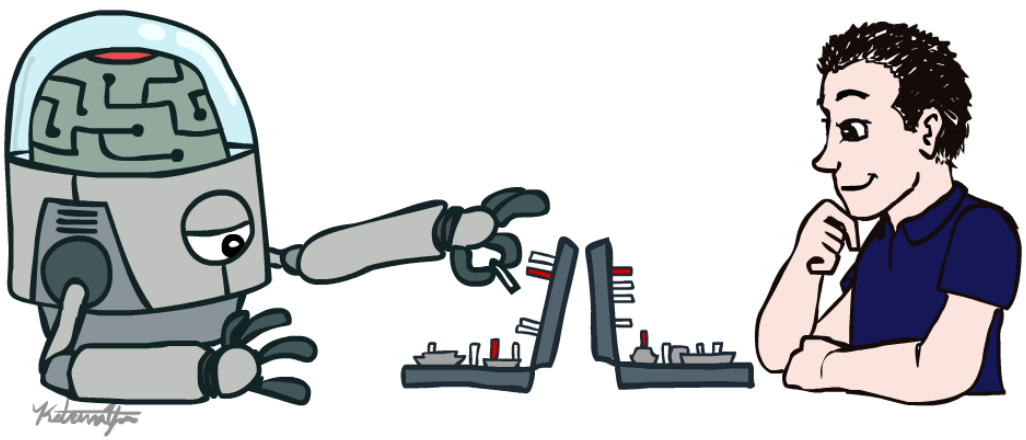
CS 188: Artificial Intelligence

Midterm Review



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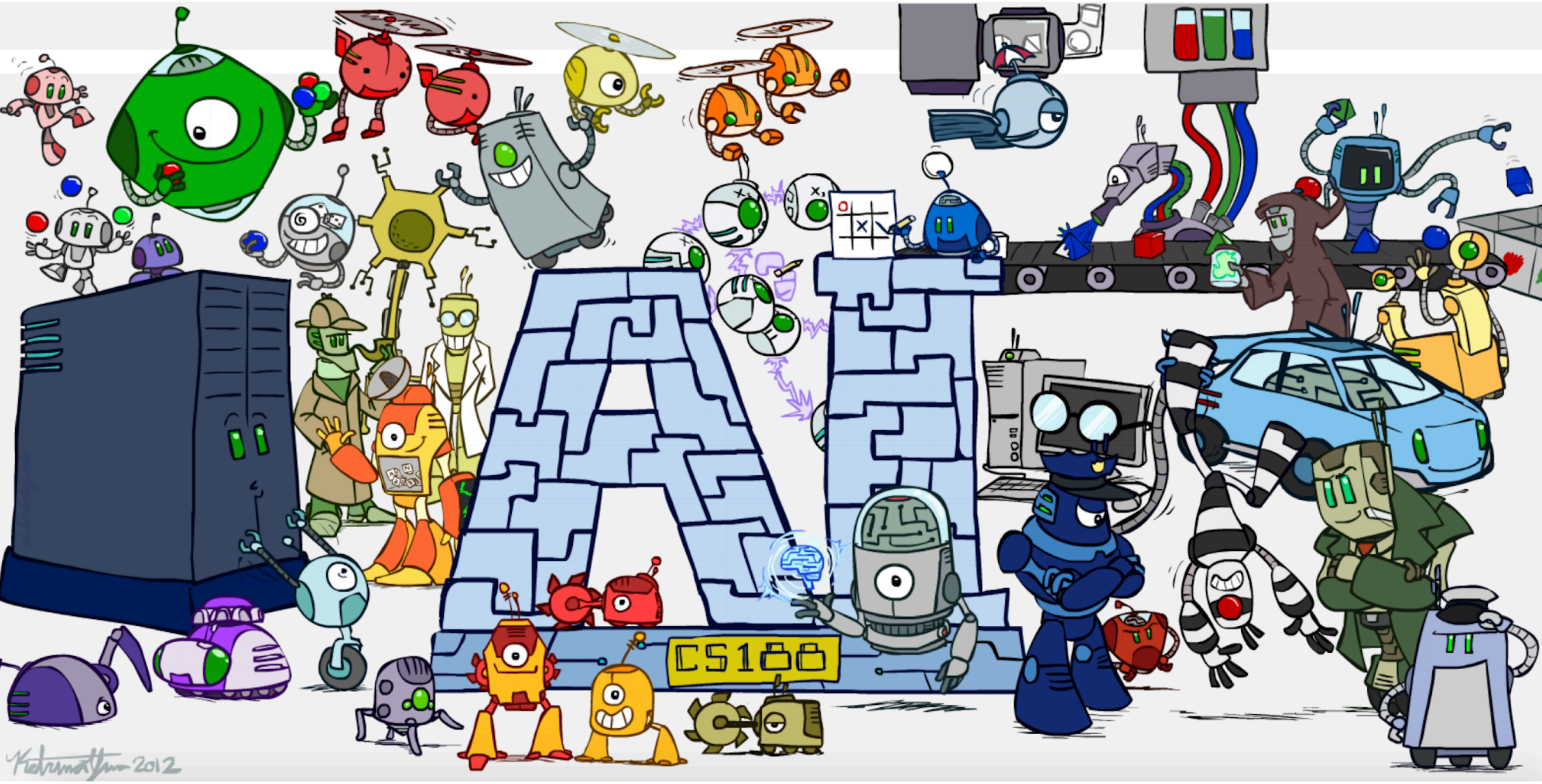
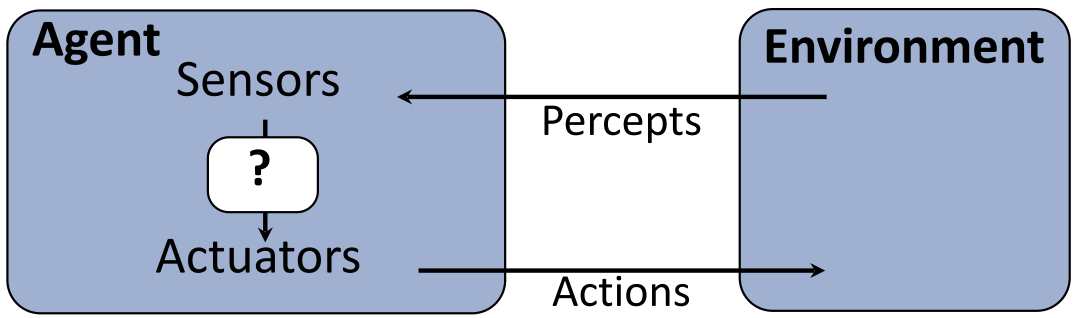
(Slides Credit: Dan Klein, Pieter Abbeel, Anca Dragan, Stuart Russell, Satish Rao, Ketrina Yim, and many others)


CS188

Midterm: Topics in Scope

- Utilities and Rationality, MEU Principle
- Search and Planning
- Constraint Satisfaction Programming
- Game Trees, Minimax, Pruning, Expectimax
- Probabilistic Inference, Bayesian Networks, Variable Elimination, D-Separation, Sampling
- Markov Models, HMMs, Viterbi Algorithm, Particle Filtering, Dynamic Bayes Nets

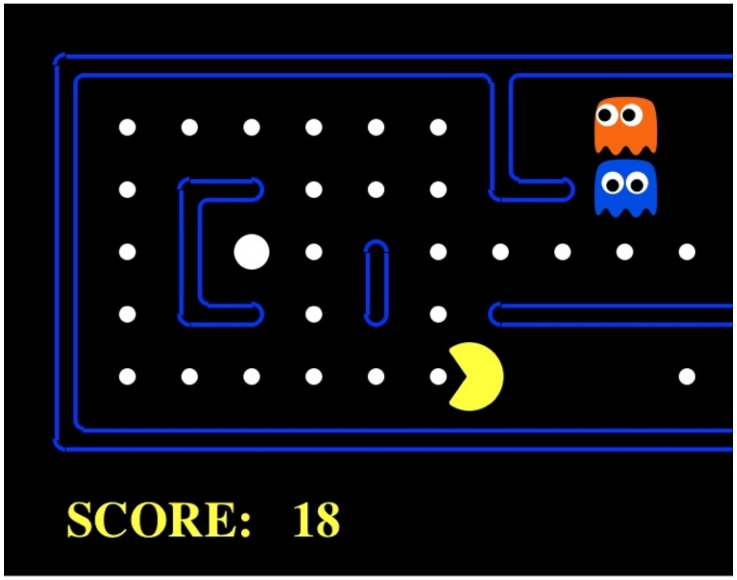
Agents and environments



- An agent ***perceives*** its environment through ***sensors*** and ***acts*** upon it through ***actuators*** (or *effectors*, depending on whom you ask)
- The ***agent function*** maps percept sequences to actions
- It is generated by an ***agent program*** running on a ***machine***

The task environment - PEAS

- Performance measure
  - -1 per step; + 10 food; +500 win; -500 die; +200 hit scared ghost
- Environment
  - Pacman dynamics (incl ghost behavior)
- Actuators
  - Left Right Up Down or NSEW
- Sensors
  - Entire state is visible (except power pellet duration)

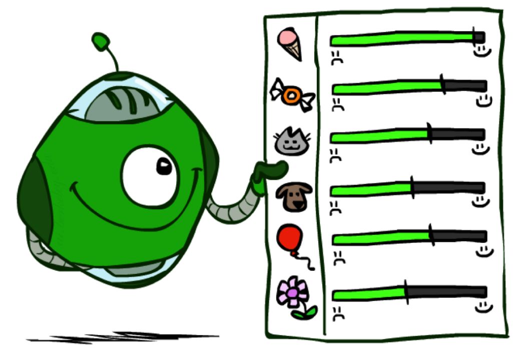


Agent design

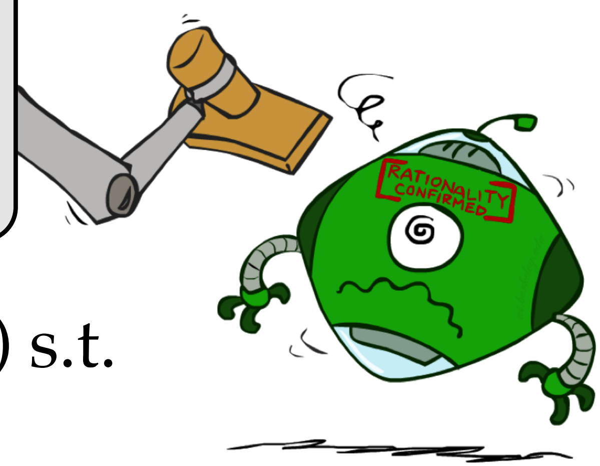
- **The environment type largely determines the agent design**
  - ***Partially observable*** => agent requires ***memory*** (internal state)
  - ***Stochastic*** => agent may have to prepare for ***contingencies***
  - ***Multi-agent*** => agent may need to behave ***randomly***
  - ***Static*** => agent has time to compute a rational decision
  - ***Continuous time*** => continuously operating ***controller***
  - ***Unknown physics*** => need for ***exploration***
  - ***Unknown perf. measure*** => observe/interact with ***human principal***

Utilities and Rationality

- Utility: map state of world to real value
- Rational Preferences



Orderability: $(A > B) \vee (B > A) \vee (A \sim B)$
Transitivity: $(A > B) \wedge (B > C) \Rightarrow (A > C)$
Continuity: $(A > B > C) \Rightarrow \exists p\ [p, A;\ 1\text{-p}, C] \sim B$
Substitutability: $(A \sim B) \Rightarrow [p, A;\ 1\text{-p}, C] \sim [p, B;\ 1\text{-p}, C]$
Monotonicity: $(A > B) \Rightarrow$
$(p \geq q) \Leftrightarrow [p, A;\ 1\text{-p}, B] \geq [q, A;\ 1\text{-q}, B]$

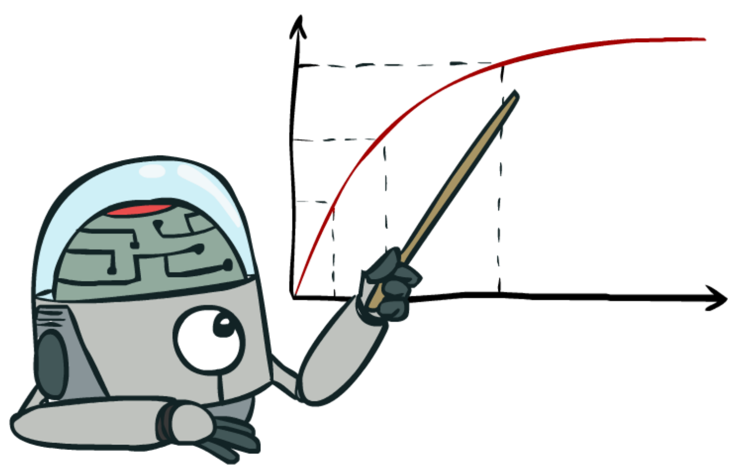


Given Rational Preferences, Exists U(X) s.t.

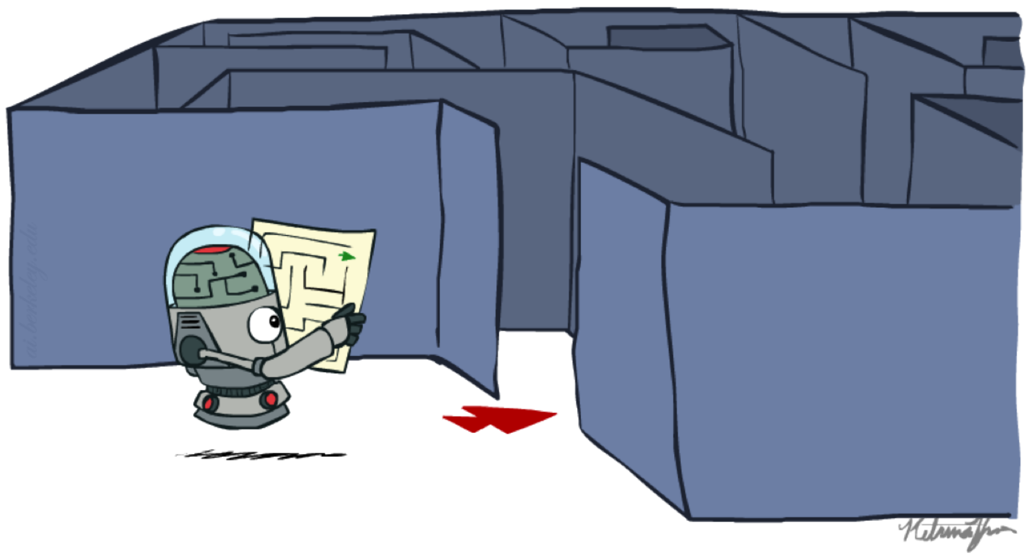
$$U(A) \geq U(B) \Leftrightarrow A \geq B$$

$$U([p_1,S_1; \ldots ; p_n,S_n]) = p_1 U(S_1) + \ldots + p_n U(S_n)$$

Maximize Your Expected Utility

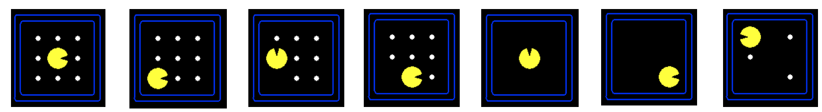


Search Problems

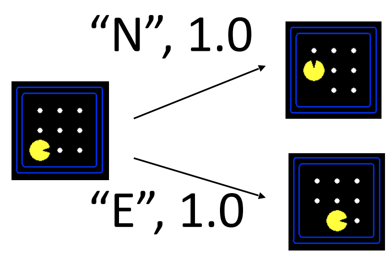


Search Problems

- A search problem consists of:
  - A state space



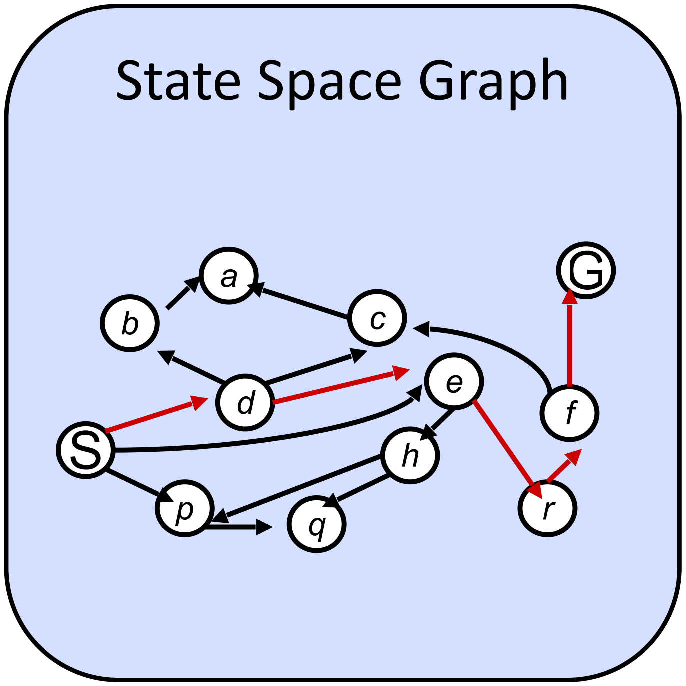
  - A successor function
    (with actions, costs)



  - A start state and a goal test

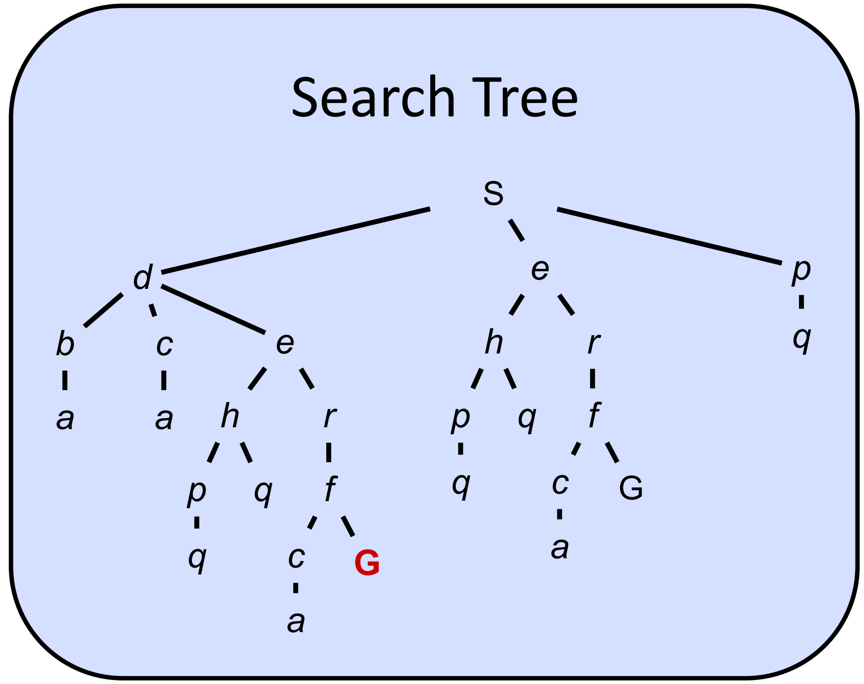
- A solution is a sequence of actions (a plan) which transforms the start state to a goal state

State Space Graphs vs. Search Trees



Each NODE in in the search tree is an entire PATH in the state space graph.

We construct only what we need on demand



General Tree Search

```
function TREE-SEARCH( problem, strategy) returns a solution, or failure
   initialize the search tree using the initial state of problem
   loop do
      if there are no candidates for expansion then return failure
      choose a leaf node for expansion according to strategy
      if the node contains a goal state then return the corresponding solution
      else expand the node and add the resulting nodes to the search tree
   end
```

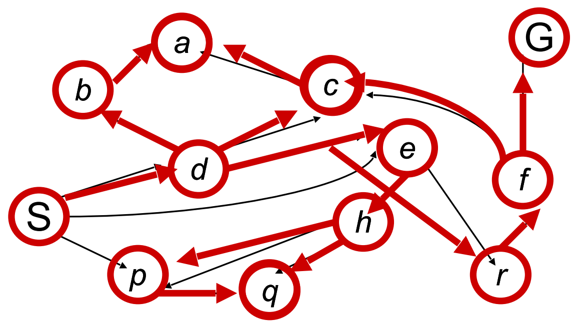
- Important ideas:
  - Fringe
  - Expansion
  - Exploration strategy
- Main question: which fringe nodes to explore?

