's for a fixed policy π?

- Idea 1: Turn recursive Bellman equations into updates (like value iteration)

$$V_0^\pi(s) = 0$$

$$V_{k+1}^\pi(s) \leftarrow \sum_{s'} T(s, \pi(s), s')[R(s, \pi(s), s') + \gamma V_k^\pi(s')]$$

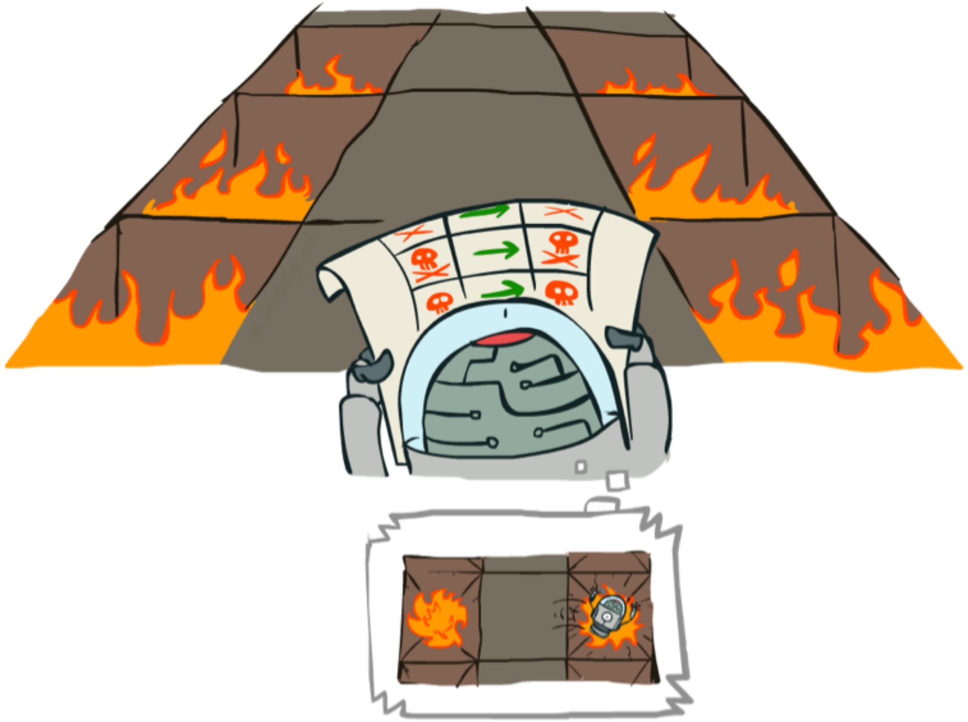



- Efficiency: O(S²) per iteration

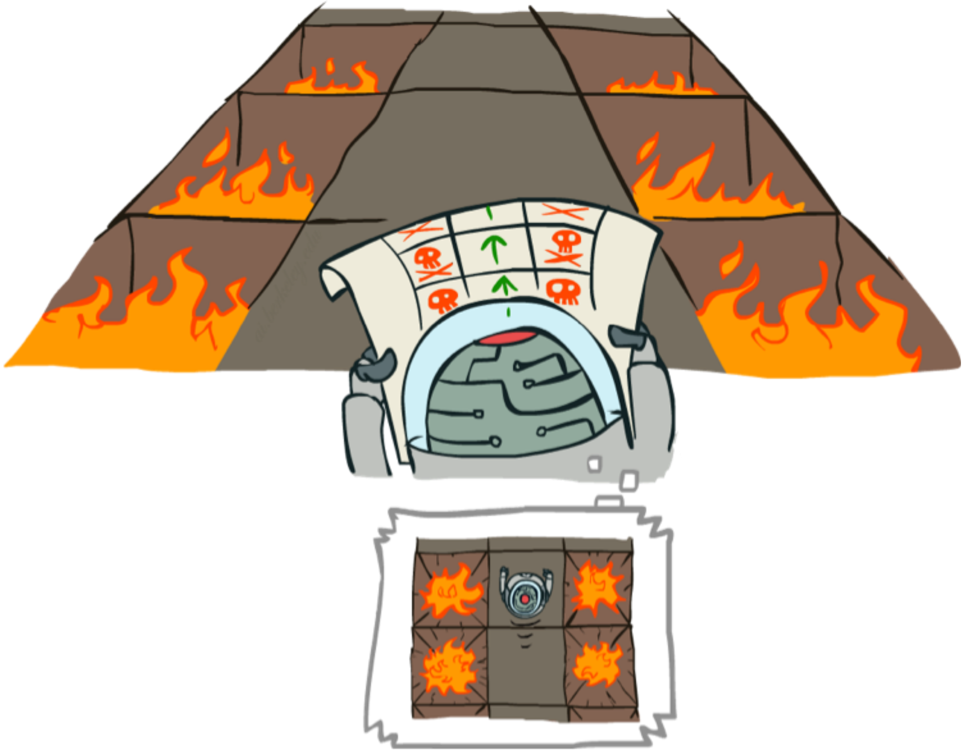
- Idea 2: Without the maxes, the Bellman equations are just a linear system
  - Solve with Matlab (or your favorite linear system solver)

