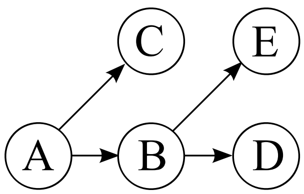
CS 188 Summer 2024 — Introduction to Artificial Intelligence — **Exam Prep 3b Solutions**

Q1. Bayes Nets: Variable Elimination



	$P(A)$
$+a$	0.25
$-a$	0.75

$P(B\|A)$	$+b$	$-b$
$+a$	0.5	0.5
$-a$	0.25	0.75

$P(C\|A)$	$+c$	$-c$
$+a$	0.2	0.8
$-a$	0.6	0.4

$P(D\|B)$	$+d$	$-d$
$+b$	0.6	0.4
$-b$	0.8	0.2

$P(E\|B)$	$+e$	$-e$
$+b$	0.25	0.75
$-b$	0.1	0.9

(a) Using the Bayes' Net and conditional probability tables above, calculate the following quantities:

(i) $P(+b|+a) =$ 0.5

(ii) $P(+a, +b) =$
$0.25 * 0.5 = 0.125 = \frac{1}{8}$

(iii) $P(+a|+b) =$
$\frac{0.25*0.5}{0.25*0.5+0.25*0.75} = 0.4 = \frac{2}{5}$

(b) Now we are going to consider variable elimination in the Bayes' Net above.

(i) Assume we have the evidence $+c$ and wish to calculate $P(E \mid +\text{c})$. What factors do we have initially?
$P(A)$, $P(B \mid A)$, $P(+\text{c} \mid A)$, $P(D \mid B)$, $P(E \mid B)$

(ii) If we eliminate variable B, we create a new factor. What probability does that factor correspond to?
$P(D, E \mid A)$

(iii) What is the equation to calculate the factor we create when eliminating variable B?
$f(A, D, E) = \sum_B P(B \mid A) * P(D \mid B) * P(E \mid B)$

(iv) After eliminating variable B, what are the new set of factors? As in (ii), write the probabilities that the factors represent. For each factor, also provide its size.

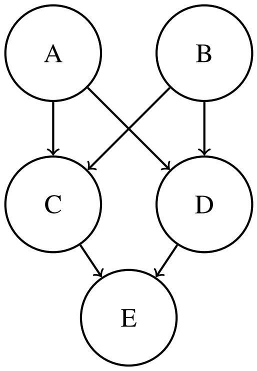
Factor	Size after elimination
$P(A)$	2
$P(+\text{c} \mid A)$	2
$P(D, E \mid A)$	2^3

(v) Now assume we have the evidence $-c$ and are trying to calculate $P(A|-c)$. What is the most efficient elimination ordering? If more than one ordering is most efficient, provide any one of them. E, D, B or D, E, B

(vi) Once we have run variable elimination and have $f(A, -c)$ how do we calculate $P(+a \mid -c)$? $\frac{f(+a,-c)}{f(+a,-c)+f(-a,-c)}$ or note that elimination is unnecessary - just use Bayes' rule

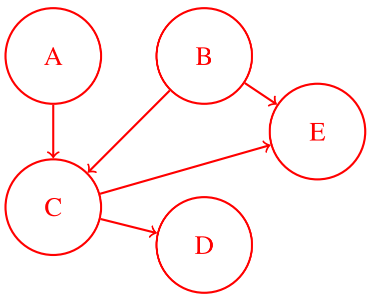
Q2. Bayes Nets and Joint Distributions

(a) Write down the joint probability distribution associated with the following Bayes Net. Express the answer as a product of terms representing individual conditional probabilities tables associated with this Bayes Net:



$P(A)P(B)P(C|A,B)P(D|A,B)P(E|C,D)$

(b) Draw the Bayes net associated with the following joint distribution:
$P(A) \cdot P(B) \cdot P(C|A,B) \cdot P(D|C) \cdot P(E|B,C)$



(c) Do the following products of factors correspond to a valid joint distribution over the variables A, B, C, D? (Circle FALSE or TRUE.)

(i) [FALSE] TRUE $P(A) \cdot P(B) \cdot P(C|A) \cdot P(C|B) \cdot P(D|C)$

(ii) FALSE [TRUE] $P(A) \cdot P(B|A) \cdot P(C) \cdot P(D|B,C)$

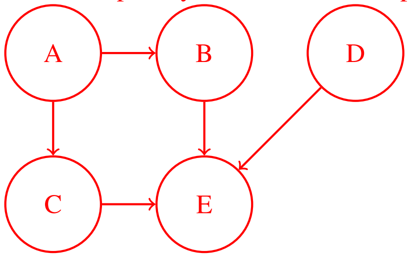
(iii) [FALSE] TRUE $P(A) \cdot P(B|A) \cdot P(C) \cdot P(C|A) \cdot P(D)$

(iv) [FALSE] TRUE $P(A|B) \cdot P(B|C) \cdot P(C|D) \cdot P(D|A)$

(d) What factor can be multiplied with the following factors to form a valid joint distribution? (Write "none" if the given set of factors can't be turned into a joint by the inclusion of exactly one more factor.)