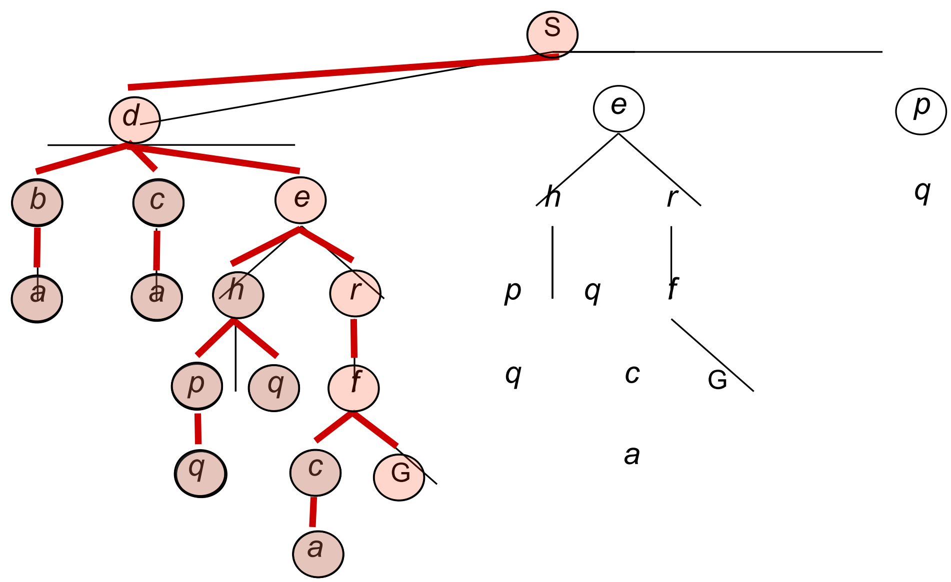
Depth-First Search

Depth-First Search

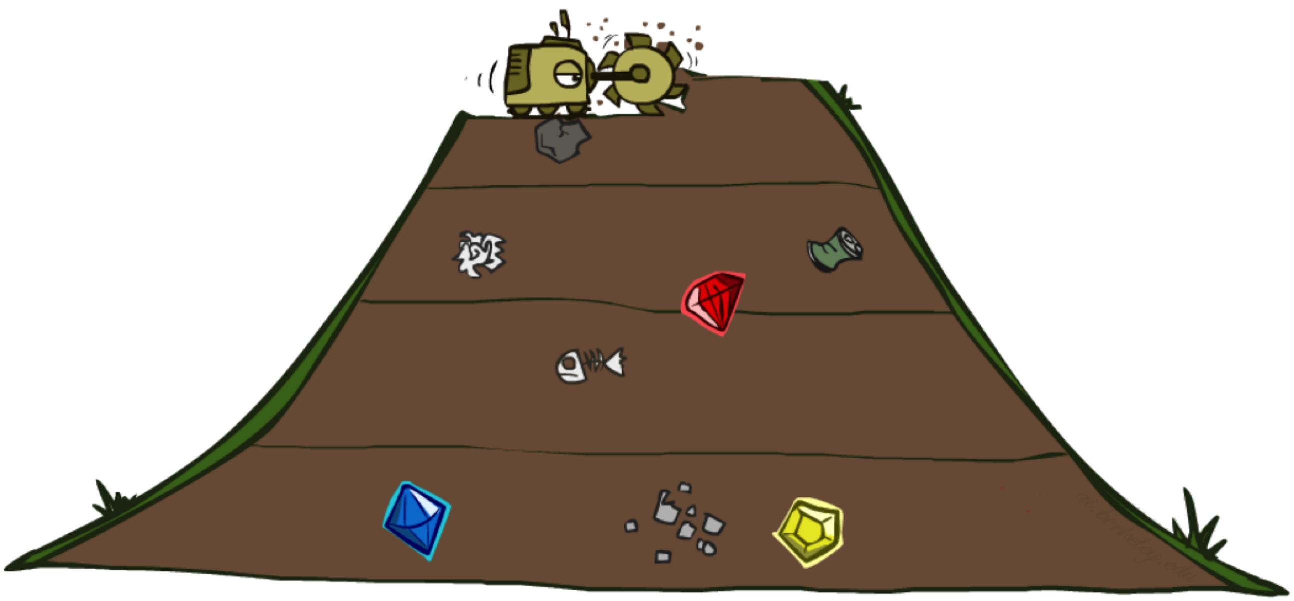
Strategy: expand a deepest node first

Implementation: Fringe is a LIFO stack





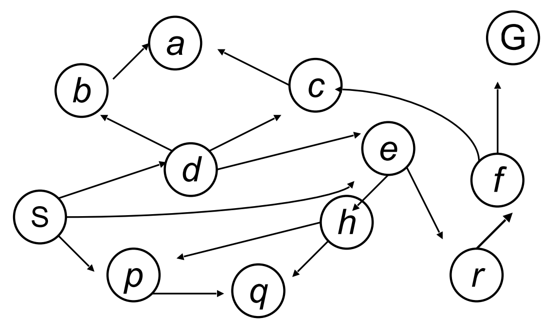
Breadth-First Search

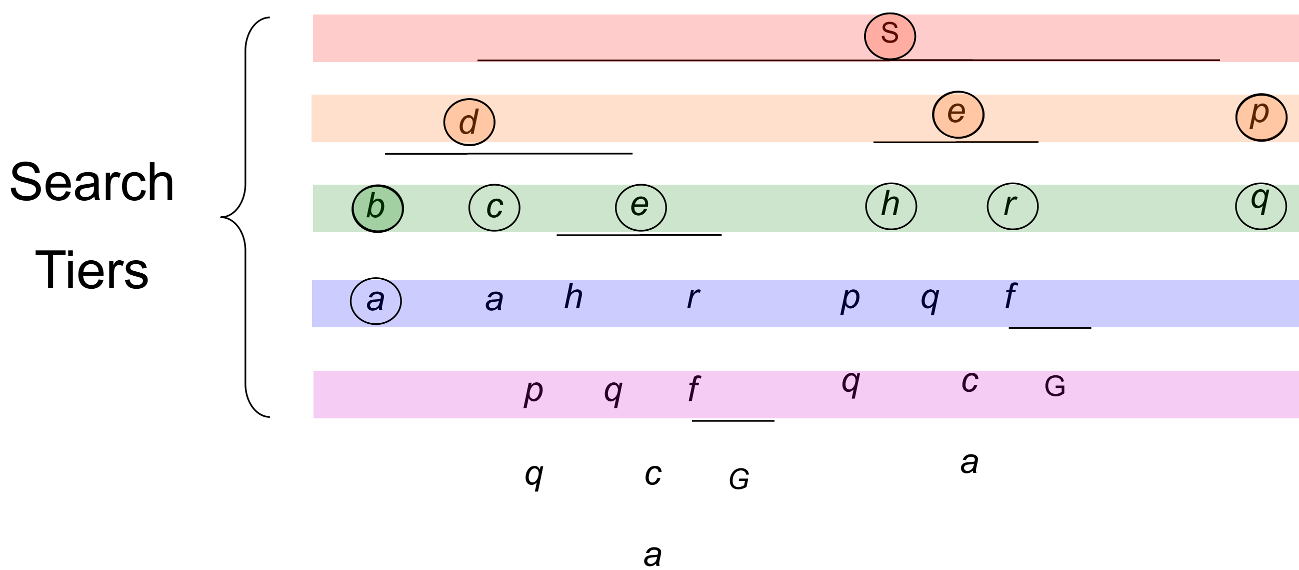


Breadth-First Search

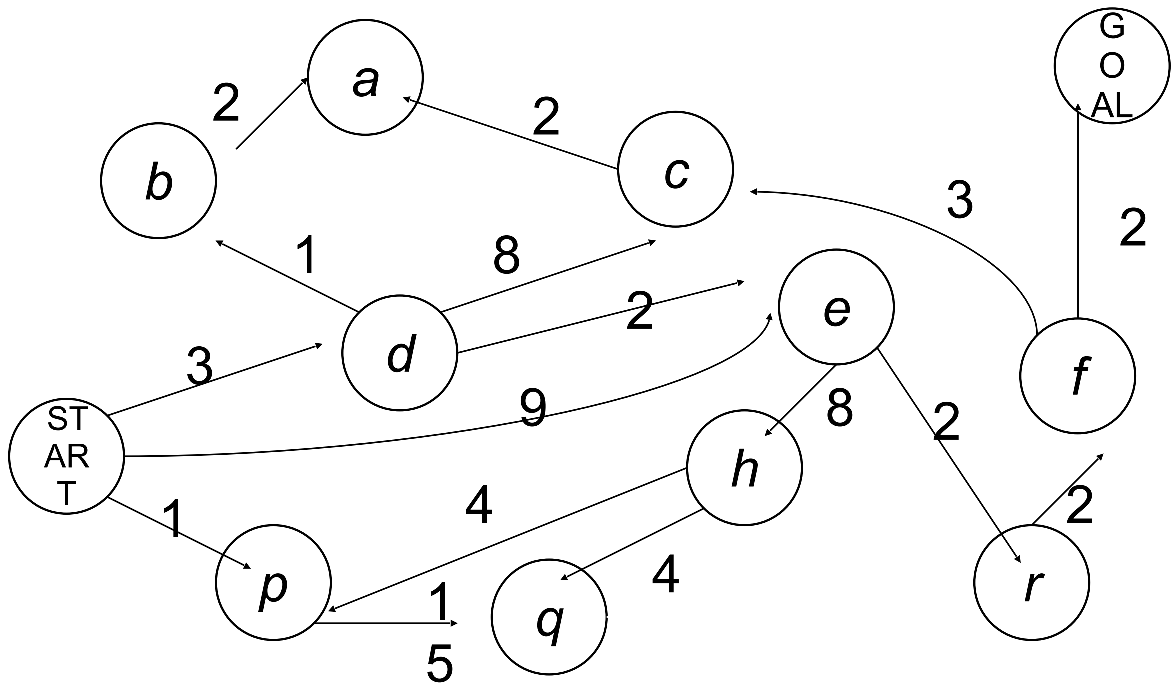
Strategy: expand a shallowest node first

Implementation: Fringe is a FIFO queue



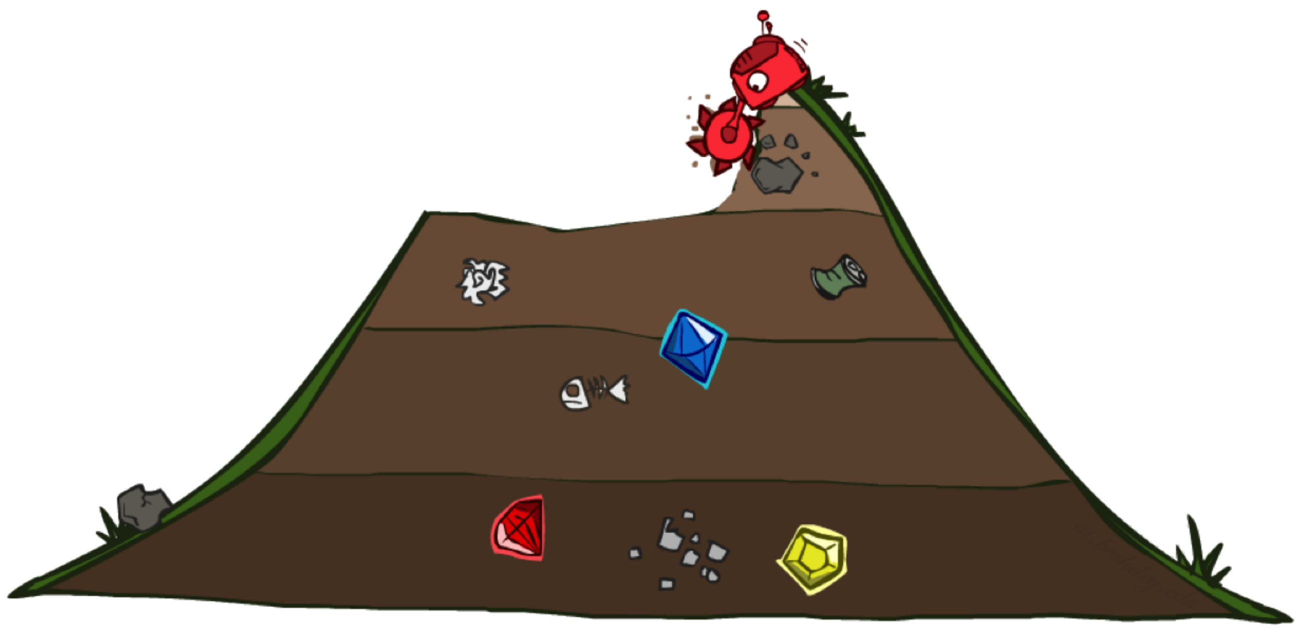


Cost-Sensitive Search



BFS finds the shortest path in terms of number of actions. It does not find the least-cost path. We will now cover a similar algorithm which does find the least-cost path.

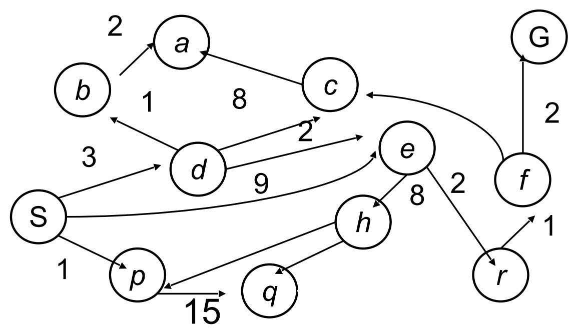
Uniform Cost Search

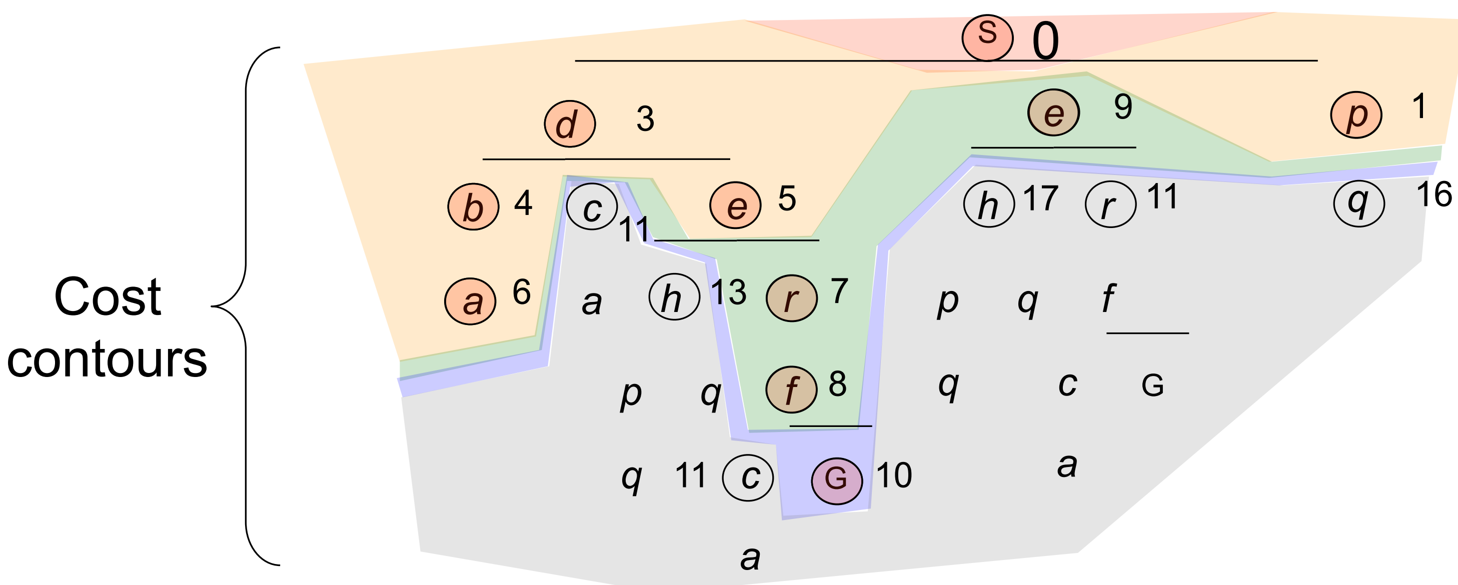


Uniform Cost Search

Strategy: expand a cheapest node first:

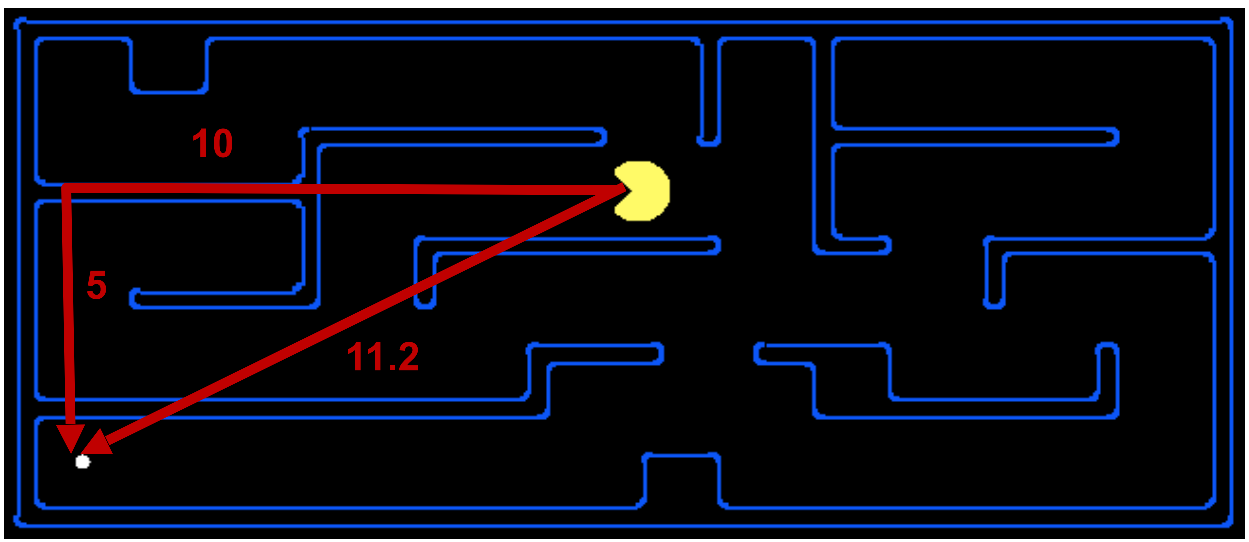
Fringe is a priority queue (priority: cumulative cost)

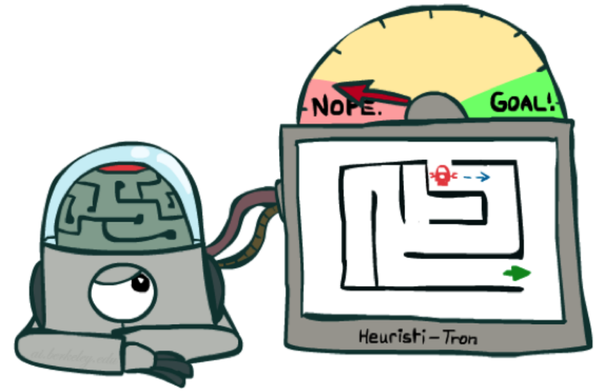


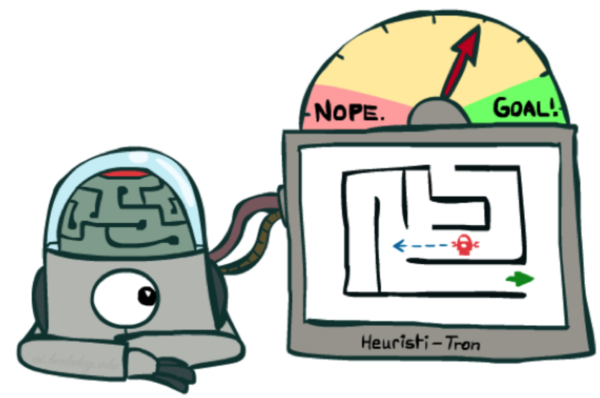


Search Heuristics

- A heuristic is:
  - A function that *estimates* how close a state is to a goal
  - Designed for a particular search problem
  - Pathing?
  - Examples: Manhattan distance, Euclidean distance