# Example: Policy Evaluation

Always Go Right



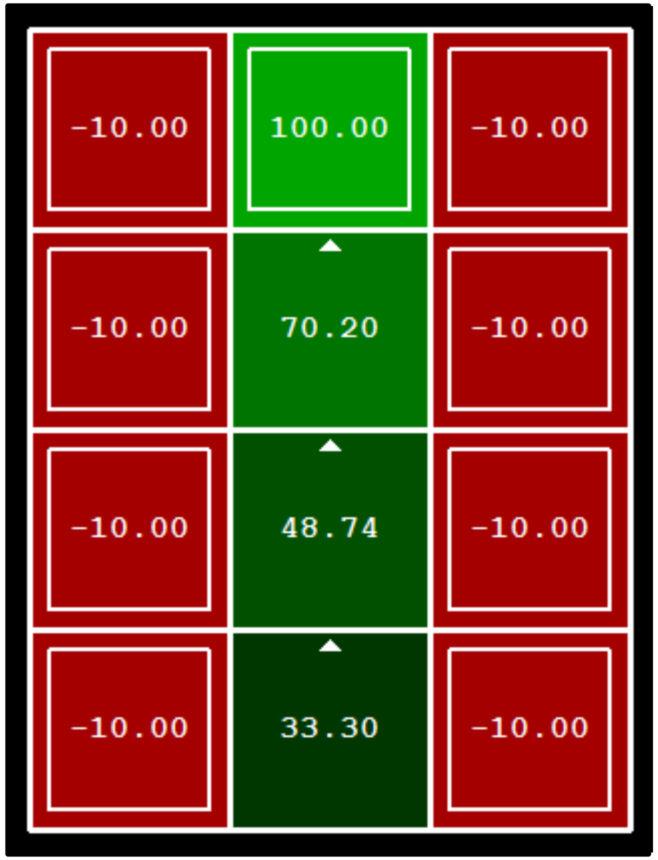
Always Go Forward



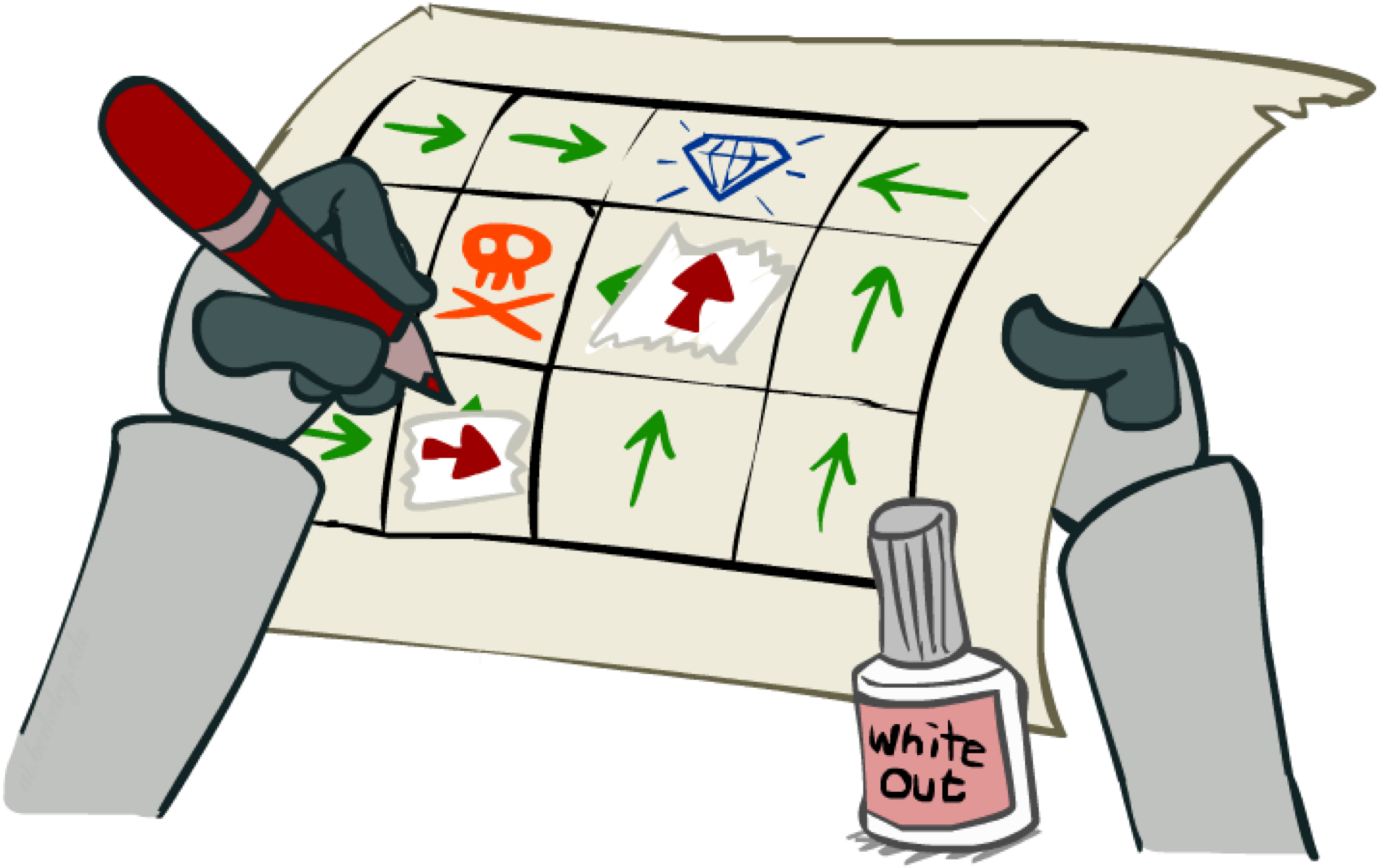
# Example: Policy Evaluation

Always Go Right


Always Go Forward



# Policy Iteration



# Policy Iteration

- Evaluation: For fixed current policy π, find values with policy evaluation:
  - Iterate until values converge:

$$V^{\pi_i}_{k+1}(s) \leftarrow \sum_{s'} T(s, \pi_i(s), s') \left[R(s, \pi_i(s), s') + \gamma V^{\pi_i}_k(s')\right]$$

- Improvement: For fixed values, get a better policy using policy extraction
  - One-step look-ahead:

$$\pi_{i+1}(s) = \arg\max_a \sum_{s'} T(s, a, s') \left[R(s, a, s') + \gamma V^{\pi_i}(s')\right]$$

# Example: Policy Iteration

$\pi_0$ | $S$ | $F$ | $\emptyset$

## Policy Evaluation:

$V^{\pi_0}(B) = 1 + 0.9 \cdot V^{\pi_0}(B) \quad \rightarrow V^{\pi_0}(B) = 10$

$V^{\pi_0}(W) = -10 + 0.9 \cdot V^{\pi_0}(O) \quad \rightarrow V^{\pi_0}(W) = -10$