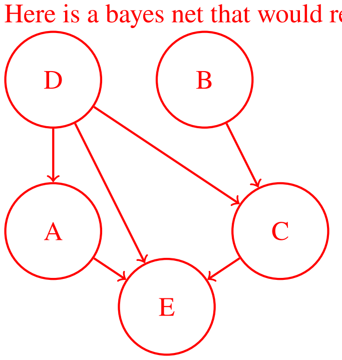
(i) $P(A) \cdot P(B|A) \cdot P(C|A) \cdot P(E|B, C, D)$

P(D) is missing. D could also be conditioned on A,B, and/or C without creating a cycle (e.g. $P(D|A, B, C)$). Here is an example bayes net that would represent the distribution after adding in $P(D)$:

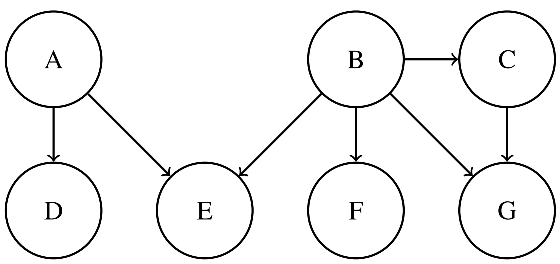


(ii) $P(D) \cdot P(B) \cdot P(C|D, B) \cdot P(E|C, D, A)$

P(A) is missing to form a valid joint distributions. A could also be conditioned on B, C, and/or D (e.g. $P(A|B, C, D)$). Here is a bayes net that would represent the distribution is $P(A|D)$ was added in.



(e) Answer the next questions based off of the Bayes Net below:
All variables have domains of {-1, 0, 1}



(i) Before eliminating any variables or including any evidence, how many entries does the factor at G have?

The factor is $P(G|B, C)$, so that gives $3^3 = 27$ entries.

(ii) Now we observe $e = 1$ and want to query $P(D|e = 1)$, and you get to pick the first variable to be eliminated.

- Which choice would create the **largest** factor f_1?
  Eliminating B first would give the largest f_1:, $f_1(A, F, G, C, e) = \sum_{B=b} P(b)P(e|A, b)P(F|b)P(G|b, C)P(C|b)$. This factor has 3^4 entries.
- Which choice would create the **smallest** factor f_1?
  eliminating F first would give smallest factors of 3 entries: $f_1(B) = \sum_f P(f|B)$. Eliminating D is not correct because D is the query variable.

Q3. Probability and Bayes Nets

(a) A, B, and C are random variables with binary domains. How many entries are in the following probability tables and what is the sum of the values in each table? Write a "?" in the box if there is not enough information given.

Table	Size	Sum
$P(A, B\|C)$	8	2
$P(A\| +b, +c)$	2	1
$P(+a\|B)$	2	?

(b) Circle true if the following probability equalities are valid and circle false if they are invalid (leave it blank if you don't wish to risk a guess). Each True/False question is worth 1 points. Leaving a question blank is worth 0 points. **Answering incorrectly is worth −1 points.**

No independence assumptions are made.

(i) [*true* or *false*] $P(A, B) = P(A|B)P(A)$
False. $P(A, B) = P(A|B)P(B)$ would be a valid example.

(ii) [*true* or *false*] $P(A|B)P(C|B) = P(A, C|B)$
False. This assumes that A and C are conditionally independent given B.

(iii) [*true* or *false*] $P(B, C) = \sum_{a \in A} P(B, C|A)$
False. $P(B, C) = \sum_{a \in A} P(A, B, C)$ would be a valid example.

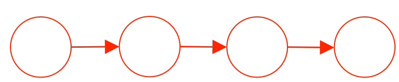
(iv) [*true* or *false*] $P(A, B, C, D) = P(C)P(D|C)P(A|C, D)P(B|A, C, D)$
True. This is a valid application of the chain rule.

(c) Space Complexity of Bayes Nets

Consider a joint distribution over N variables. Let k be the domain size for all of these variables, and let d be the maximum indegree of any node in a Bayes net that encodes this distribution.

(i) What is the space complexity of storing the entire joint distribution? Give an answer of the form $O(\cdot)$.
$O(k^N)$ was the intended answer. Because of the potentially misleading wording, we also allowed $O(Nk^{d+1})$, one possible bound on the space complexity of storing the Bayes net ($O((N-d)k^{d+1})$ is an asymptotically tighter bound, but this requires considerably more effort to prove).

(ii) Draw an example of a Bayes net over four binary variables such that it takes less space to store the Bayes net than to store the joint distribution.
A simple Markov chain works. Size $2+4+4+4 = 14$, which is less than $2^4 = 16$. Less edges, less inbound edges (v-shape), or no edges would work too.



(iii) Draw an example of a Bayes net over four binary variables such that it takes more space to store the Bayes net than to store the joint distribution.
Size $2+2+2+2^4 = 22$, which is more than $2^4 = 16$. Other configurations could work too, especially any with a node with indegree 3.