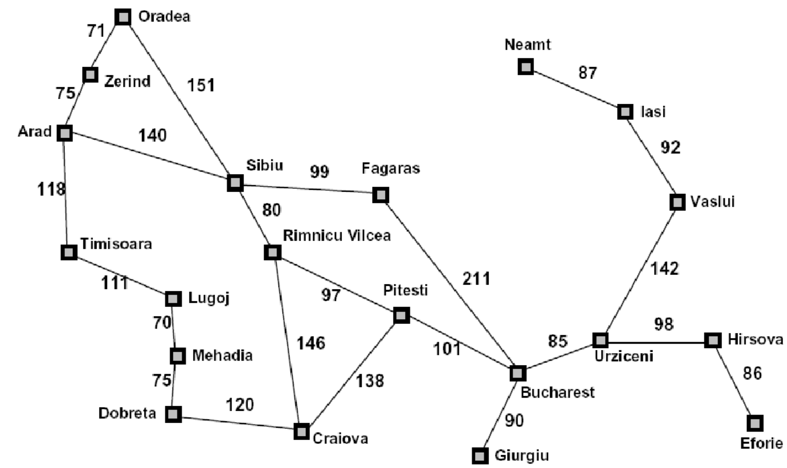


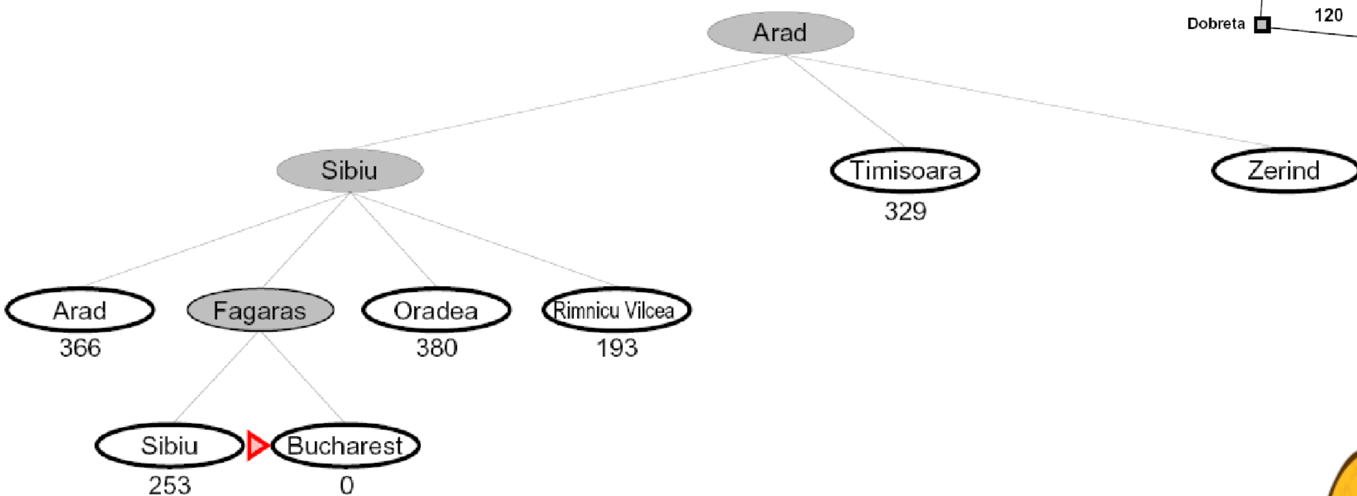


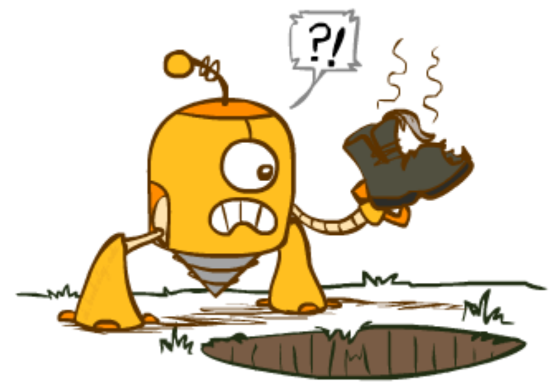
Greedy Search

Greedy Search

- Expand the node that seems closest…
  - Move to smallest heuristic value





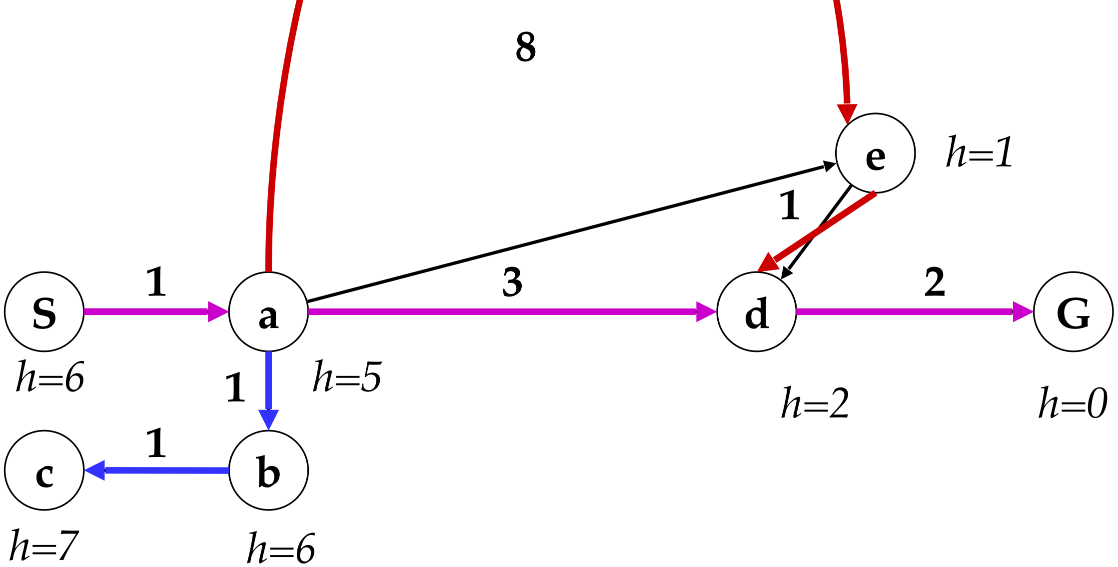


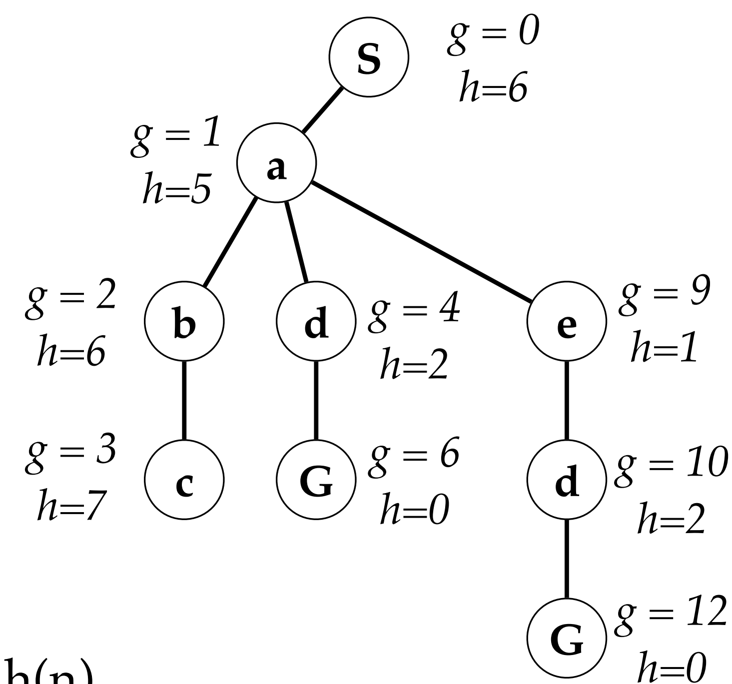
- Is it optimal?
  - No. Resulting path to Bucharest is not the shortest!

A* Search

Combining UCS and Greedy

- Uniform-cost orders by path cost, or *backward cost* g(n)
- Greedy orders by goal proximity, or *forward cost* h(n)



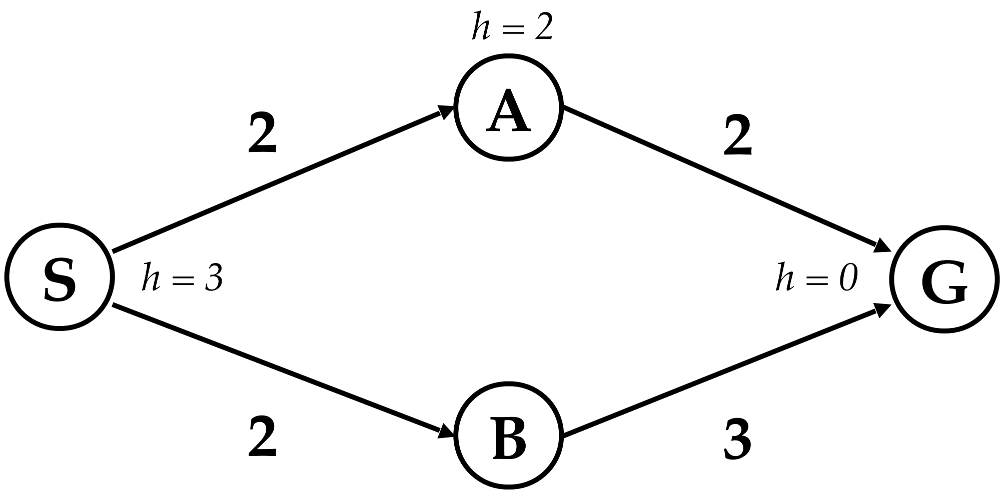


- A* Search orders by the sum: f(n) = g(n) + h(n)

Example: Teg Grenager

When should A* terminate?

- Should we stop when we enqueue a goal?



- No: only stop when we dequeue a goal

	g	h	+
~~S~~	~~0~~	~~3~~	~~3~~
~~S->A~~	~~2~~	~~2~~	~~4~~
~~S->B~~	~~2~~	~~1~~	~~3~~
S->B->G	5	0	5
S->A->G	4	0	4

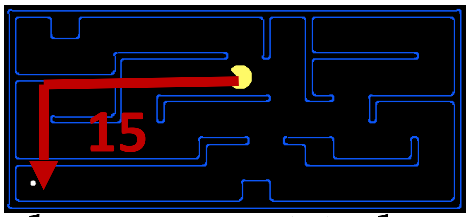
Admissible Heuristics

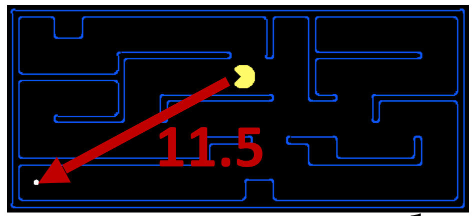
- A heuristic *h* is *admissible* (optimistic) iff:

$$0 \leq h(n) \leq h^*(n)$$

  where $h^*(n)$ the true cost to a nearest goal

- Examples:





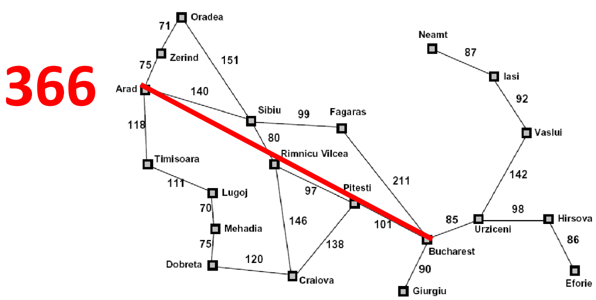
0.0

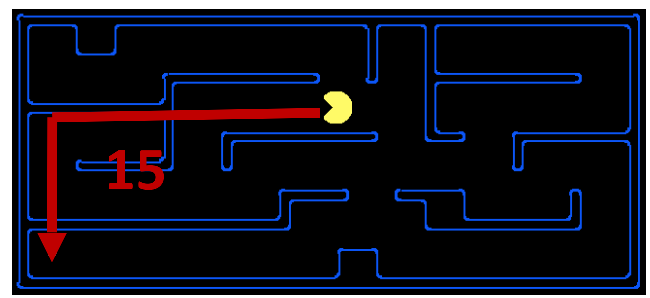
- Coming up with admissible heuristics is most of what's involved in using A* in practice.

Creating Admissible Heuristics

- Most of the work in solving hard search problems optimally is in coming up with admissible heuristics

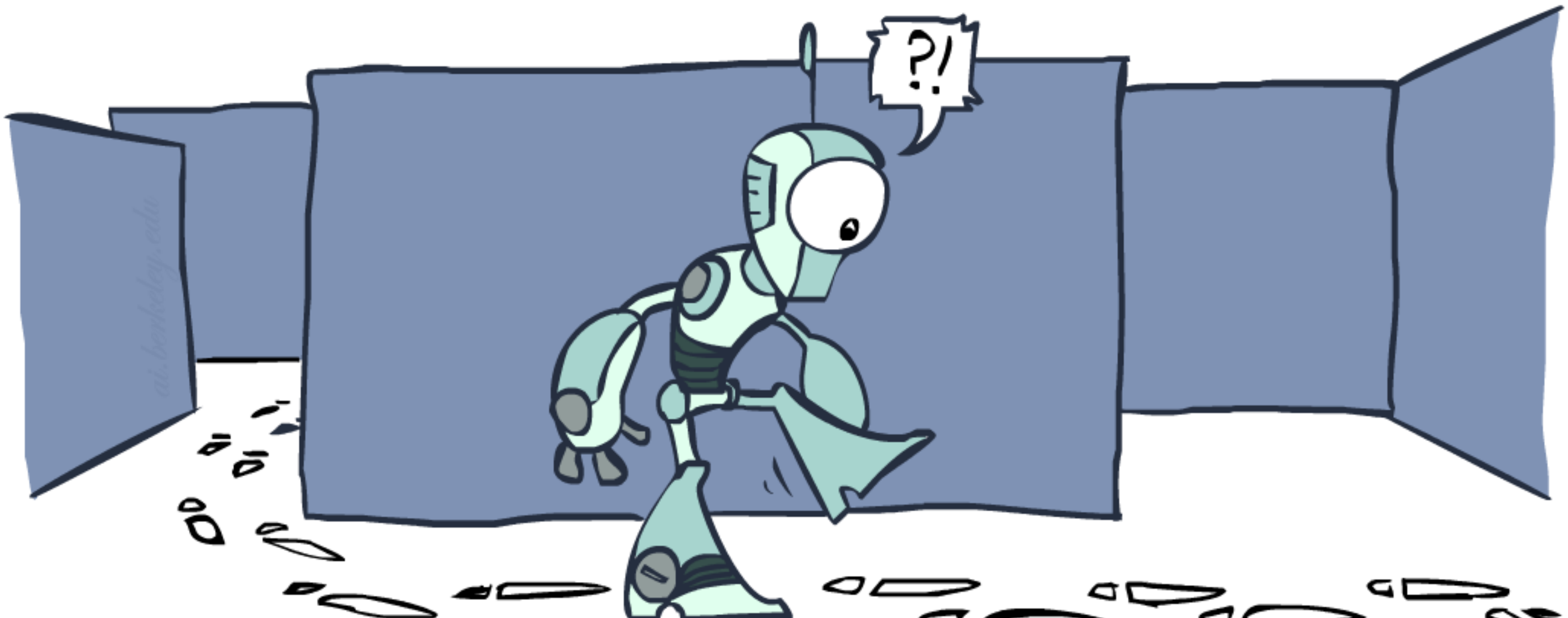
- Often, admissible heuristics are solutions to *relaxed problems*, where new actions are available





- Inadmissible heuristics are often useful too

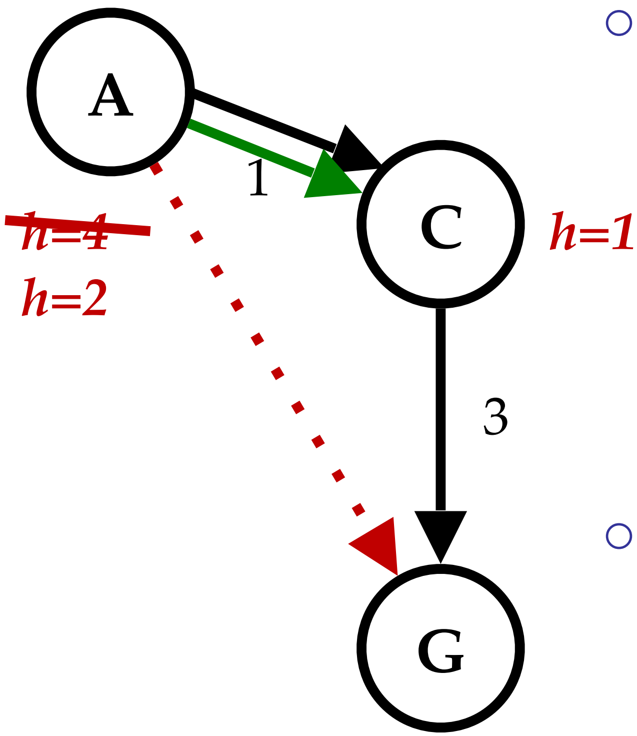
Graph Search



Graph Search Pseudo-Code

```
function GRAPH-SEARCH(problem, fringe) return a solution, or failure
    closed ← an empty set
    fringe ← INSERT(MAKE-NODE(INITIAL-STATE[problem]), fringe)
    loop do
        if fringe is empty then return failure
        node ← REMOVE-FRONT(fringe)
        if GOAL-TEST(problem, STATE[node]) then return node
        if STATE[node] is not in closed then
            add STATE[node] to closed
            for child-node in EXPAND(STATE[node], problem) do
                fringe ← INSERT(child-node, fringe)
            end
    end
```

Consistency of Heuristics



- Main idea: estimated heuristic costs ≤ actual costs
  - Admissibility: heuristic cost ≤ actual cost to goal

    $h(v) \le h^*(v)$ for all $v \in V$

    Underestimate the true cost to the goal!
  - Consistency: heuristic "arc" cost ≤ actual cost for each arc

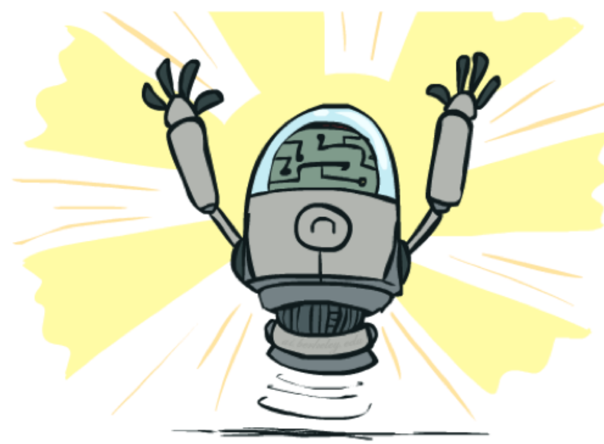
    $h(u) - h(v) \le d(u, v)$ for all $(u, v) \in E$

    Underestimate the weight of every edge!
- Consequences of consistency:
  - The f value along a path never decreases

    $h(A) \le \text{cost}(A \text{ to } C) + h(C)$
  - A* graph search is optimal

Optimality of A* Search

- With a admissible heuristic, Tree A* is optimal.
- With a consistent heuristic, Graph A* is optimal.
  - With h=0, the same proof shows that UCS is optimal.



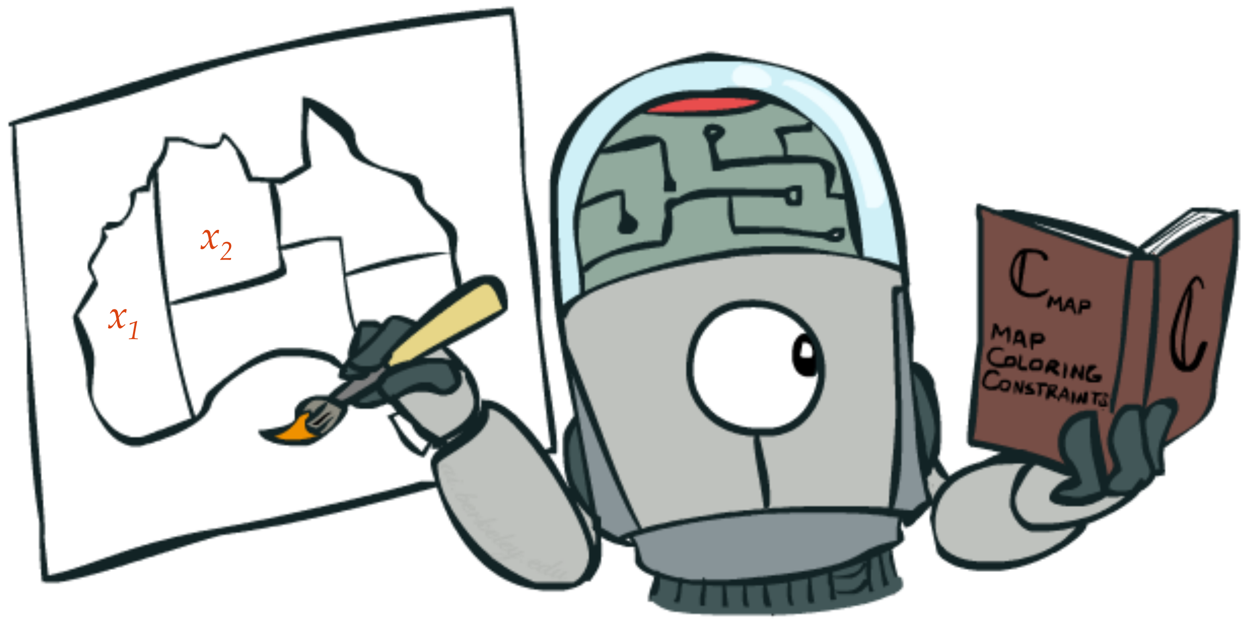
Constraint Satisfaction Problems

Constraint Satisfaction Problems

N variables

domain D

constraints



states

partial assignment

goal test

complete; satisfies constraints

successor function

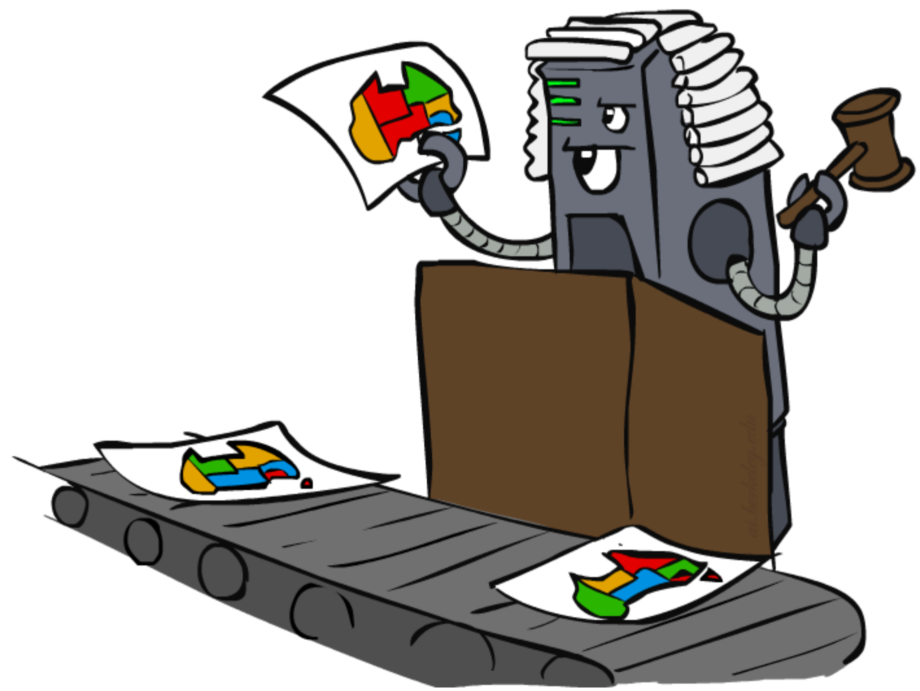
assign an unassigned variable

Standard Search Formulation

- Standard search formulation of CSPs

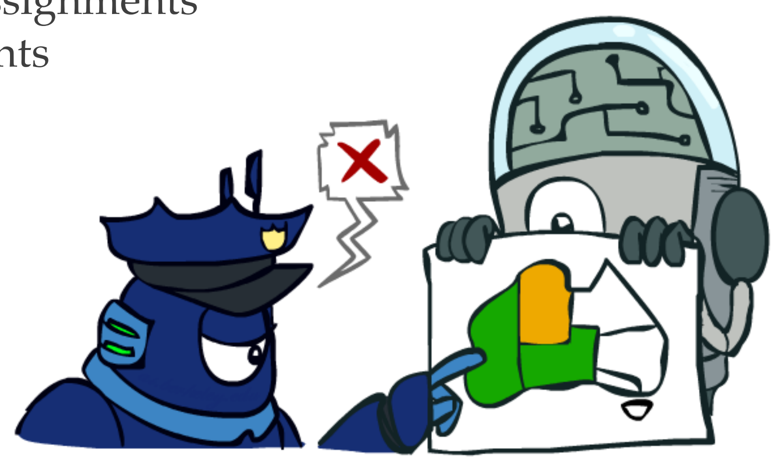
- States defined by the values assigned so far (partial assignments)
  - Initial state: the empty assignment, {}
  - Successor function: assign a value to an unassigned variable
  - Goal test: the current assignment is complete and satisfies all constraints