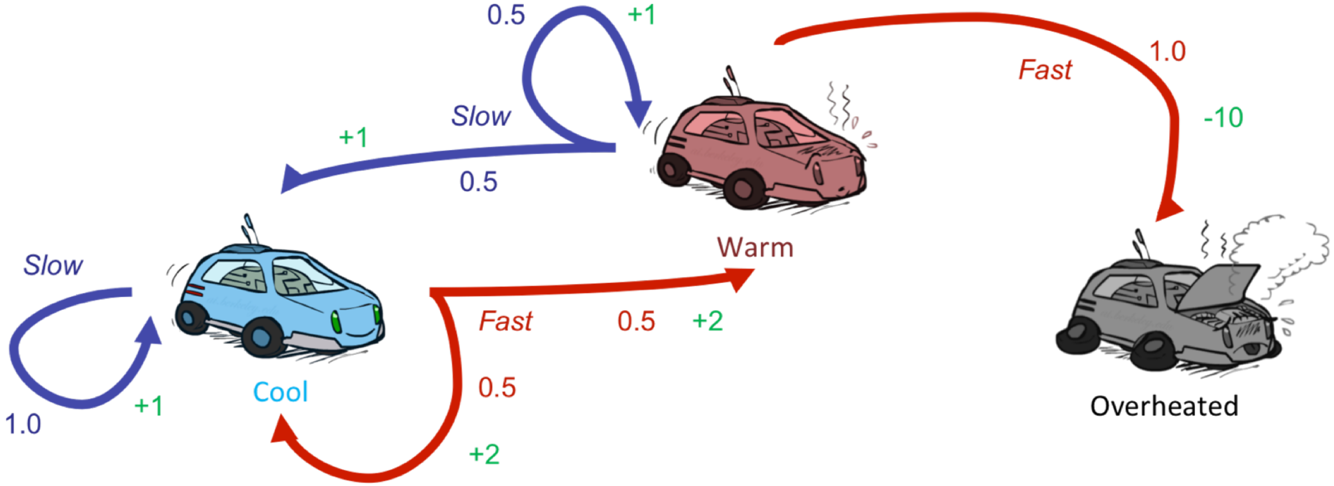
$V^{\pi_0}(O) = 0$

## Policy Improvement:

$\pi_1$

$S$: $1 + 0.9 \cdot 10 = 10$

$F$: $0.5(2 + 0.9 \cdot 10) + 0.5(2 + 0.9 \cdot -10) = 0$



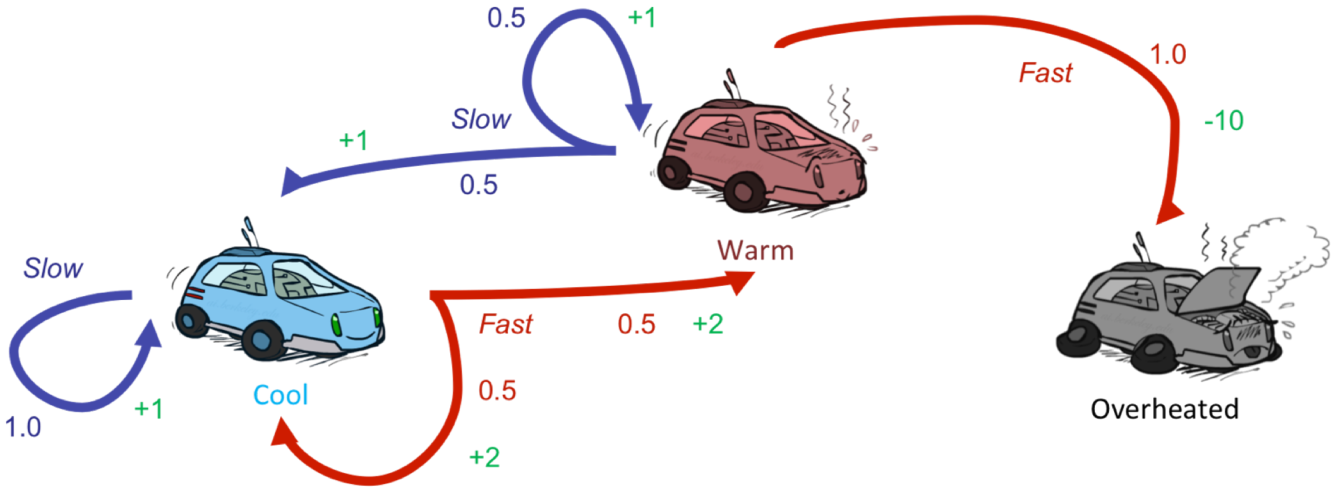

*Assume discount = 0.9*

$$V^{\pi}(s) = \sum_{s'} T(s, \pi(s), s')[R(s, \pi(s), s') + \gamma V^{\pi}(s')]$$

$$\pi_{i+1}(s) = \arg\max_a \sum_{s'} T(s, a, s') \left[R(s, a, s') + \gamma V^{\pi_i}(s')\right]$$

# Example: Policy Iteration

| | | | |
|---|---|---|---|
| $\pi_0$ | $S$ | $F$ | $\emptyset$ |




## Policy Evaluation:

$V^{\pi_0}(B) = 1 + 0.9 \cdot V^{\pi_0}(B) \quad \rightarrow V^{\pi_0}(B) = 10$

$V^{\pi_0}(W) = -10 + 0.9 \cdot V^{\pi_0}(O) \quad \rightarrow V^{\pi_0}(W) = -10$

$V^{\pi_0}(O) = 0$

*Assume discount = 0.9*

## Policy Improvement:

$$V^{\pi}(s) = \sum_{s'} T(s, \pi(s), s')[R(s, \pi(s), s') + \gamma V^{\pi}(s')]$$

$$\pi_{i+1}(s) = \arg\max_a \sum_{s'} T(s, a, s') \left[R(s, a, s') + \gamma V^{\pi_i}(s')\right]$$

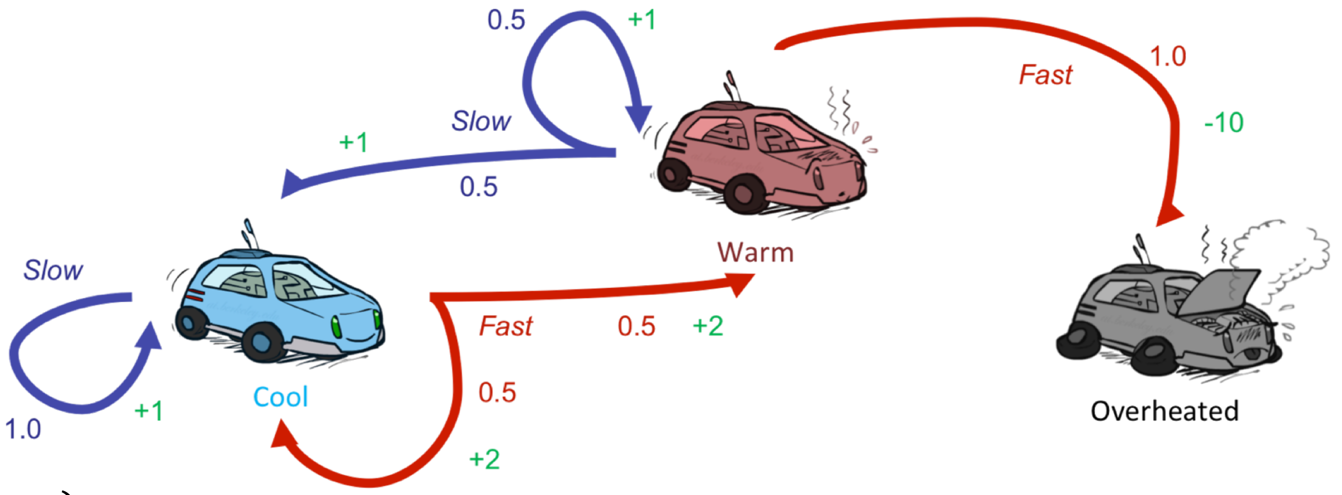
| | | | |
|---|---|---|---|
| $\pi_1$ | $S$ | $S$ | $\emptyset$ |

$S$: $0.5(1 + 0.9 \cdot 10) + 0.5(1 + 0.9 \cdot -10) = 0$

$F$: $-10 + 0.9 \cdot 0 = -10$

# Example: Policy Iteration

| | | | |
|---|---|---|---|
| $\pi_1$ | $S$ | $S$ | $\emptyset$ |




## Policy Evaluation:

$V^{\pi_0}(B) = 1 + 0.9 \cdot V^{\pi_0}(B) \quad \rightarrow V^{\pi_0}(B) = 10$

$V^{\pi_0}(W) = 0.5\big(1 + 0.9 \cdot V^{\pi_0}(B)\big) + 0.5\big(1 + 0.9 \cdot V^{\pi_0}(W)\big) \rightarrow V^{\pi_0}(W) = 10$

$V^{\pi_0}(O) = 0$

*Assume discount = 0.9*

$$V^{\pi}(s) = \sum_{s'} T(s, \pi(s), s')[R(s, \pi(s), s') + \gamma V^{\pi}(s')]$$

## Policy Improvement:

$$\pi_{i+1}(s) = \arg\max_a \sum_{s'} T(s, a, s') \left[R(s, a, s') + \gamma V^{\pi_i}(s')\right]$$