- We'll start with the straightforward, naïve approach, then improve it

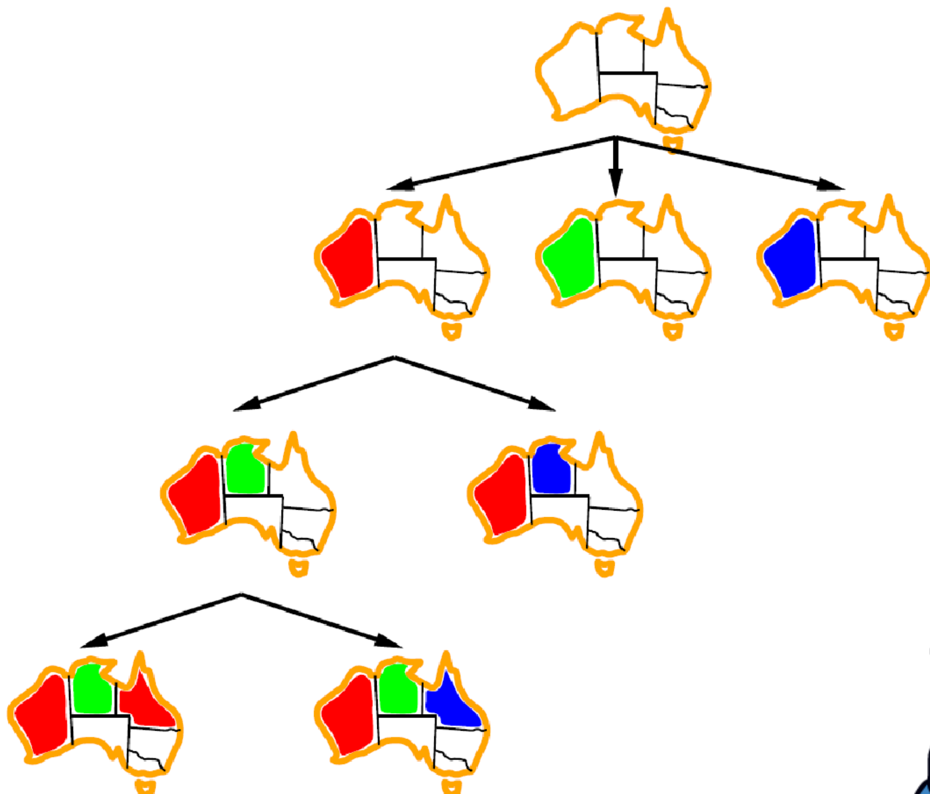


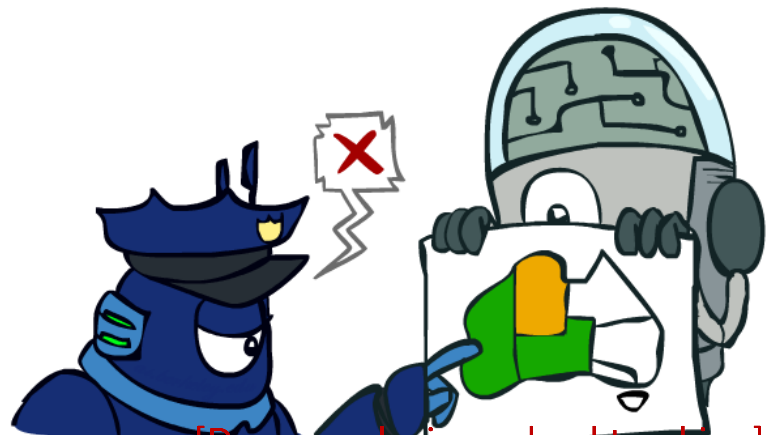
Backtracking Search

- Backtracking search is the basic uninformed algorithm for solving CSPs
- Idea 1: One variable at a time
  - Variable assignments are commutative, so fix ordering -> better branching factor!
  - I.e., [WA = red then NT = green] same as [NT = green then WA = red]
  - Only need to consider assignments to a single variable at each step
- Idea 2: Check constraints as you go
  - I.e. consider only values which do not conflict previous assignments
  - Might have to do some computation to check the constraints
  - "Incremental goal test"
- Depth-first search with these two improvements is called *backtracking search* (not the best name)
- Can solve n-queens for $n \approx 25$



Backtracking Example





[Demo: coloring -- backtracking]

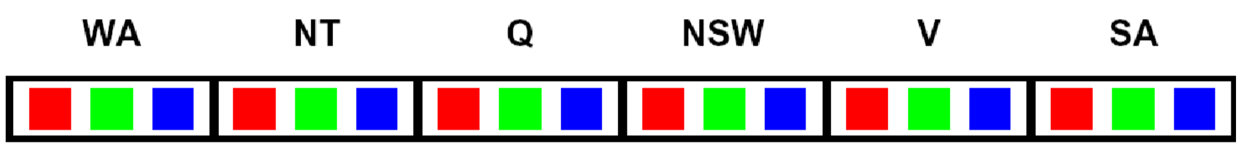
Backtracking Search

```
function BACKTRACKING-SEARCH(csp) returns solution/failure
   return RECURSIVE-BACKTRACKING({ }, csp)
function RECURSIVE-BACKTRACKING(assignment, csp) returns soln/failure
   if assignment is complete then return assignment
   var ← SELECT-UNASSIGNED-VARIABLE(VARIABLES[csp], assignment, csp)
   for each value in ORDER-DOMAIN-VALUES(var, assignment, csp) do
      if value is consistent with assignment given CONSTRAINTS[csp] then
         add {var = value} to assignment
         result ← RECURSIVE-BACKTRACKING(assignment, csp)
         if result ≠ failure then return result
         remove {var = value} from assignment
   return failure
```

- Backtracking = DFS + variable-ordering + fail-on-violation
- What are the choice points?

Filtering: Forward Checking

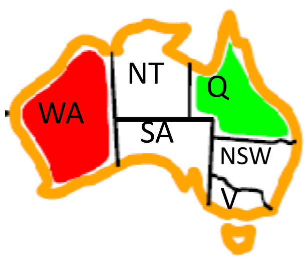
- Filtering: Keep track of domains for unassigned variables and cross off bad options
- Forward checking: Cross off values that violate a constraint when added to the existing assignment

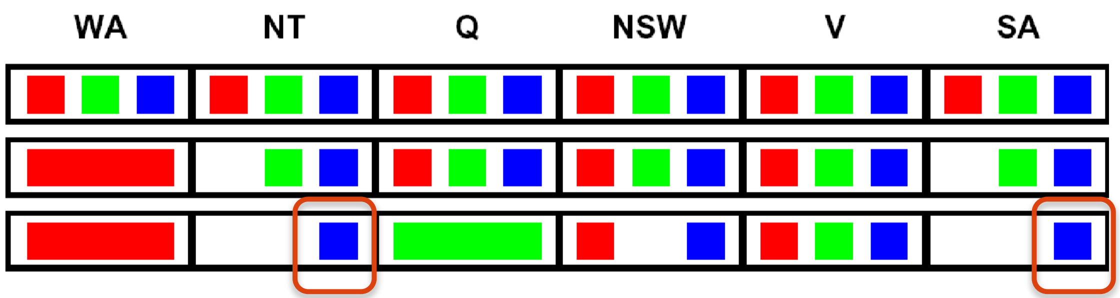


[Demo: coloring -- forward checking]

Filtering: Constraint Propagation

- Forward checking propagates information from assigned to unassigned variables, but doesn't provide early detection for all failures:

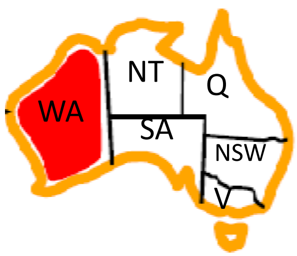


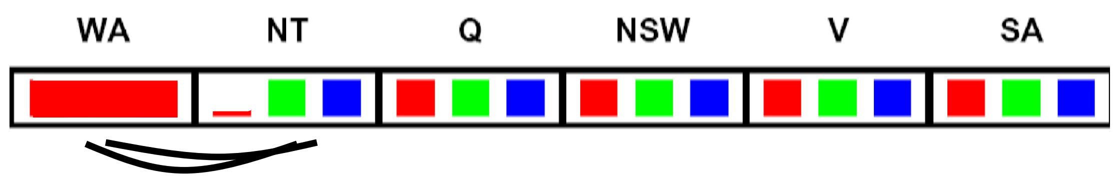


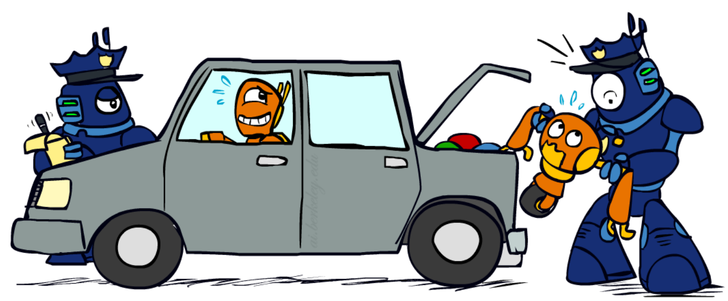
- NT and SA cannot both be blue!
- Why didn't we detect this yet?
- *Constraint propagation:* reason from constraint to constraint

Consistency of A Single Arc

- An arc X → Y is consistent iff for *every* x in the tail there is *some* y in the head which could be assigned without violating a constraint







Delete from the tail!

Forward checking?
Enforcing consistency of arcs pointing to each new assignment

Enforcing Arc Consistency in a CSP

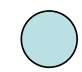
```
function AC-3( csp) returns the CSP, possibly with reduced domains
   inputs: csp, a binary CSP with variables {X1, X2, ..., Xn}
   local variables: queue, a queue of arcs, initially all the arcs in csp

   while queue is not empty do
      (Xi, Xj) ← REMOVE-FIRST(queue)
      if REMOVE-INCONSISTENT-VALUES(Xi, Xj) then
         for each Xk in NEIGHBORS[Xi] do
            add (Xk, Xi) to queue
```

```
function REMOVE-INCONSISTENT-VALUES( Xi, Xj) returns true iff succeeds
   removed ← false
   for each x in DOMAIN[Xi] do
      if no value y in DOMAIN[Xj] allows (x,y) to satisfy the constraint Xi ↔ Xj
         then delete x from DOMAIN[Xi]; removed ← true
   return removed
```

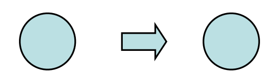
- Runtime: $O(n^2d^3)$, can be reduced to $O(n^2d^2)$
- … but detecting all possible future problems is NP-hard – why?

K-Consistency

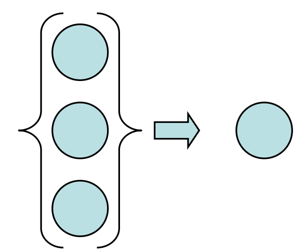
- Increasing degrees of consistency
  - 1-Consistency (Node Consistency): Each single node's domain has a value which meets that node's unary constraints



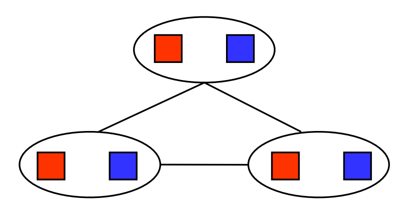
  - 2-Consistency (Arc Consistency): For each pair of nodes, any consistent assignment to one can be extended to the other



  - K-Consistency: For each k nodes, any consistent assignment to k-1 can be extended to the k^{th} node.



- Higher k more expensive to compute
- (You need to know the k=2 case: arc consistency)



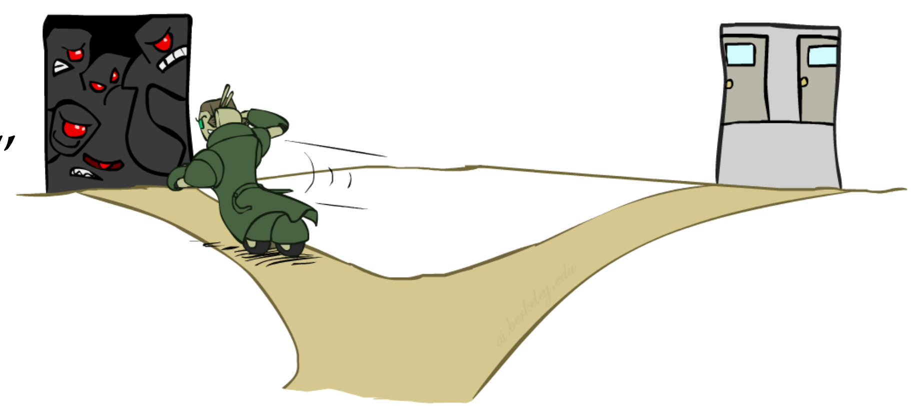
Strong K-Consistency

- Strong k-consistency: also k-1, k-2, … 1 consistent
- Claim: strong n-consistency means we can solve without backtracking!
- Why?
  - Choose any assignment to any variable
  - Choose a new variable
  - By 2-consistency, there is a choice consistent with the first
  - Choose a new variable
  - By 3-consistency, there is a choice consistent with the first 2
  - …
- Lots of middle ground between arc consistency and n-consistency! (e.g. k=3, called path consistency)

Ordering: Minimum Remaining Values

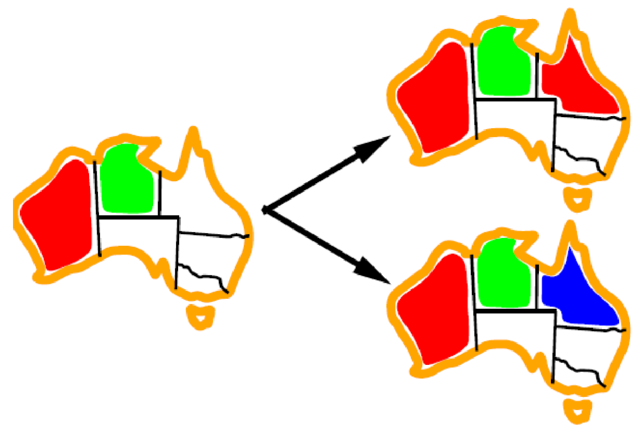
- Variable Ordering: Minimum remaining values (MRV):
  - Choose the variable with the fewest legal left values in its domain

- Why min rather than max?
- Also called "most constrained variable"
- "Fail-fast" ordering

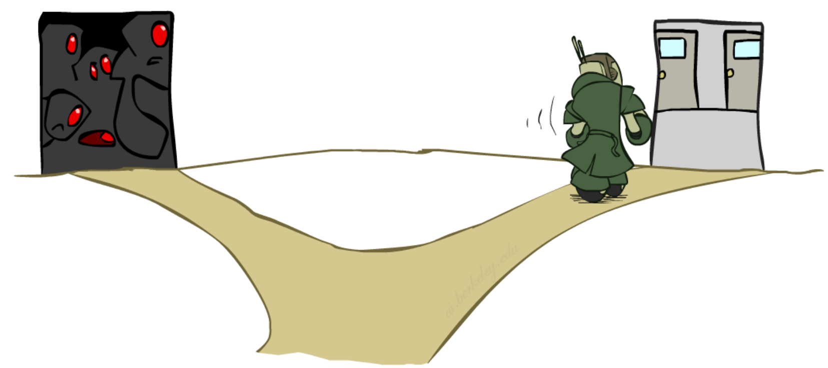


Ordering: Least Constraining Value

- Value Ordering: Least Constraining Value
  - Given a choice of variable, choose the *least constraining value*
  - I.e., the one that rules out the fewest values in the remaining variables
  - Note that it may take some computation to determine this! (E.g., rerunning filtering)



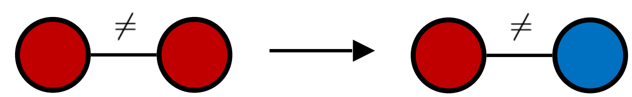
- Why least rather than most?
- Combining these ordering ideas makes 1000 queens feasible



Iterative Algorithms for CSPs

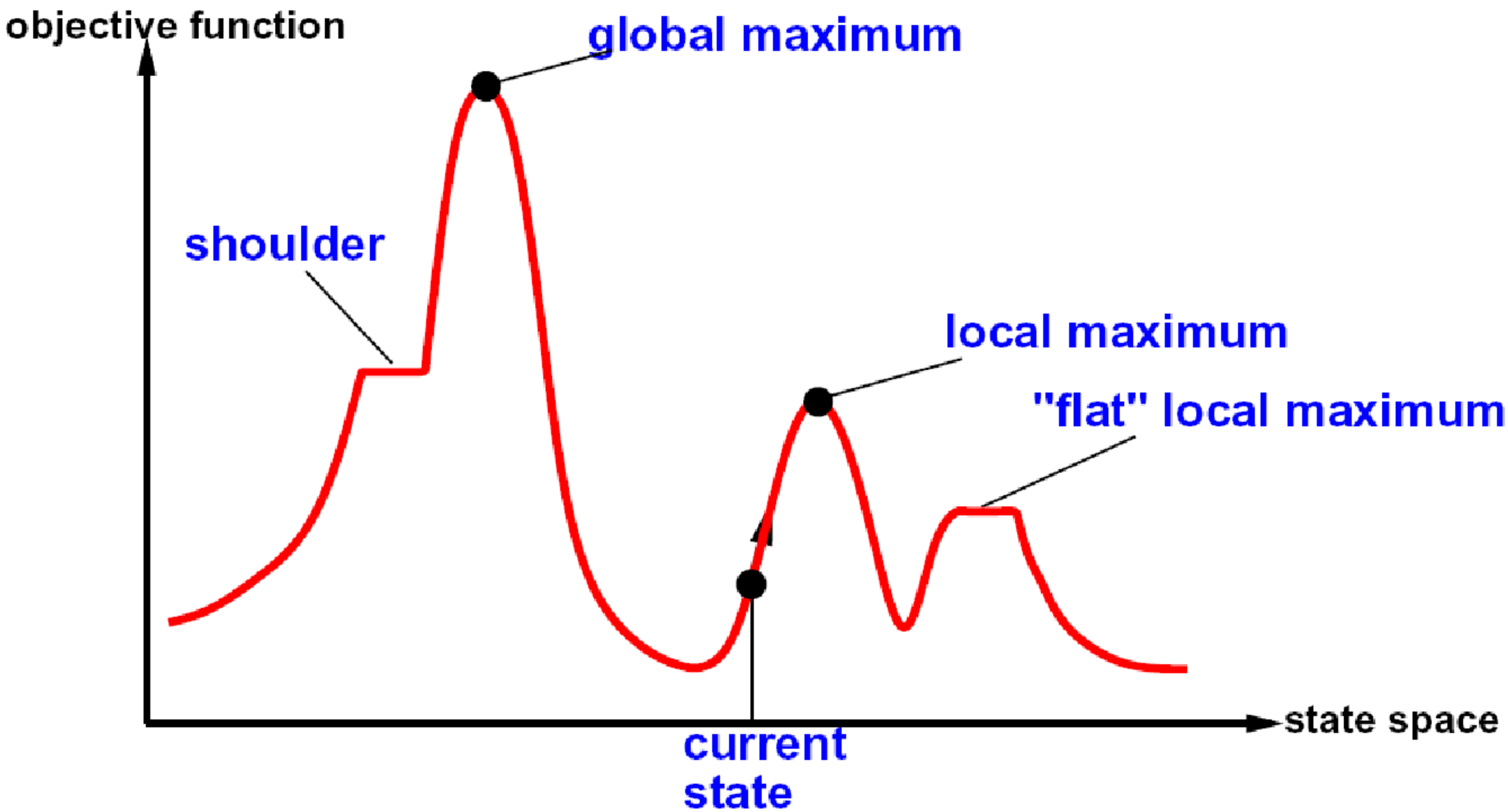
- Local search methods typically work with "complete" states, i.e., all variables assigned

- To apply to CSPs:
  - Take an assignment with unsatisfied constraints
  - Operators *reassign* variable values
  - No fringe! Live on the edge.



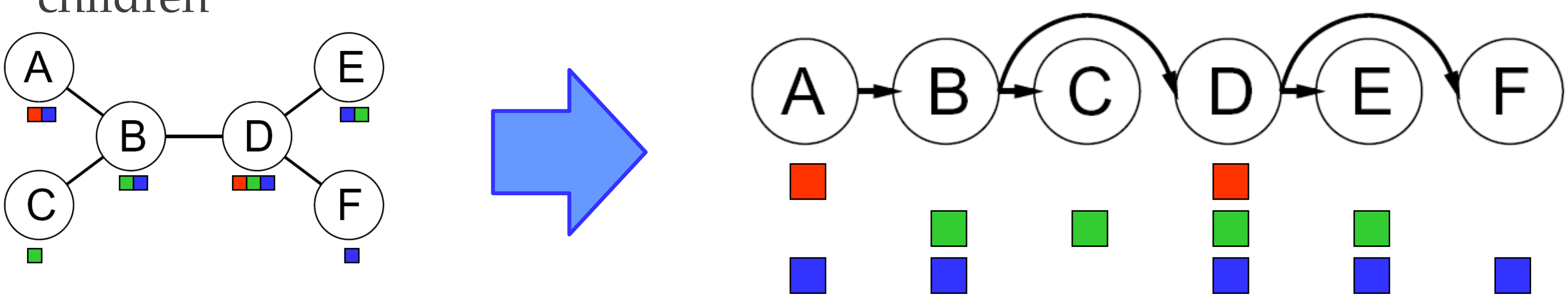
- Algorithm: While not solved,
  - Variable selection: randomly select any conflicted variable
  - Value selection: min-conflicts heuristic:
    - Choose a value that violates the fewest constraints
    - I.e., hill climb with h(x) = total number of violated constraints

Hill Climbing



Tree-Structured CSPs

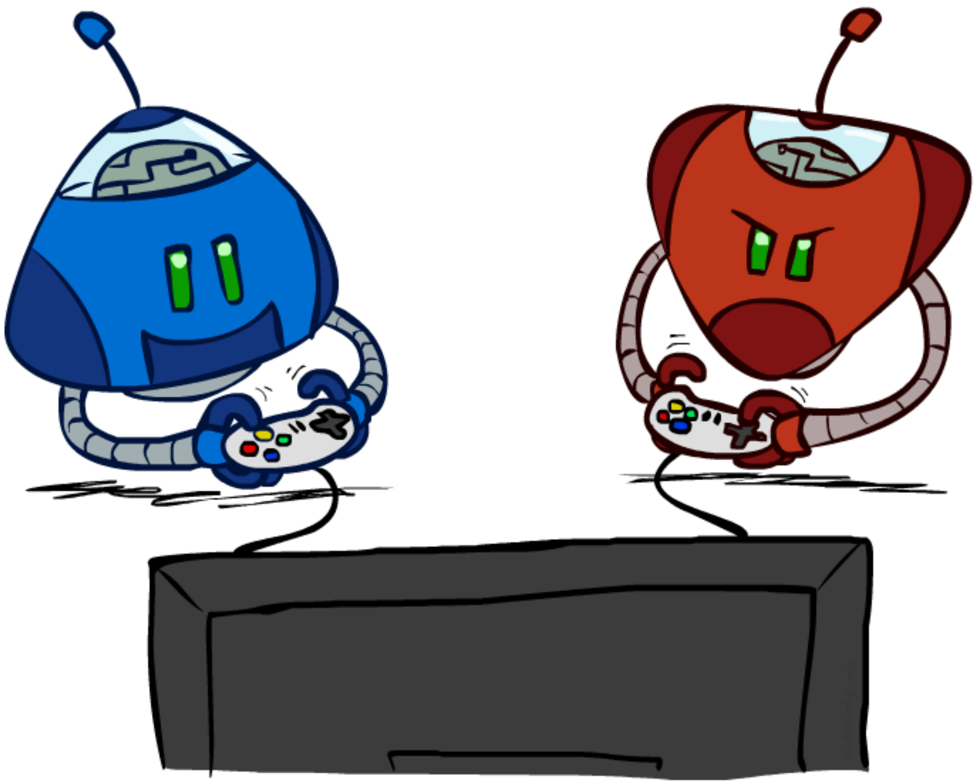
- Algorithm for tree-structured CSPs:
  - Order: Choose a root variable, order variables so that parents precede children



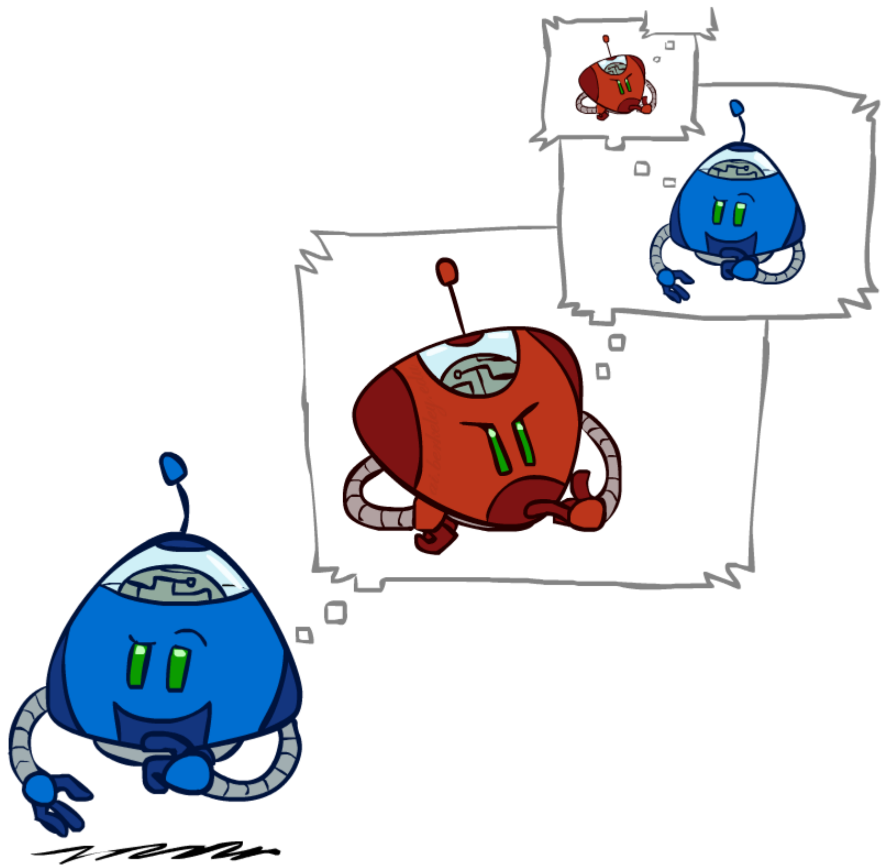
  - Remove backward: For $i = n : 2$, apply RemoveInconsistent(Parent(X_i),X_i)
  - Assign forward: For $i = 1 : n$, assign X_i consistently with Parent(X_i)

- Runtime: $O(n\ d^2)$ (why?)

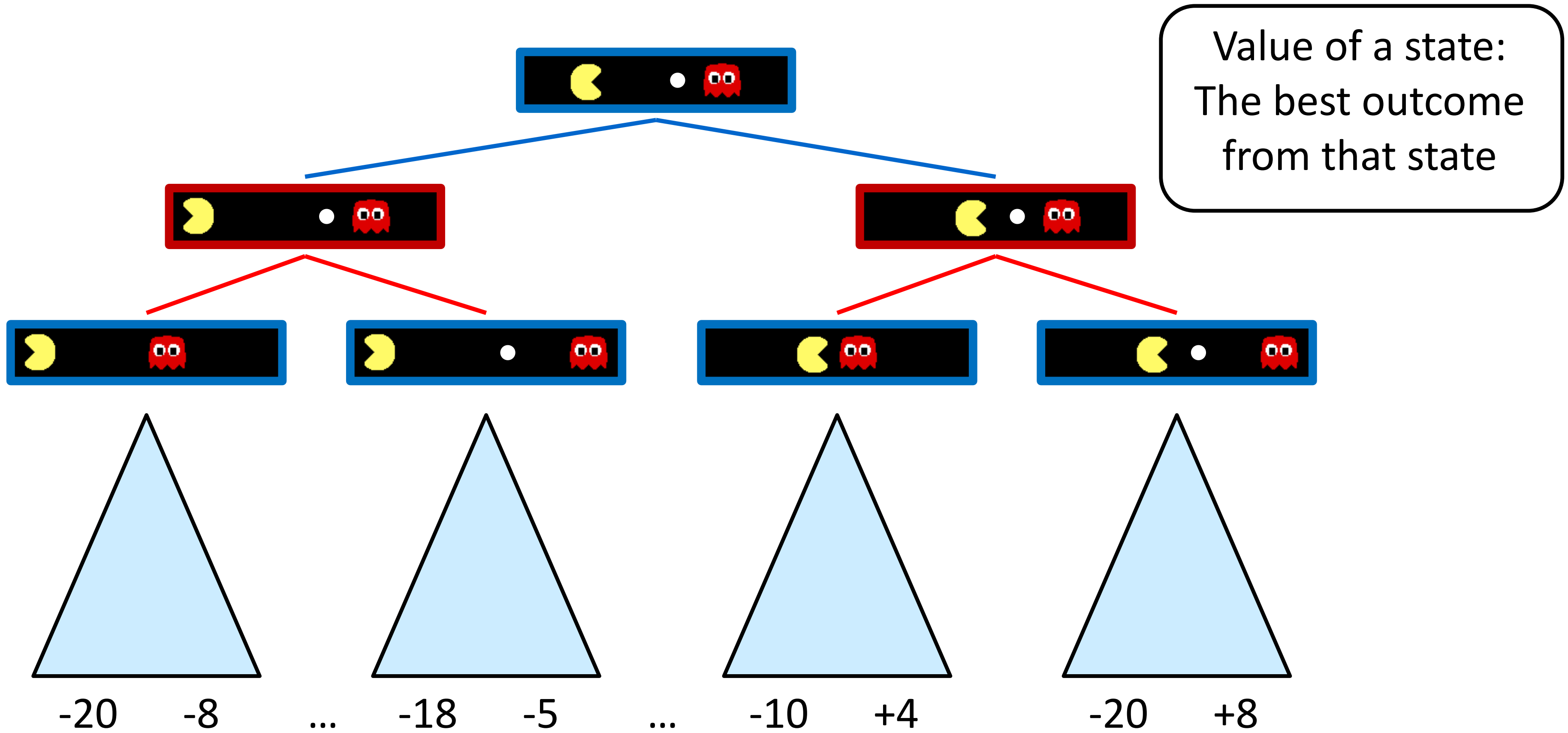
Game Playing: Search with other agents



Adversarial Search



Adversarial Game Trees



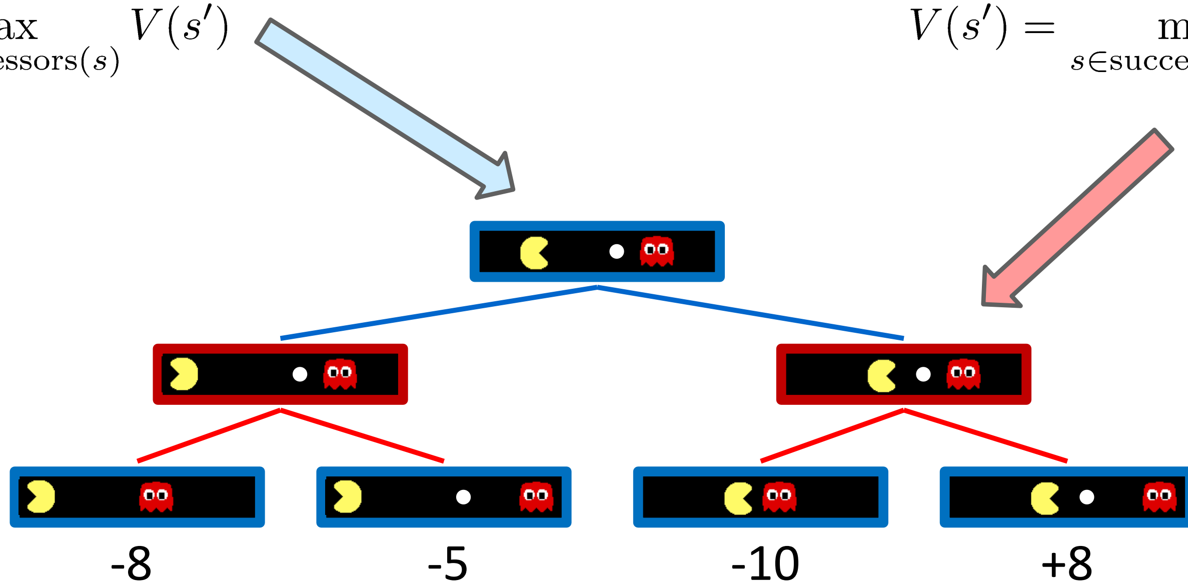
Minimax Values

States Under Agent's Control:

$$V(s) = \max_{s' \in \text{successors}(s)} V(s')$$

States Under Opponent's Control:

$$V(s') = \min_{s \in \text{successors}(s')} V(s)$$



Terminal States:

$$V(s) = \text{known}$$

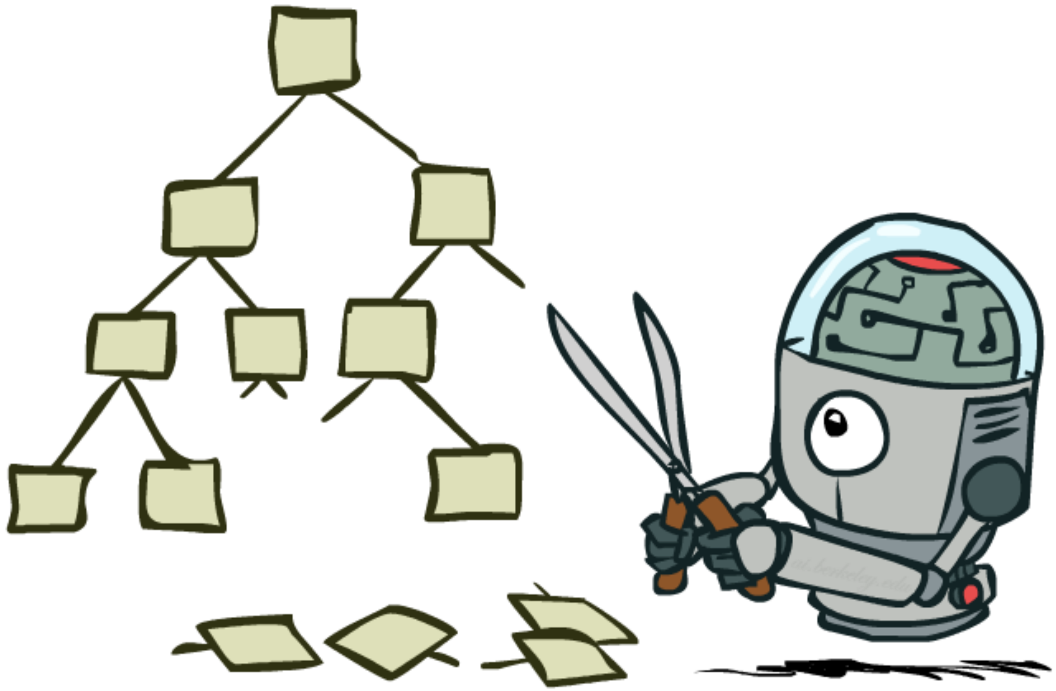
Minimax Implementation (Dispatch)

```
def value(state):
    if the state is terminal: return the state's utility
    if the next agent is MAX: return max-value(state)
    if the next agent is MIN: return min-value(state)
```

```
def max-value(state):
    initialize v = -∞
    for each successor of state:
        v = max(v, value(successor))
    return v
```

```
def min-value(state):
    initialize v = +∞
    for each successor of state:
        v = min(v, value(successor))
    return v
```

Game Tree Pruning



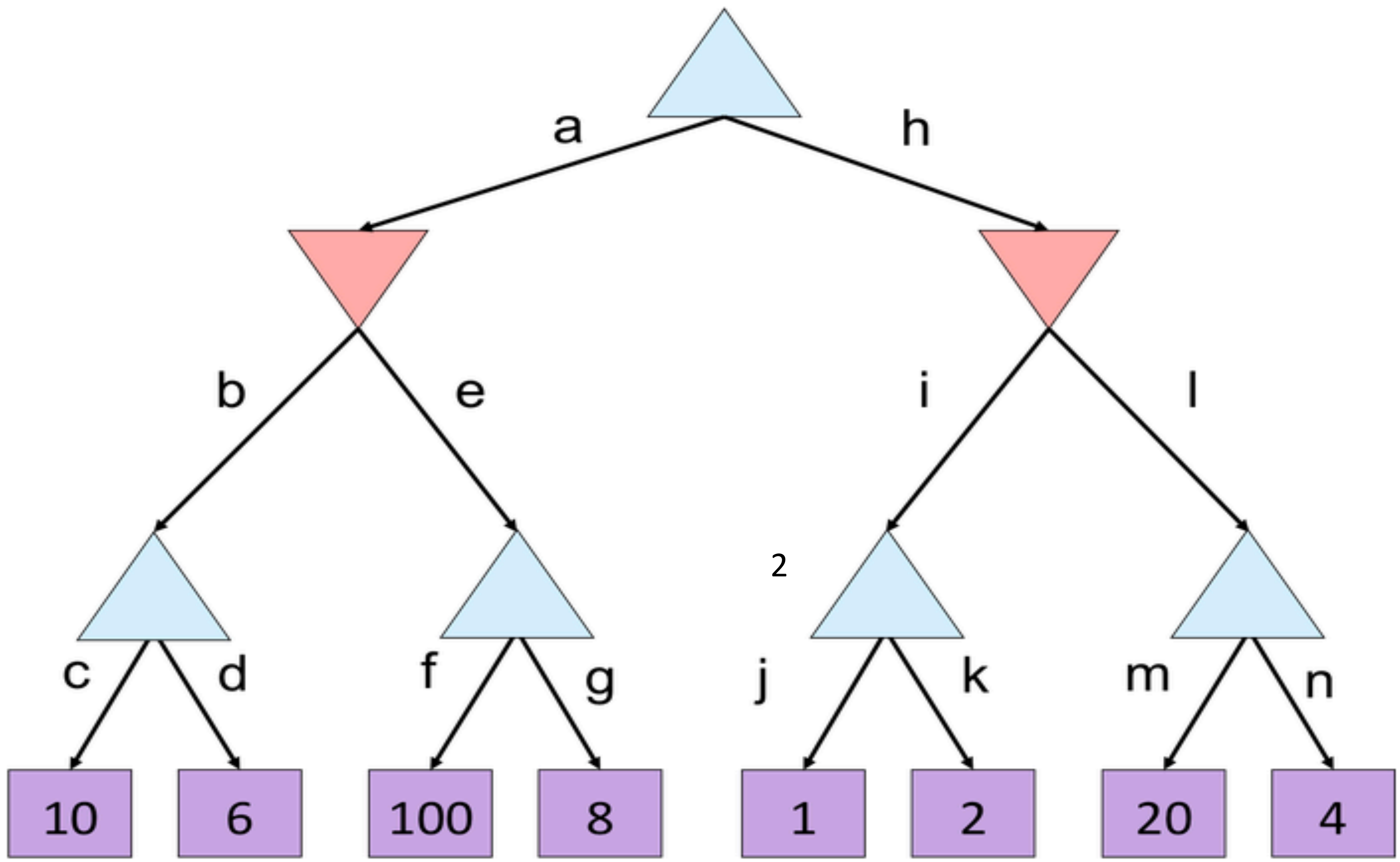
Alpha-Beta Implementation

α: MAX's best option on path to root
β: MIN's best option on path to root

```
def max-value(state, α, β):
    initialize v = -∞
    for each successor of state:
        v = max(v, value(successor, α, β))
        if v ≥ β return v
        α = max(α, v)
    return v
```

```
def min-value(state , α, β):
    initialize v = +∞
    for each successor of state:
        v = min(v, value(successor, α, β))
        if v ≤ α return v
        β = min(β, v)
    return v
```

Alpha-Beta Example



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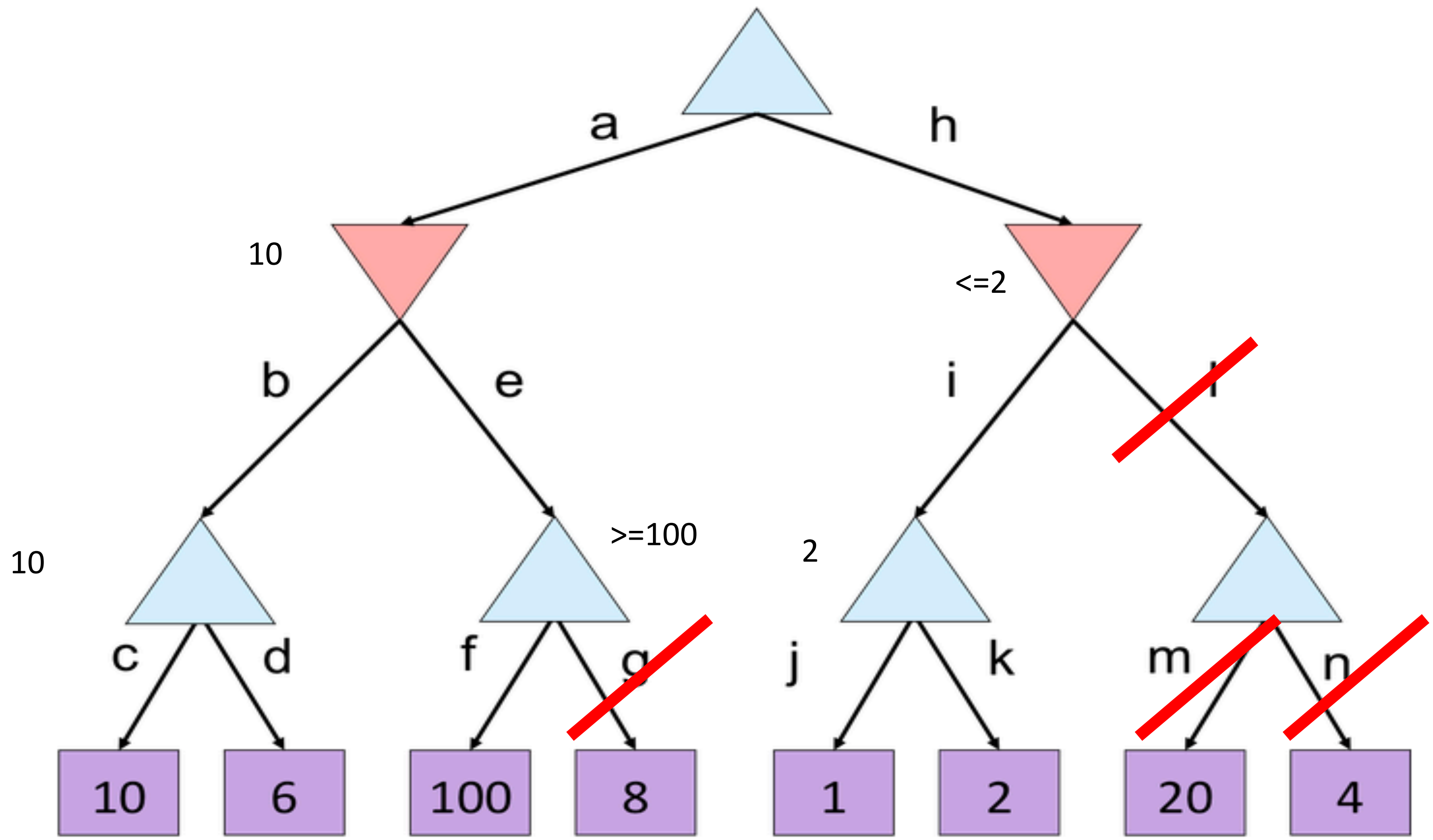
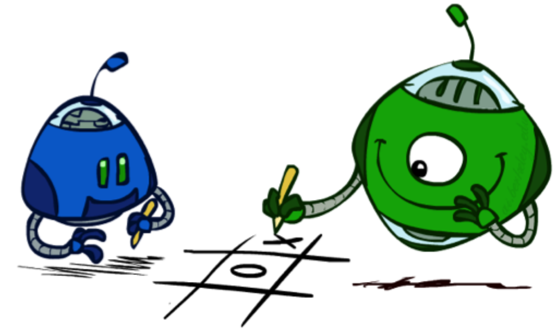
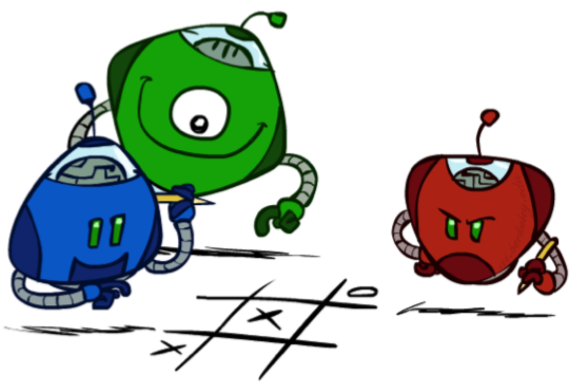
Alpha-Beta Quiz 2