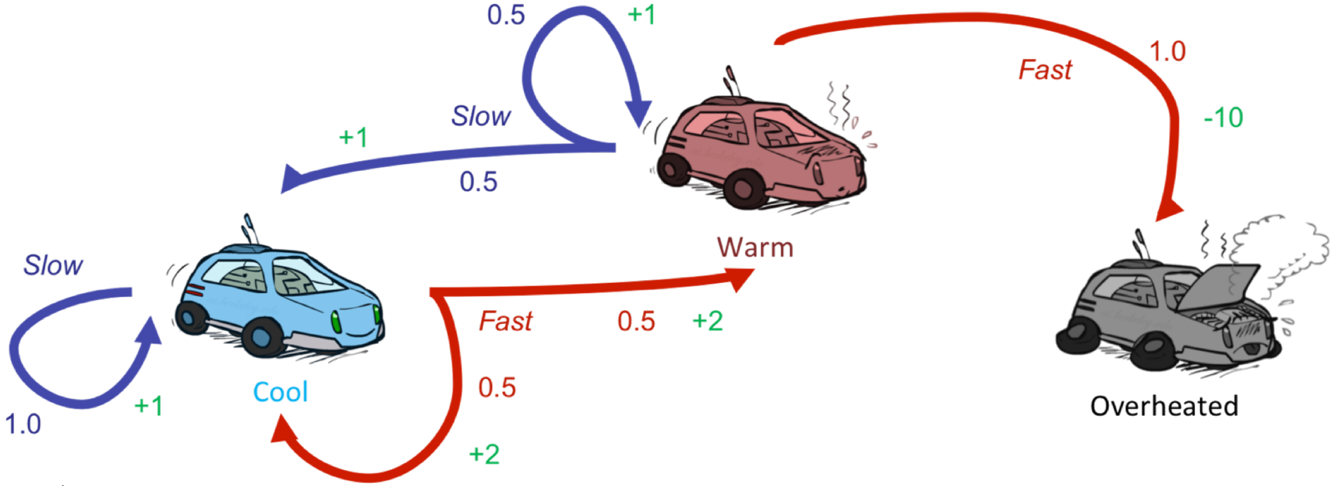
$\pi_2$

$S\text{: } 1 + 0.9 \cdot 10 = 10$

$F\text{: } 0.5(2 + 0.9 \cdot 10) + 0.5(2 + 0.9 \cdot 10) = 11$

# Example: Policy Iteration

| | | | |
|---|---|---|---|
| $\pi_1$ | $S$ | $S$ | $\emptyset$ |




## Policy Evaluation:

$$V^{\pi_0}(B) = 1 + 0.9 \cdot V^{\pi_0}(B) \quad \rightarrow V^{\pi_0}(B) = 10$$

$$V^{\pi_0}(W) = 0.5\big(1 + 0.9 \cdot V^{\pi_0}(B)\big) + 0.5\big(1 + 0.9 \cdot V^{\pi_0}(W)\big) \rightarrow V^{\pi_0}(W) = 10$$

$$V^{\pi_0}(O) = 0$$

*Assume discount = 0.9*

$$V^\pi(s) = \sum_{s'} T(s, \pi(s), s')[R(s, \pi(s), s') + \gamma V^\pi(s')]$$

## Policy Improvement:

$$\pi_{i+1}(s) = \arg\max_a \sum_{s'} T(s, a, s') \left[R(s, a, s') + \gamma V^{\pi_i}(s')\right]$$

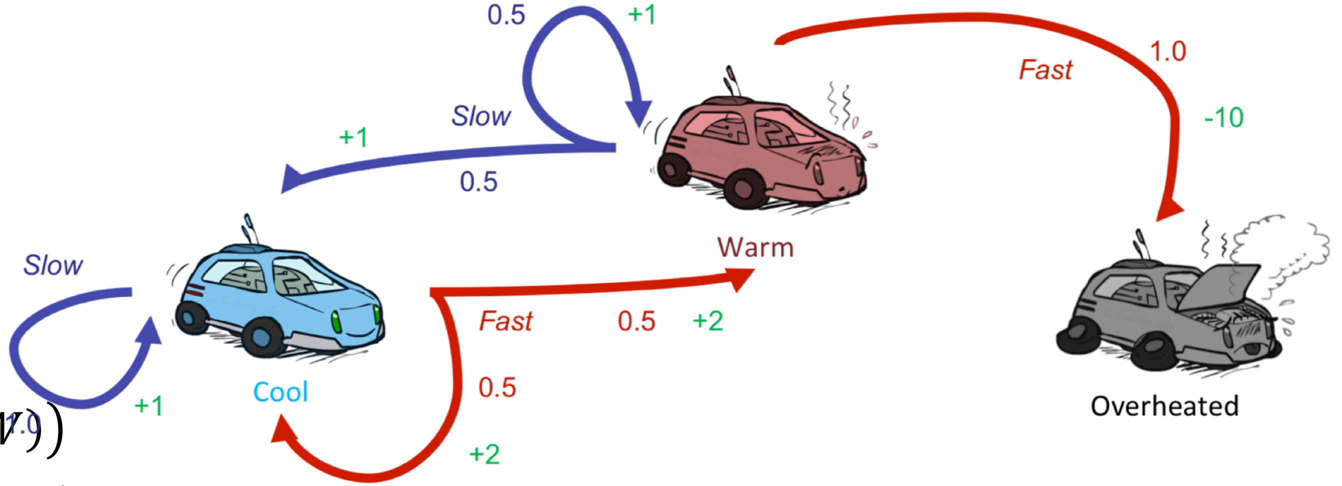