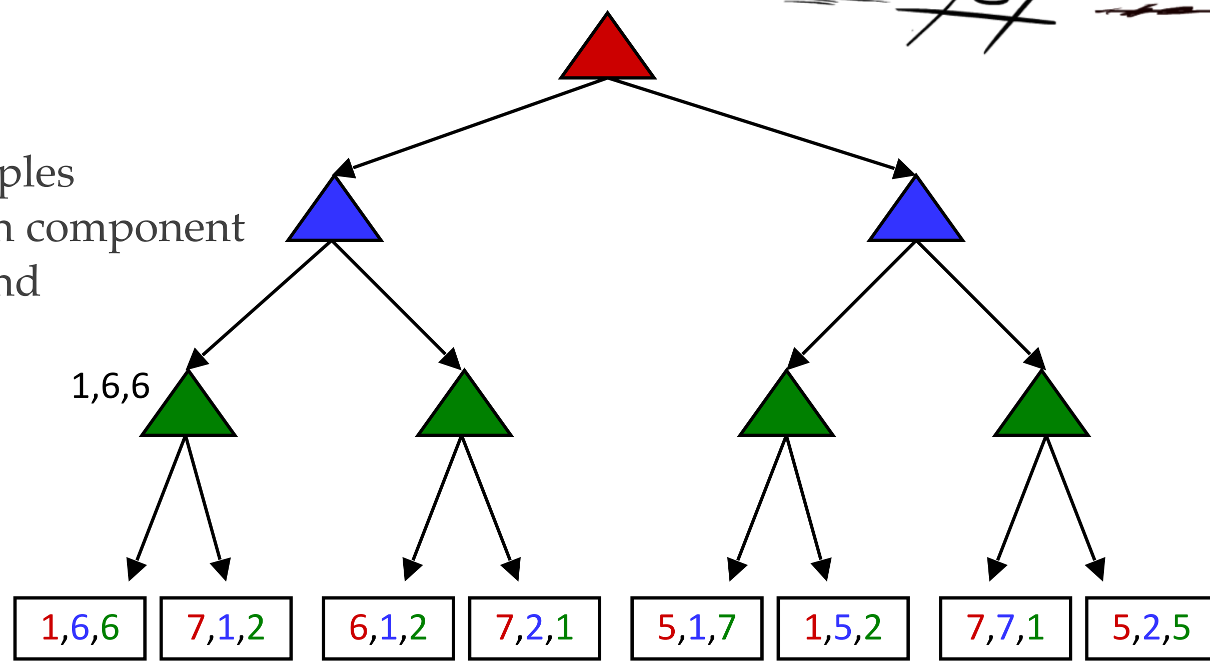


Multi-Agent Utilities



- What if the game is not zero-sum, or has multiple players?
- Generalization of minimax:
  - Terminals have utility tuples
  - Node values are also utility tuples
  - Each player maximizes its own component
  - Can give rise to cooperation and competition dynamically…



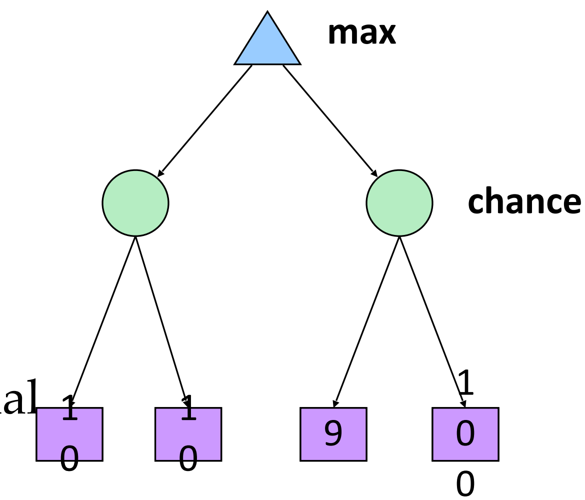


Chance Nodes

- We don't know what the result of an action will be:
  - Explicit randomness: rolling dice
  - Unpredictable opponents
  - Actions can fail

- Values should now reflect average-case (expectimax) outcomes, not worst-case (minimax) outcomes

- Expectimax search: compute the average score under optimal play
  - Max nodes as in minimax search
  - Chance nodes: calculate expected utilities



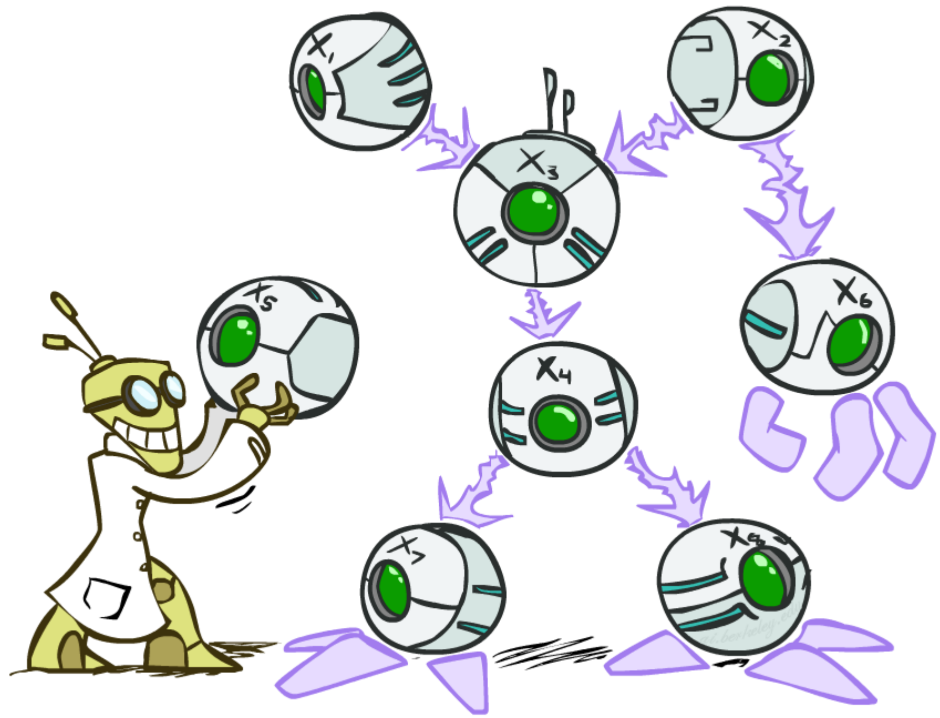
Expectimax Pseudocode

```
def value(state):
    if the state is a terminal state: return the state's utility
    if the next agent is MAX: return max-value(state)
    if the next agent is EXP: return exp-value(state)
```

```
def max-value(state):
    initialize v = -∞
    for each successor of state:
        v = max(v, value(successor))
    return v
```

```
def exp-value(state):
    initialize v = 0
    for each successor of state:
        p = probability(successor)
        v += p * value(successor)
    return v
```

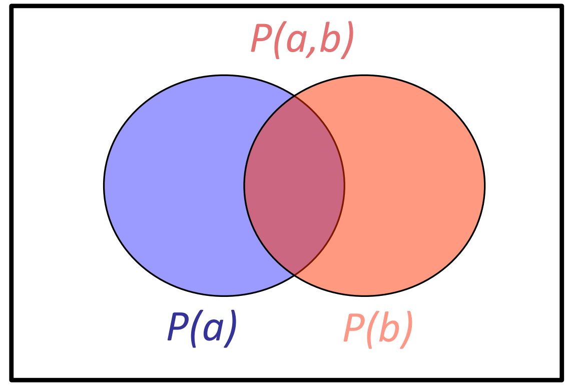
Bayesian Networks



Conditional Probabilities

- Bayes Rule

$$P(a|b) = \frac{P(a,b)}{P(b)}$$



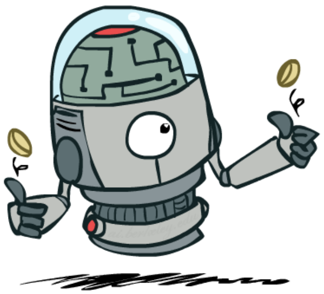
$$P(T,W)$$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

$$P(W=s|T=c) = \frac{P(W=s,T=c)}{P(T=c)} = \frac{0.2}{0.5} = 0.4$$

$$= P(W=s,T=c) + P(W=r,T=c)$$
$$= 0.2 + 0.3 \quad = 0.5$$

Conditional Independence



- X and Y are independent iff

$$\forall x, y \;\; P(x, y) = P(x)P(y) \qquad\qquad X \perp\!\!\!\perp Y$$

- Given Z, we say X and Y are conditionally independent iff

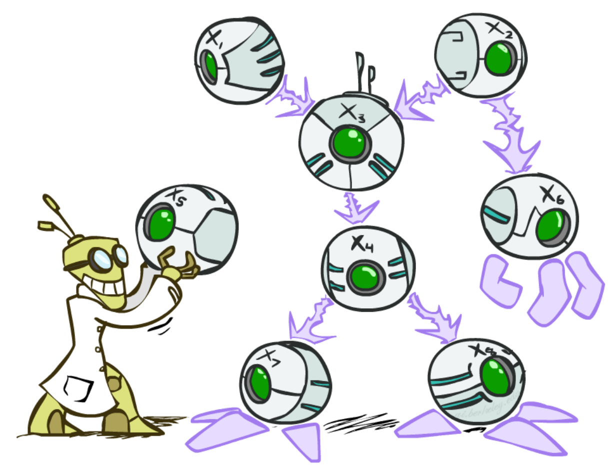
$$\forall x, y, z \;\; P(x, y|z) = P(x|z)P(y|z) \dashrightarrow X \perp\!\!\!\perp Y|Z$$

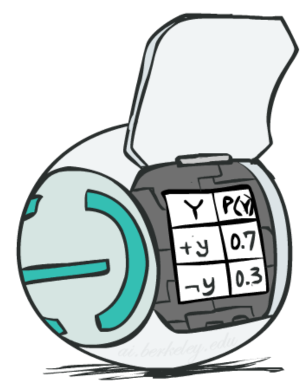
- (Conditional) independence is a property of a distribution

Bayesian Networks

- A directed acyclic graph (DAG), one node per random variable
- A conditional probability table (CPT) for each node
  - Probability of X, given a combination of values for parents.
    $$P(X|a_1 \dots a_n)$$
- Bayes nets implicitly encode joint distributions as a product of local conditional distributions
  - To see what probability a BN gives to a full assignment, multiply all the relevant conditionals together:

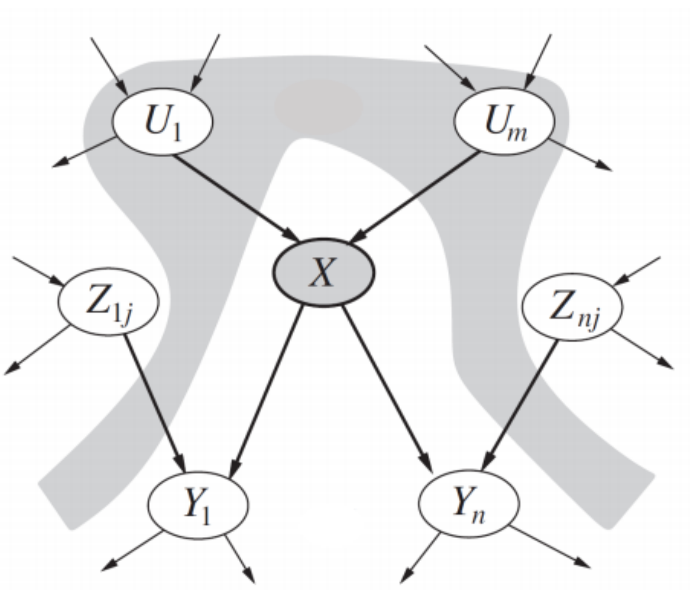
$$P(x_1, x_2, \dots x_n) = \prod_{i=1}^{n} P(x_i|\mathit{parents}(X_i))$$



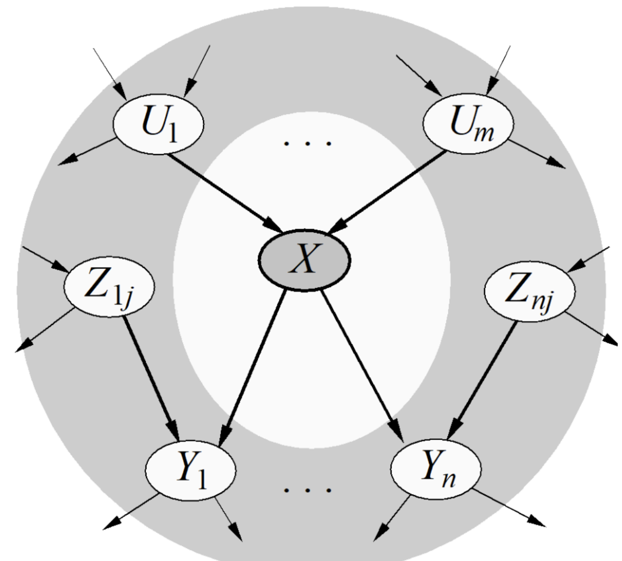


Independence Assumptions

- Definition: Each node, given its parents, is conditionally independent of all its non-descendants in the graph



Each node, given its MarkovBlanket, is conditionally independent of all other nodes in the graph



MarkovBlanket refers to the parents, children, and children's other parents.

Inference by Enumeration

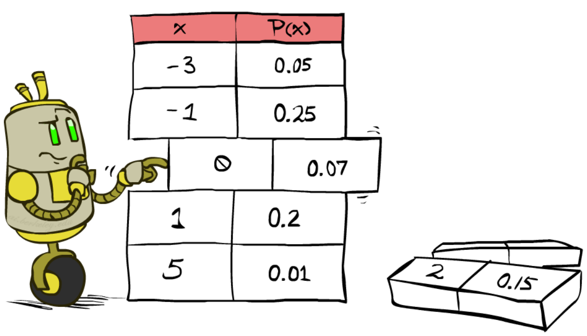
- General case:
  - Evidence variables: $E_1 \dots E_k = e_1 \dots e_k$
  - Query* variable: Q
  - Hidden variables: $H_1 \dots H_r$

$\left.\begin{matrix} E_1 \dots E_k = e_1 \dots e_k \\ Q \\ H_1 \dots H_r \end{matrix}\right\} X_1, X_2, \dots X_n$ *All variables*

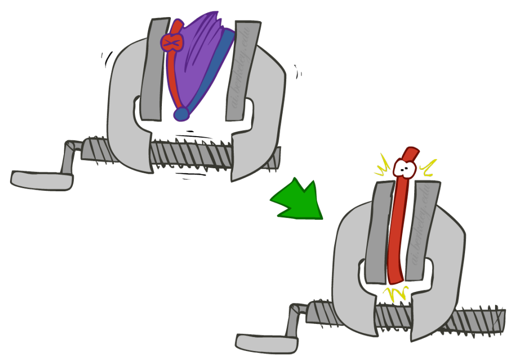
- We want:

$$P(Q|e_1 \dots e_k)$$

- Step 1: Select the entries consistent with the evidence



- Step 2: Sum out H to get joint of Query and evidence



$$P(Q, e_1 \dots e_k) = \sum_{h_1 \dots h_r} \underbrace{P(Q, h_1 \dots h_r, e_1 \dots e_k)}_{X_1, X_2, \dots X_n}$$

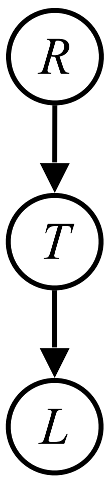
- Step 3: Normalize

$$\times \frac{1}{Z}$$

$$Z = \sum_q P(Q, e_1 \cdots e_k)$$

$$P(Q|e_1 \cdots e_k) = \frac{1}{Z} P(Q, e_1 \cdots e_k)$$

Inference on Bayes Nets



$$P(L) = ?$$

- Inference by Enumeration

$$= \sum_t \sum_r P(L|t)P(r)P(t|r)$$

Join on r

Join on t

Eliminate r

Eliminate t

- Variable Elimination

$$= \sum_t P(L|t) \sum_r P(r)P(t|r)$$

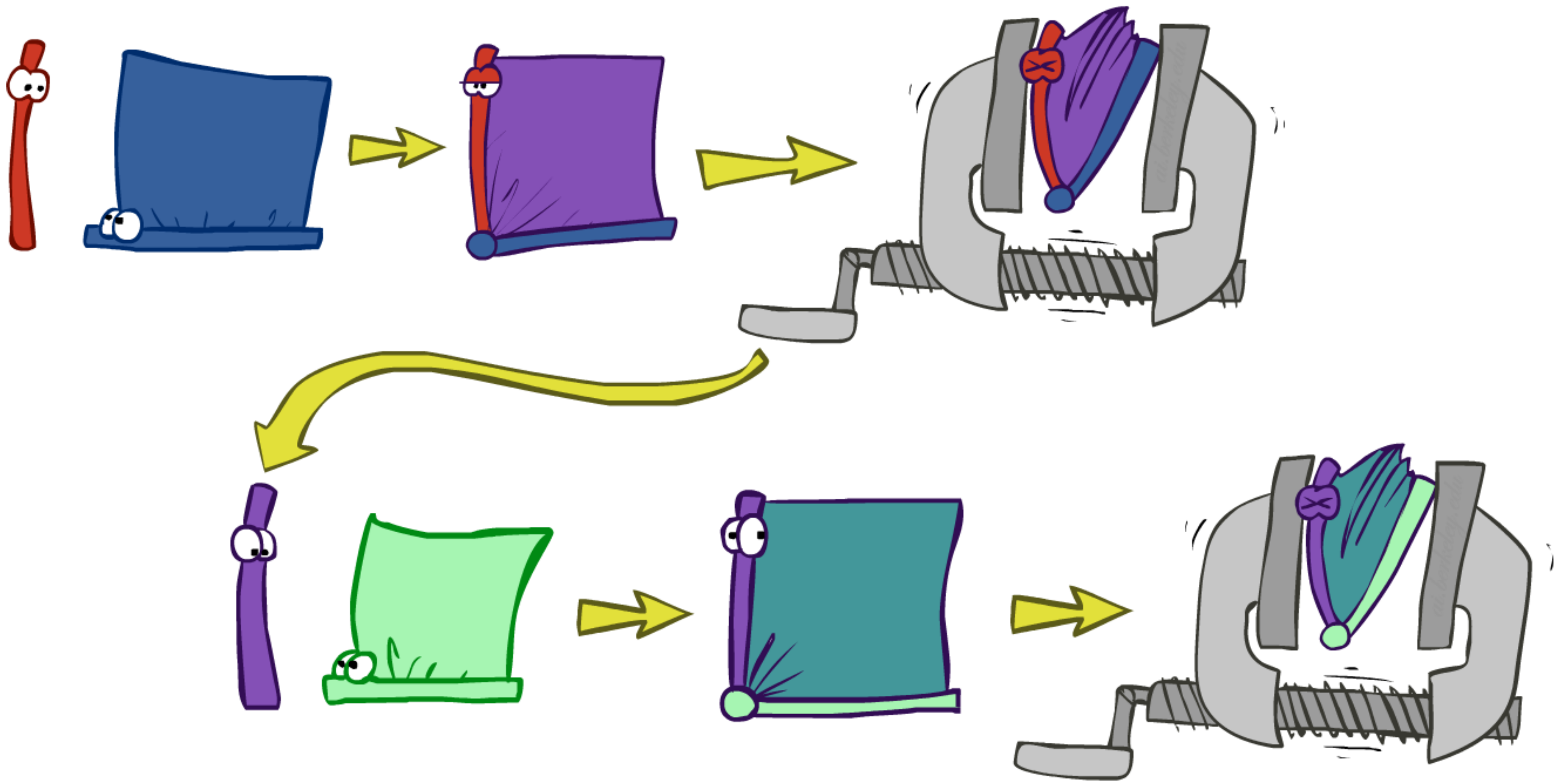
Join on r

Eliminate r

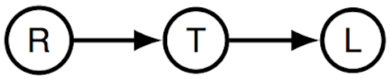
Join on t

Eliminate t

Marginalizing Early (Variable Elimination)



Variable Elimination



$P(R)$

T	L
+r	0.1
-r	0.9

$P(T|R)$

+r	+t	0.8
+r	-t	0.2
-r	+t	0.1
-r	-t	0.9

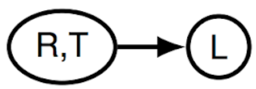
$P(L|T)$

+t	+l	0.3
+t	-l	0.7
-t	+l	0.1
-t	-l	0.9

Join R

$P(R, T)$

+r	+t	0.08
+r	-t	0.02
-r	+t	0.09
-r	-t	0.81

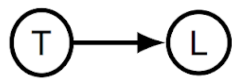


$P(L|T)$

+t	+l	0.3
+t	-l	0.7
-t	+l	0.1
-t	-l	0.9

Sum out R

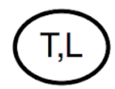
+t	0.17
-t	0.83



$P(L|T)$

+t	+l	0.3
+t	-l	0.7
-t	+l	0.1
-t	-l	0.9

Join T



$P(T, L)$

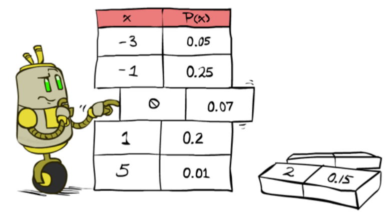
+t	+l	0.051
+t	-l	0.119
-t	+l	0.083
-t	-l	0.747

Sum out T

+l	0.134
-l	0.866

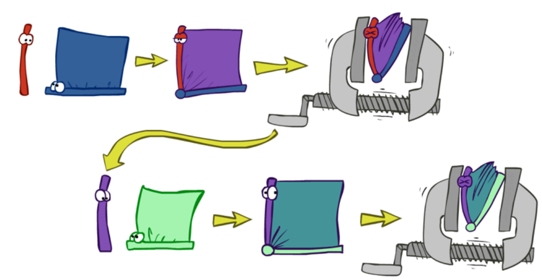
General Variable Elimination

- Query: $P(Q|E_1 = e_1, \dots E_k = e_k)$



- Start with initial factors:
  - Local CPTs (but instantiated by evidence)

- While there are still hidden variables (not Q or evidence):
  - Pick a hidden variable H
  - Join all factors mentioning H
  - Eliminate (sum out) H



- Join all remaining factors and normalize

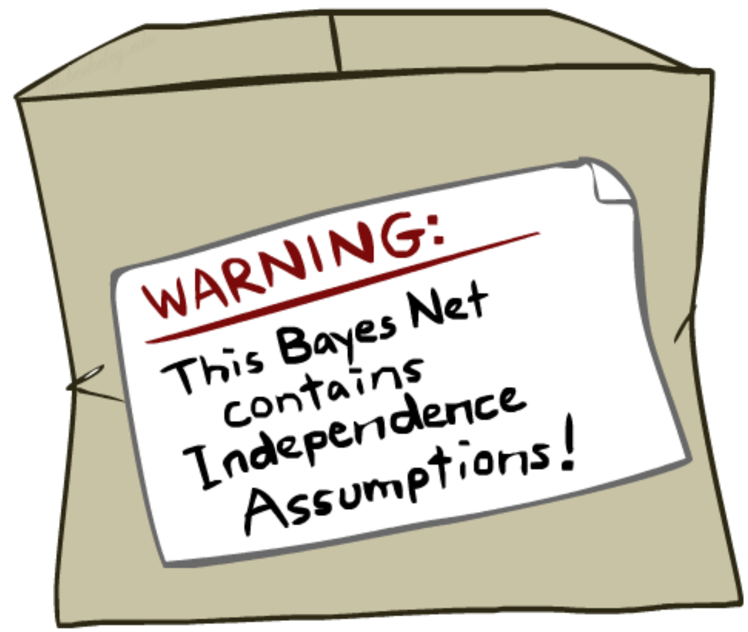
$\times \frac{1}{Z}$

Independence Assumptions in a Bayes Net

- Assumptions we are required to make to define the Bayes net when given the graph:

$$P(x_i|x_1 \cdots x_{i-1}) = P(x_i|parents(X_i))$$

- Important for modeling: understand assumptions made when choosing a Bayes net graph

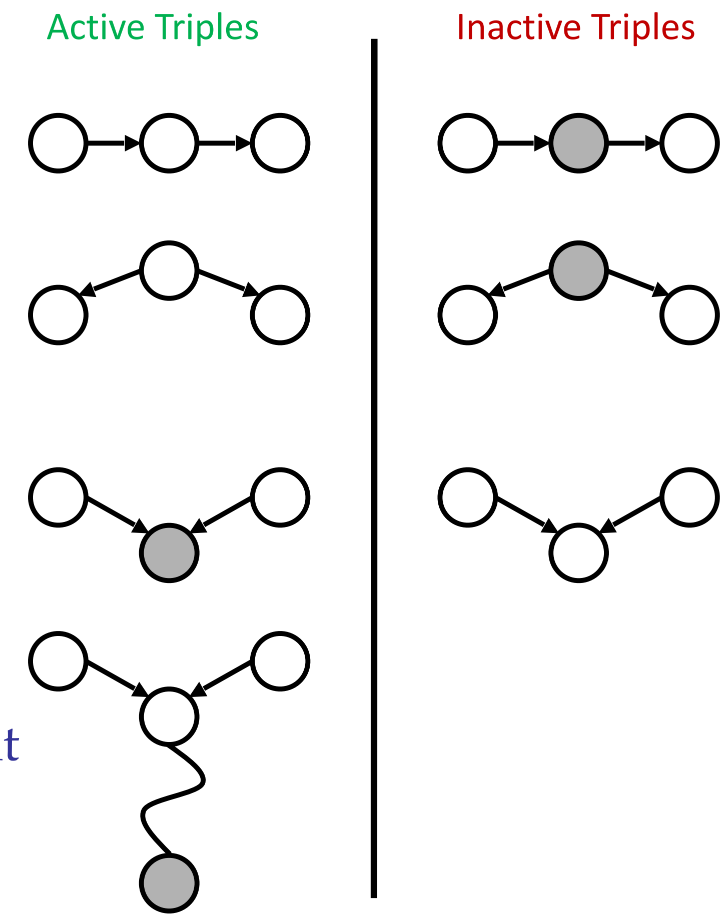


Active / Inactive Paths

- Question: Are X and Y conditionally independent given evidence variables {Z}?
  - Yes, if X and Y "d-separated" by Z
  - Consider all (undirected) paths from X to Y
  - No active paths = independence!

- A path is active if each triple is active:
  - Causal chain A -> B -> C where B is unobserved (either direction)
  - Common cause A <- B -> C where B is unobserved
  - Common effect (aka v-structure)
    A -> B <- C where B *or one of its descendants* is observed

- All it takes to block a path is a single inactive segment



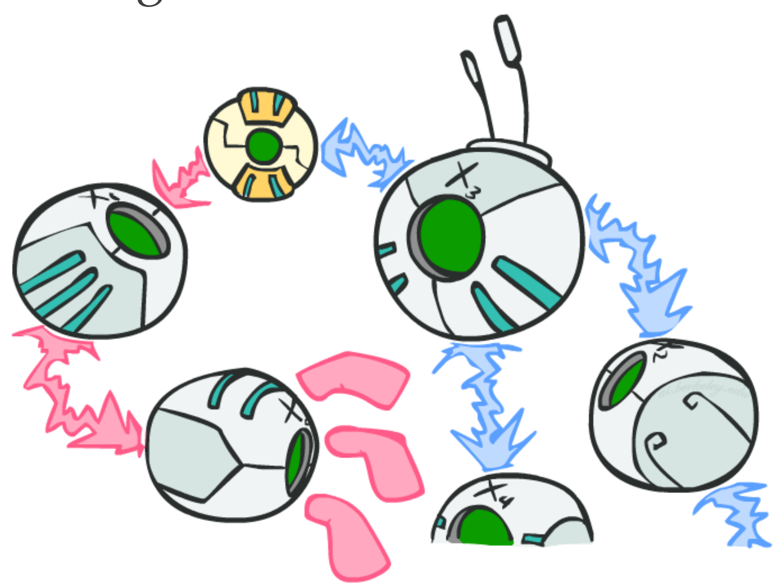
D-Separation

- Query: $X_i \perp\!\!\!\perp X_j | \{X_{k_1}, ..., X_{k_n}\}$?
- Check all (undirected!) paths between X_i and X_j
  - If one or more active paths, then independence not guaranteed

$$X_i \not\perp\!\!\!\perp X_j | \{X_{k_1}, ..., X_{k_n}\}$$

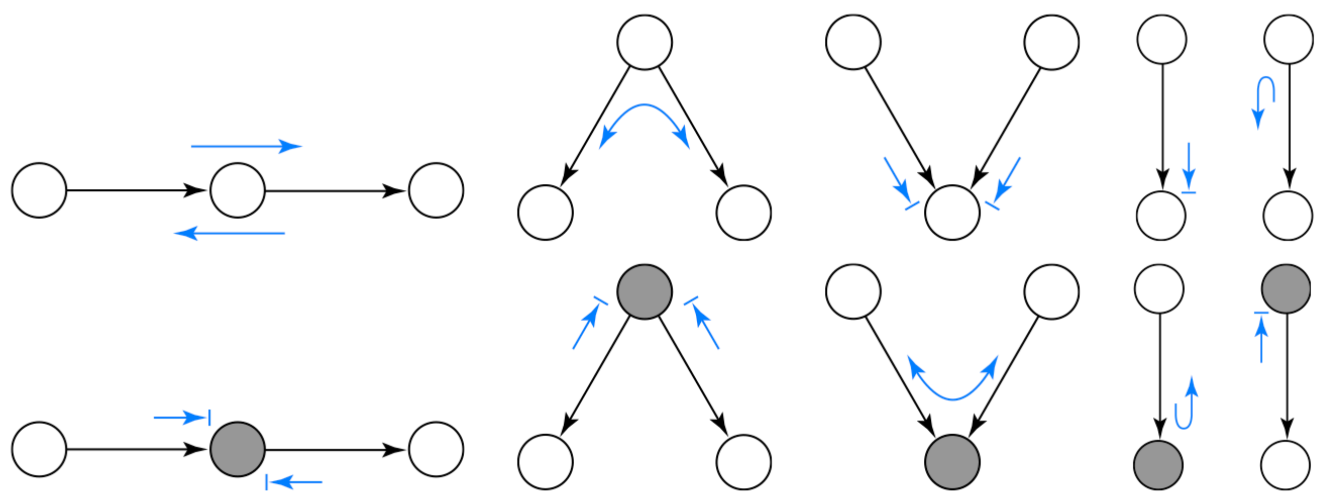
  - Otherwise (i.e. if all paths are inactive), then independence is guaranteed

$$X_i \perp\!\!\!\perp X_j | \{X_{k_1}, ..., X_{k_n}\}$$



Another Perspective: Bayes Ball

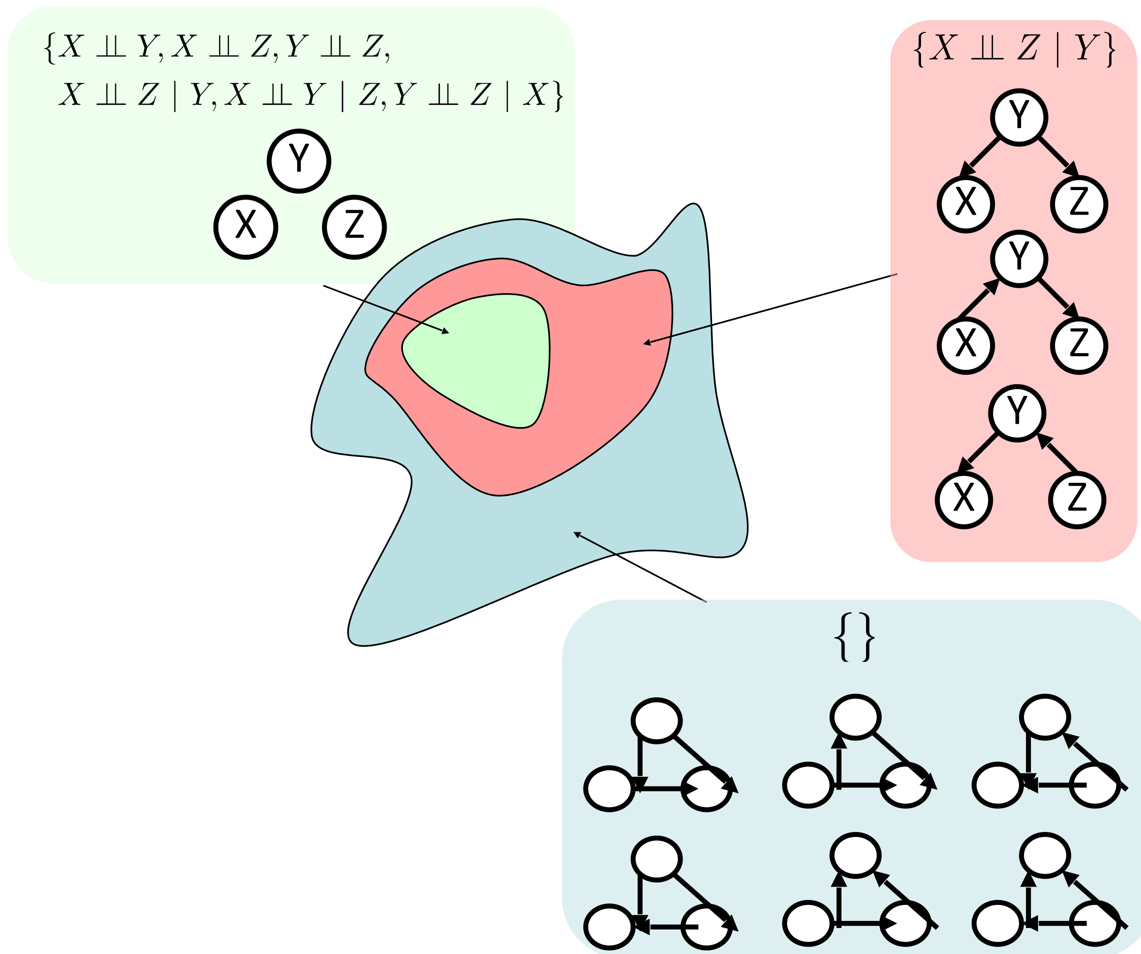
An undirected path is active if a Bayes ball travelling along it never encounters the "stop" symbol: $\longrightarrow\!\!|$



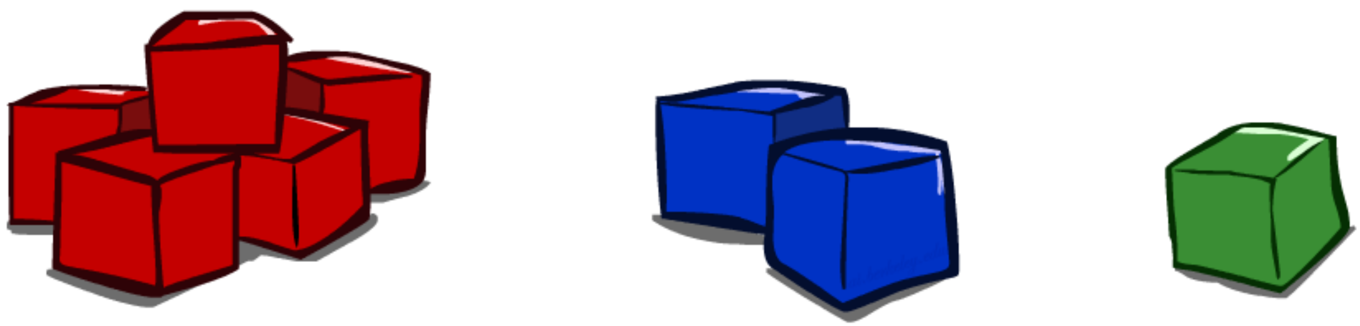
If there are no active paths from X to Y when $\{Z_1, \ldots, Z_k\}$ are shaded, then $X \perp\!\!\!\perp Y \mid \{Z_1, \ldots, Z_k\}$.

Topology Limits Distributions

- Given some graph topology G, only certain joint distributions can be encoded
- The graph structure guarantees certain (conditional) independences
- (There might be more independence)
- Adding arcs increases the set of distributions, but has several costs
- Full conditioning can encode any distribution

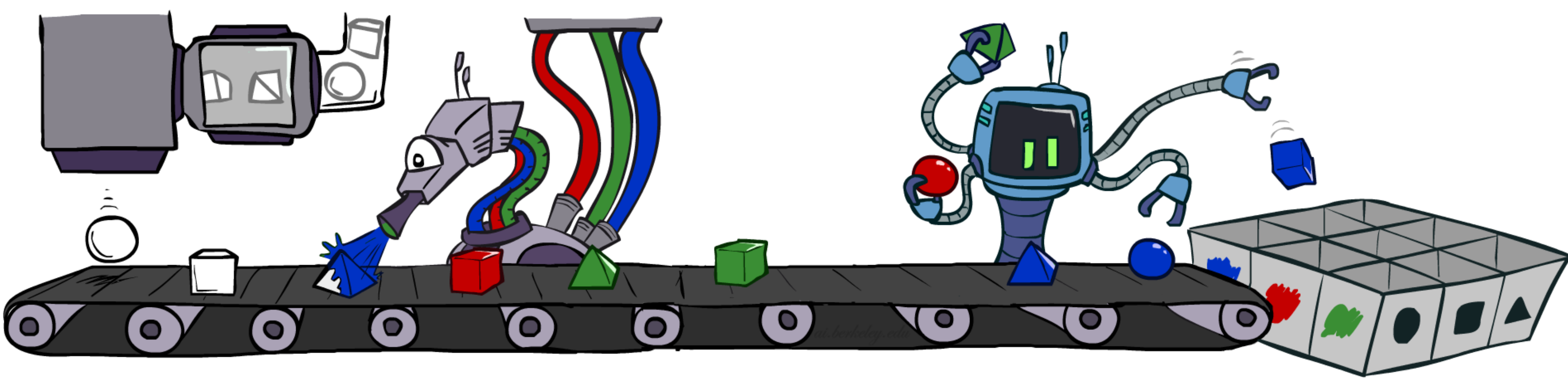


Approximate Inference: Sampling



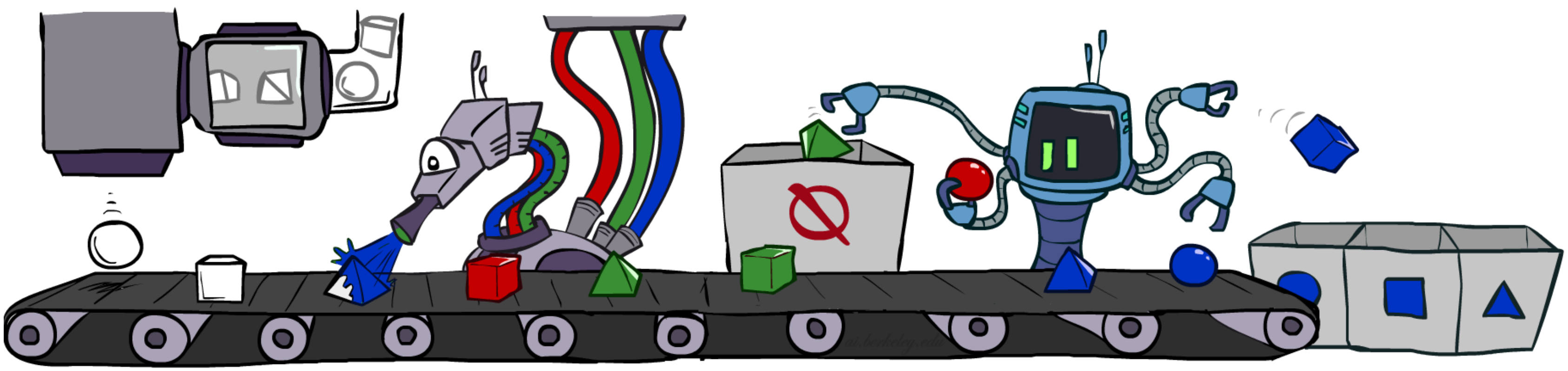
Prior Sampling

- For $i = 1, 2, \ldots, n$ in topological order
  - Sample x_i from $P(X_i \mid Parents(X_i))$
- Return $(x_1, x_2, \ldots, x_n)$



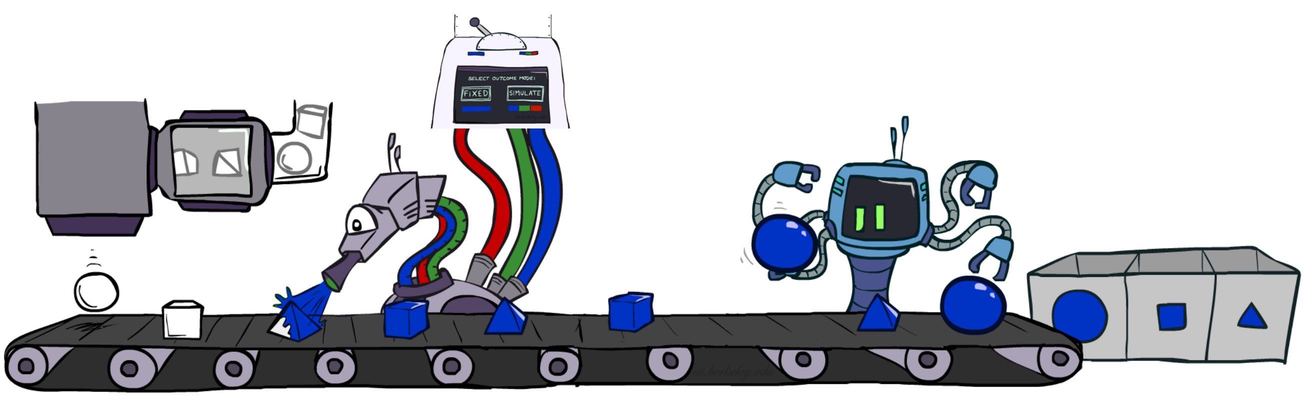
Rejection Sampling

- Input: evidence instantiation
- For $i = 1, 2, \ldots, n$ in topological order
  - Sample x_i from $P(X_i \mid Parents(X_i))$
  - If x_i not consistent with evidence
    - Reject: return – no sample is generated in this cycle
- Return $(x_1, x_2, \ldots, x_n)$



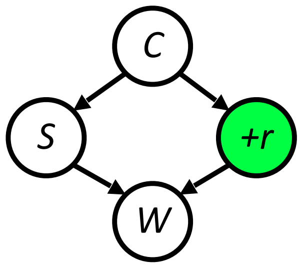
Likelihood Weighting

- Input: evidence instantiation
- w = 1.0
- for i = 1, 2, …, n in topological order
  - if X_i is an evidence variable
    - X_i = observation x_i for X_i
    - Set w = w * $P(x_i \mid \text{Parents}(X_i))$
  - else
    - Sample x_i from $P(X_i \mid \text{Parents}(X_i))$
- return $(x_1, x_2, \ldots, x_n)$, w

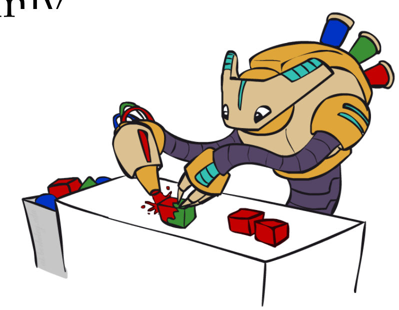


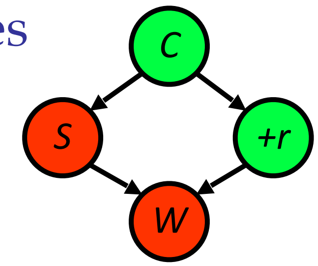
Gibbs Sampling

- Step 1: Fix evidence
  - R = +r

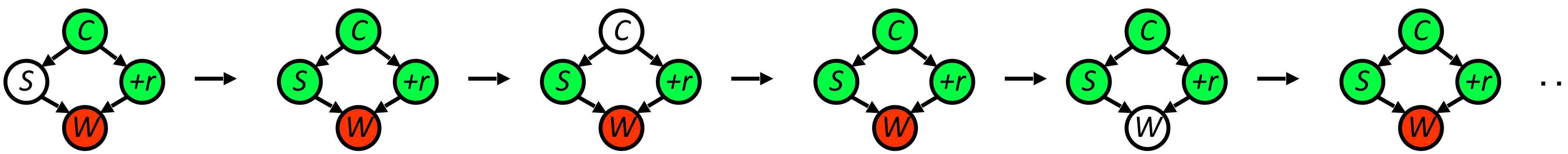


- Step 2: Initialize other variables
  - Randomly





- Steps 3: Repeat:
  - Choose a non-evidence variable X
  - Resample X from P(X | MarkovBlanket(X))



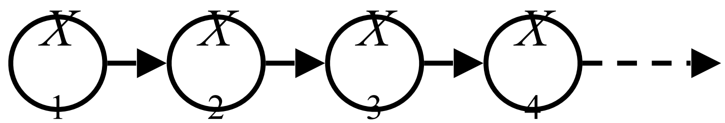
Sample from $P(S|+c,-w,+r)$ Sample from $P(C|+s,-w,+r)$ Sample from $P(W|+s,+c,+r)$

Hidden Markov Models

Markov Chains (Review from EE 16A, CS 70)