| | | | |
|---|---|---|---|
| $\pi_2$ | $F$ | $S$ | $\emptyset$ |

$S$: $0.5(1 + 0.9 \cdot 10) + 0.5(1 + 0.9 \cdot 10) = 10$

$F$: $-10 + 0.9 \cdot 0 = -10$

# Example: Policy Iteration

| | | | |
|---|---|---|---|
| $\pi_2$ | $F$ | $S$ | $\emptyset$ |




## Policy Evaluation:

$\rightarrow V^{\pi_0}(B) = 15.5$

$$V^{\pi_0}(B) = 0.5\big(2 + 0.9 \cdot V^{\pi_0}(B)\big) + 0.5\big(2 + 0.9 \cdot V^{\pi_0}(W)\big)$$

$$V^{\pi_0}(W) = 0.5\big(1 + 0.9 \cdot V^{\pi_0}(B)\big) + 0.5\big(1 + 0.9 \cdot V^{\pi_0}(W)\big) \rightarrow V^{\pi_0}(W) = 14.5$$

$$V^{\pi_0}(O) = 0$$

*Assume discount = 0.9*

$$V^{\pi}(s) = \sum_{s'} T(s, \pi(s), s')[R(s, \pi(s), s') + \gamma V^{\pi}(s')]$$

$$\pi_{i+1}(s) = \arg\max_a \sum_{s'} T(s, a, s') \left[R(s, a, s') + \gamma V^{\pi_i}(s')\right]$$