- Value of X at a given time is called the state

P(X$_0$)	
sun	rain
1	0.0

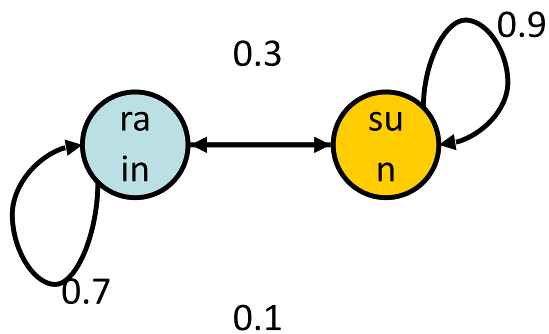


$P(X_1)$ $\quad$ $P(X_t|X_{t-1})$

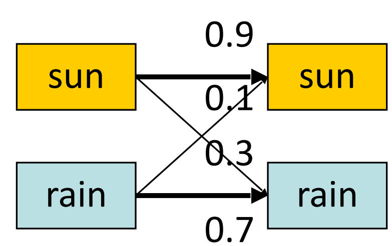
$$P(X_t) = ?$$

X_{t-1}	X_t	$P(X_t \mid X_{t-1})$
sun	sun	0.9
sun	rain	0.1
rain	sun	0.3
rain	rain	0.7

State Transition Diagram
(Flow Graph)

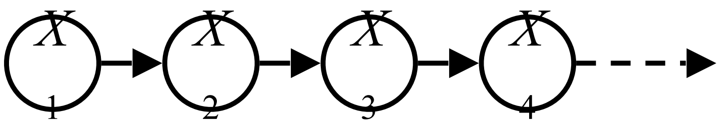


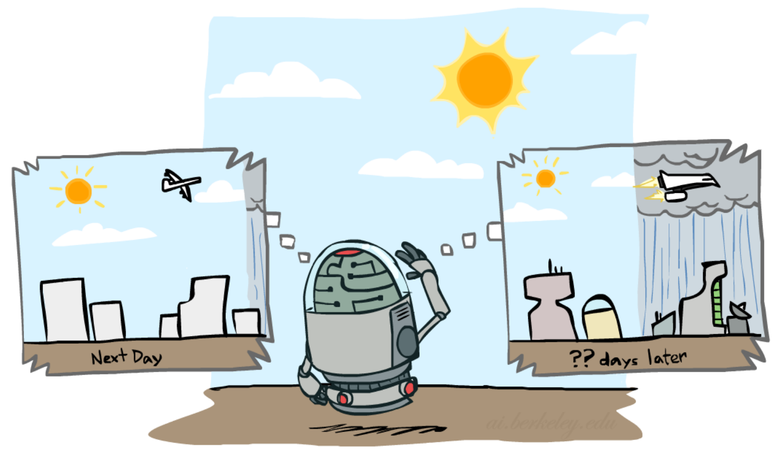
State Trellis



Mini-Forward Algorithm

- Question: What's P(X) on some day t?





$$P(x_1) = \text{known}$$

$$P(x_t) = \sum_{x_{t-1}} P(x_{t-1}, x_t)$$
$$= \sum_{x_{t-1}} P(x_t \mid x_{t-1}) P(x_{t-1})$$

Forward simulation

Stationary Distribution

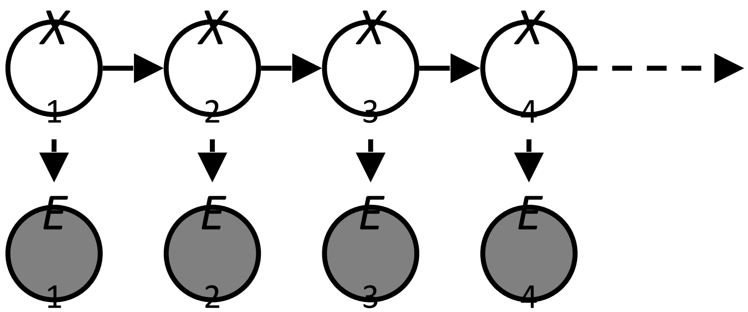
- For most chains:
  - Influence of the initial distribution gets less and less over time.
  - The distribution we end up in is independent of the initial distribution

- Stationary distribution:
  - The distribution we end up with is called the stationary distribution P_∞ of the chain
  - It satisfies

$$P_\infty(X) = P_{\infty+1}(X) = \sum_x P(X|x) P_\infty(x)$$

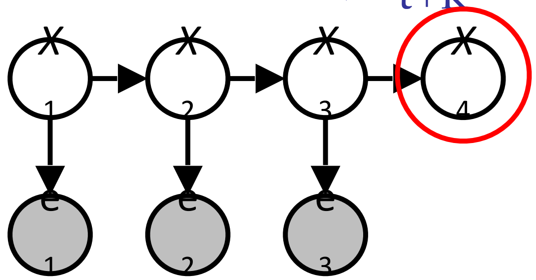
Hidden Markov Models

- Markov chains not so useful for most agents
  - Need observations to update your beliefs
- Hidden Markov models (HMMs)
  - Underlying Markov chain over states X_i
  - You observe outputs (effects) at each time step

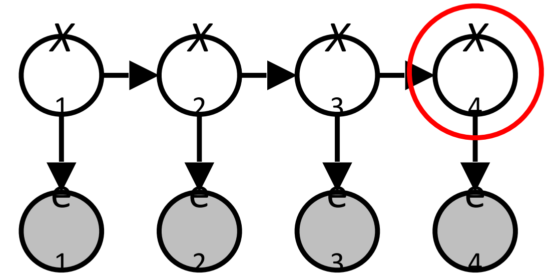


Inference tasks

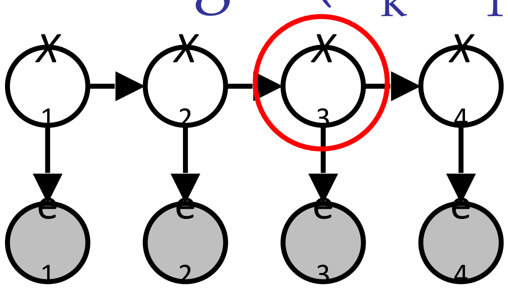
Prediction: $P(X_{t+k} \mid e_{1:t})$



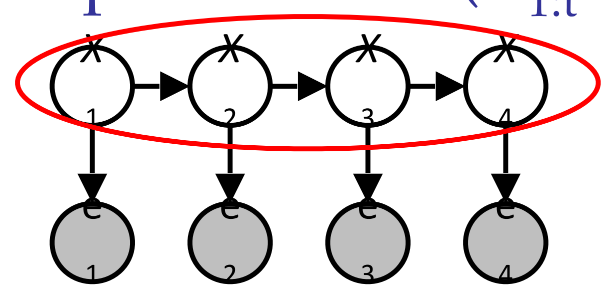
Filtering: $P(X_t \mid e_{1:t})$



Smoothing: $P(X_k \mid e_{1:t})$, $k<t$



Explanation: $P(X_{1:t} \mid e_{1:t})$

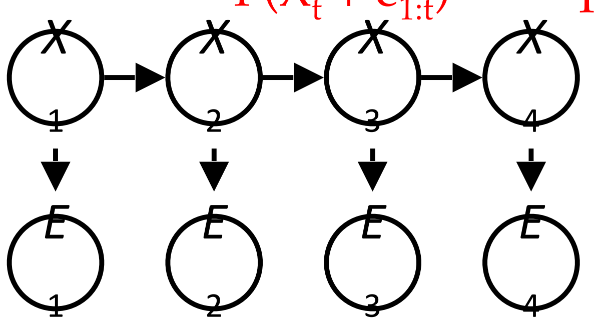


Inference: Find State Given Evidence

- We are given evidence at each time and want to know $P(X_t|e_{1:t})$
- Idea: start with $P(X_1)$ and derive $P(X_t \mid e_{1:t})$ in terms of $P(X_{t-1} \mid e_{1:t-1})$
- Two steps: Passage of time + Incorporate Evidence

$P(X_{t+1} \mid e_{1:t})$

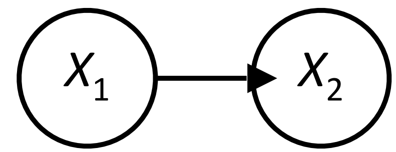
$P(X_t \mid e_{1:t})$ $P(X_{t+1} \mid e_{1:t+1})$



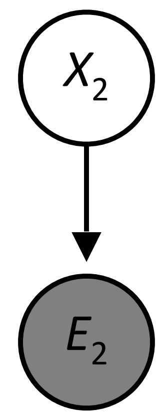
Forward Algorithm

- Every time step, we start with current P(X | evidence)
- We update for time:

$$P(x_t|e_{1:t-1}) = \sum_{x_{t-1}} P(x_{t-1}|e_{1:t-1}) \cdot P(x_t|x_{t-1})$$



- We update for evidence:

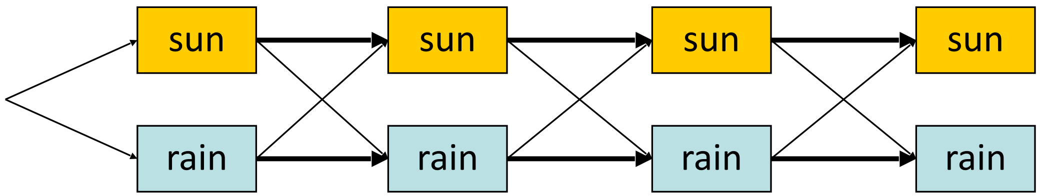


$$P(x_t|e_{1:t}) \propto_X P(x_t|e_{1:t-1}) \cdot P(e_t|x_t)$$

- The forward algorithm does both at once

Most likely explanation = most probable path

- **State trellis**: graph of states and transitions over time



X_1 X_2 … X_T

$\text{argmax}_{x1:t} P(x_{1:t} \mid e_{1:t})$

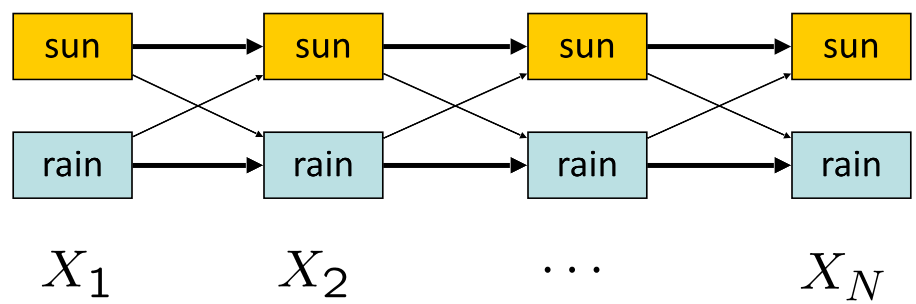
$= \text{argmax}_{x1:t} P(x_{1:t}, e_{1:t})$

$= \text{argmax}_{x1:t} \; P(x_{1:t}, e_{1:t})$

$= \text{argmax}_{x1:t} P(x_0) \prod_t P(x_t \mid x_{t\text{-}1}) P(e_t \mid x_t)$

- Each arc represents some transition $X_{t-1} \rightarrow X_t$
- Each arc has weight $P(x_t \mid x_{t-1}) \, P(e_t \mid x_t)$ (arcs to initial states have weight $P(x_0)$)
- The **product** of weights on a path is proportional to that state seq's probability
- Forward algorithm: sums of paths
- **Viterbi algorithm:** best paths
  - Dynamic Programming: solve subproblems, combine them as you go along

Forward / Viterbi Algorithms



Forward Algorithm (Sum)

For each state at time t, keep track of the ***total probability of all paths*** to it

$$f_t[x_t] = P(x_t, e_{1:t})$$

$$= P(e_t|x_t) \sum_{x_{t-1}} P(x_t|x_{t-1}) f_{t-1}[x_{t-1}]$$

Viterbi Algorithm (Max)

For each state at time t, keep track of the ***maximum probability of any path*** to it

$$m_t[x_t] = \max_{x_{1:t-1}} P(x_{1:t-1}, x_t, e_{1:t})$$

$$= P(e_t|x_t) \max_{x_{t-1}} P(x_t|x_{t-1}) m_{t-1}[x_{t-1}]$$

Viterbi Algorithm Pseudocode

```
function VITERBI(O, S, Π, Y, A, B) : X
    for each state i = 1, 2, ..., K do
        T_1[i, 1] ← π_i · B_{iy_1}
        T_2[i, 1] ← 0
    end for
    for each observation j = 2, 3, ..., T do
        for each state i = 1, 2, ..., K do
            T_1[i, j] ← max_k (T_1[k, j−1] · A_{ki} · B_{iy_j})
            T_2[i, j] ← argmax_k (T_1[k, j−1] · A_{ki} · B_{iy_j})
        end for
    end for
    z_T ← argmax_k (T_1[k, T])
    x_T ← s_{z_T}
    for j = T, T−1, ..., 2 do
        z_{j−1} ← T_2[z_j, j]
        x_{j−1} ← s_{z_{j−1}}
    end for
    return X
end function
```

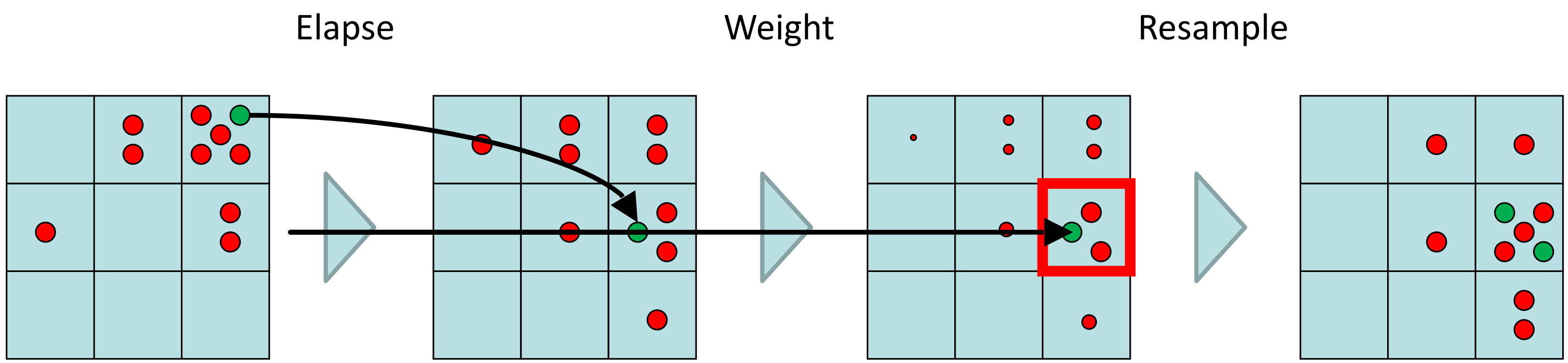
Observation Space	$O = \{o_1, o_2, \ldots, o_N\}$
State Space	$S = \{s_1, s_2, \ldots, s_K\}$
Initial probabilities	$\Pi = (\pi_1, \pi_2, \ldots, \pi_K)$
Observations	$Y = (y_1, y_2, \ldots, y_T)$
Transition Matrix	$A \in \mathbb{R}^{K \times K}$
Emission Matrix	$B \in \mathbb{R}^{K \times N}$

Matrix $T_1[i, j]$ stores probabilities of most likely path so far with $x_j = s_i$

Matrix $T_2[i, j]$ stores x_{j-1} of most likely path so far with $x_j = s_i$

Particle Filtering: Approximate Inference on HMMs

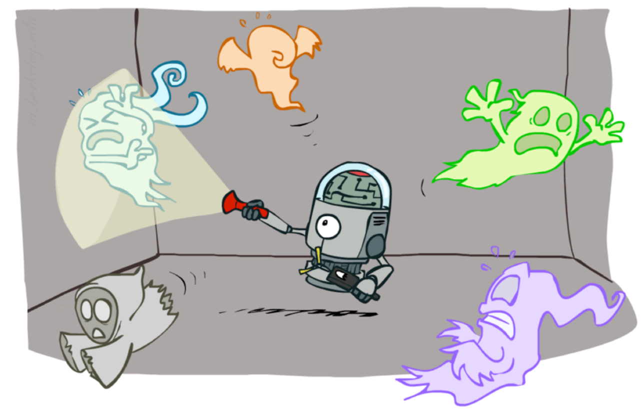
- Particles: track samples of states rather than an explicit distribution

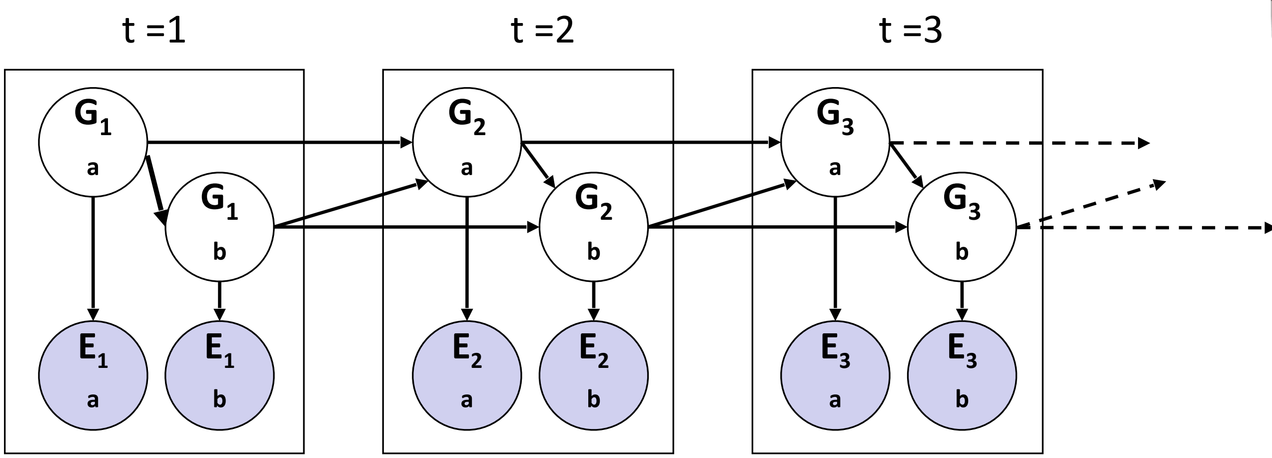


Particles:	Particles:	Particles:	(New) Particles:
(3,3)	(3,2)	(3,2) w=.9	(3,2)
(2,3)	(2,3)	(2,3) w=.2	(2,2)
(3,3)	(3,2)	(3,2) w=.9	(3,2)
(3,2)	(3,1)	(3,1) w=.4	(2,3)
(3,3)	(3,3)	(3,3) w=.4	(3,3)
(3,2)	(3,2)	(3,2) w=.9	(3,2)
(1,2)	(1,3)	(1,3) w=.1	(1,3)
(3,3)	(2,3)	(2,3) w=.2	(2,3)
(3,3)	(3,2)	(3,2) w=.9	(3,2)
(2,3)	(2,2)	(2,2) w=.4	(3,2)

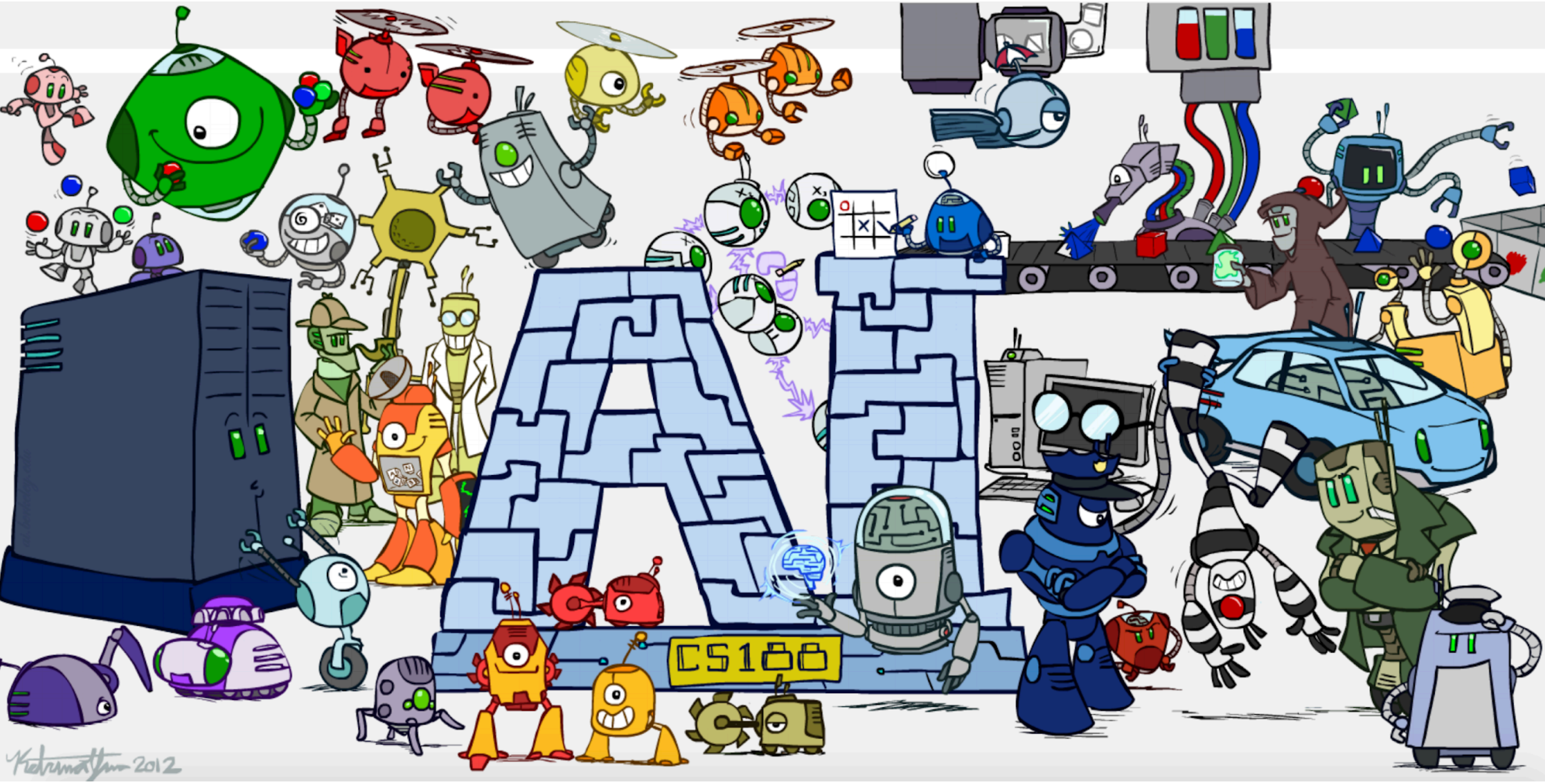
Dynamic Bayes Nets (DBNs)

- We want to track multiple variables over time, using multiple sources of evidence
- Idea: Repeat a fixed Bayes net structure at each time
- Variables from time *t* can condition on those from *t-1*





- Dynamic Bayes nets are a generalization of HMMs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