## Policy Improvement:

$\pi_3$

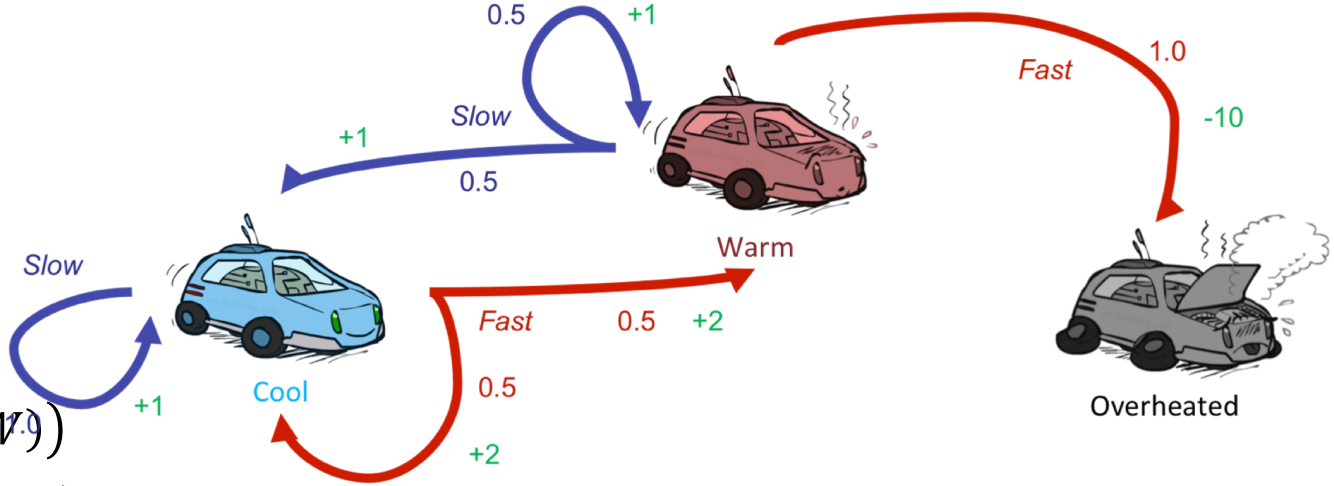
$S\!: \; 1 + 0.9 \cdot 15.5 = 14.95$

$F\!: \; 0.5(2 + 0.9 \cdot 15.5) + 0.5(2 + 0.9 \cdot 14.5) = 15.5$

# Example: Policy Iteration

$\pi_2$

| | | |
|---|---|---|
| $F$ | $S$ | $\emptyset$ |




## Policy Evaluation:

$\rightarrow V^{\pi_0}(B) = 15.5$

$$V^{\pi_0}(B) = 0.5\big(2 + 0.9 \cdot V^{\pi_0}(B)\big) + 0.5\big(2 + 0.9 \cdot V^{\pi_0}(W)\big)$$

$$V^{\pi_0}(W) = 0.5\big(1 + 0.9 \cdot V^{\pi_0}(B)\big) + 0.5\big(1 + 0.9 \cdot V^{\pi_0}(W)\big) \rightarrow V^{\pi_0}(W) = 14.5$$

$$V^{\pi_0}(O) = 0$$

*Assume discount = 0.9*

## Policy Improvement:

$$V^{\pi}(s) = \sum_{s'} T(s, \pi(s), s')[R(s, \pi(s), s') + \gamma V^{\pi}(s')]$$

$$\pi_{i+1}(s) = \arg\max_{a} \sum_{s'} T(s, a, s') \left[R(s, a, s') + \gamma V^{\pi_i}(s')\right]$$

$\pi_3$

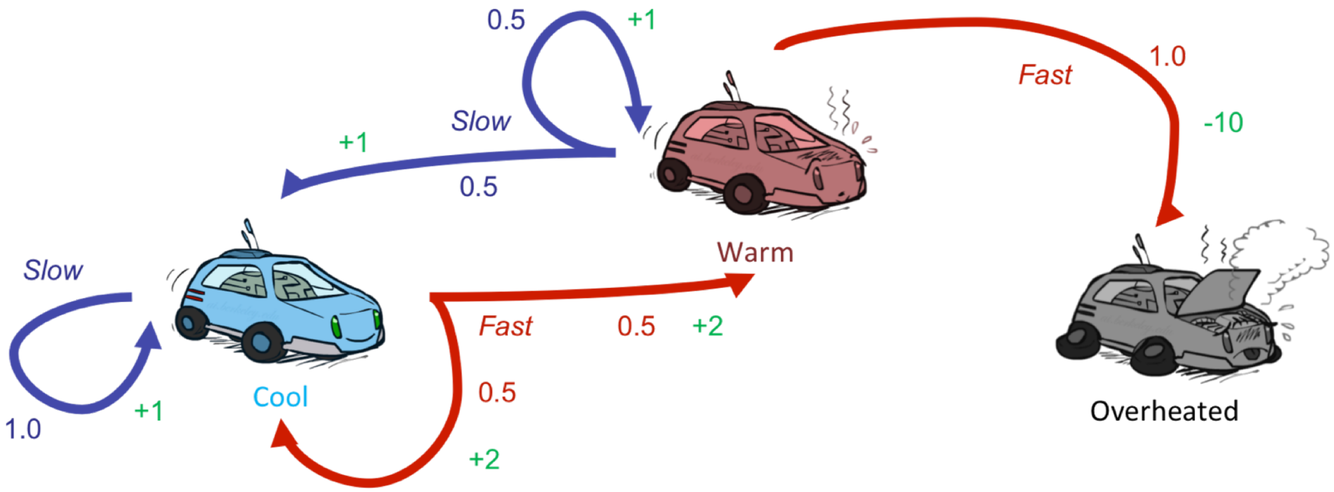
| | | |
|---|---|---|
| $F$ | $S$ | $\emptyset$ |

$S$: $0.5(1 + 0.9 \cdot 15.5) + 0.5(1 + 0.9 \cdot 14.5) = 14.5$

$F$: $-10 + 0.9 \cdot 0 = -10$

# Example: Value Iteration

| |  |  |  |
|---|---|---|---|
| $V_0$ | 0 | 0 | 0 |
| $V_1$ | 2 | 1 | 0 |
| $V_2$ | 3.35 | 2.35 | 0 |
| $V_3$ | 4.565 | 3.565 | 0 |
| $V_4$ | 5.6585 | 4.6585 | 0 |




*Assume discount = 0.9*

$$V_{k+1}(s) \leftarrow \max_a \sum_{s'} T(s,a,s') \left[ R(s,a,s') + \gamma V_k(s') \right]$$

…

| | | | |
|---|---|---|---|
| $V_{91}$ | 15.499 | 14.499 | 0 |

# Convergence*

- Proof Sketch

$$V^\pi(s) = \sum_{s'} T(s,\pi(s),s')[R(s,\pi(s),s') + \gamma V^\pi(s')]$$

$$\pi_{i+1}(s) = \arg\max_a \sum_{s'} T(s,a,s')\left[R(s,a,s') + \gamma V^{\pi_i}(s')\right]$$

  - Monotonic improvement: $\forall s \; V^{\pi_{i+1}}(s) \geq V^{\pi_i}(s)$
  - Termination: $\pi_i$ is optimal if $\forall s \; \pi_i(s) = \pi_{i+1}(s)$
    - $\pi_{i+1}(s)$ chooses the best action to take under $V^{\pi_i}(s)$
    - If $\forall s \; \pi_i(s) = \pi_{i+1}(s)$, then $\pi_i(s)$ was already the best action for all states
  - Guaranteed termination: only finite number of policies

# Comparison

- Both value iteration and policy iteration compute the same thing (all optimal values)
- In value iteration:
  - Every iteration updates both the values and (implicitly) the policy
  - We don't track the policy, but taking the max over actions implicitly recomputes it
  - Runtime per iteration: $O(|S|^2|A|)$
- In policy iteration:
  - We do several passes that update utilities with fixed policy (each pass is fast because we consider only one action, not all of them)
    - Runtime per value iteration update: $O(|S|^2)$ → total runtime to get fixed policy values: $O(|S|^3)$
  - After policy is evaluated, a new policy is chosen (slow like a value iteration pass → $O(|S|^2|A|)$)
  - The new policy will be better (or we're done)
  - Runtime per iteration: $O(|S|^3) + O(|S|^2|A|)$ → slower but can take much fewer iterations
- Both are dynamic programs for solving MDPs

# Summary: MDP Algorithms

- So you want to….
  - Compute optimal values: use **value iteration** or **policy iteration**
  - Compute values for a particular policy: use **policy evaluation**
  - Turn your values into a policy: use **policy extraction** (one-step lookahead)
- These all look the same!
  - They basically are – they are all variations of Bellman updates
  - They all use one-step lookahead expectimax fragments
  - They differ only in whether we plug in a fixed policy or max over actions