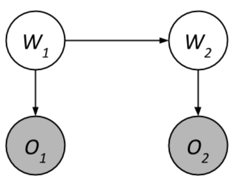
CS 188
Summer 2024

Regular Discussion 7 Solutions

1 HMMs

Consider the following Hidden Markov Model. O_1 and O_2 are supposed to be shaded.



W_1	$P(W_1)$
0	0.3
1	0.7

W_t	W_{t+1}	$P(W_{t+1}\|W_t)$
0	0	0.4
0	1	0.6
1	0	0.8
1	1	0.2

W_t	O_t	$P(O_t\|W_t)$
0	a	0.9
0	b	0.1
1	a	0.5
1	b	0.5

Suppose that we observe $O_1 = a$ and $O_2 = b$.
Using the forward algorithm, compute the probability distribution $P(W_2|O_1 = a, O_2 = b)$ one step at a time.

(a) Compute $P(W_1, O_1 = a)$.

$P(W_1, O_1 = a) = P(W_1)P(O_1 = a|W_1)$
$P(W_1 = 0, O_1 = a) = (0.3)(0.9) = 0.27$
$P(W_1 = 1, O_1 = a) = (0.7)(0.5) = 0.35$

(b) Using the previous calculation, compute $P(W_2, O_1 = a)$.

$P(W_2, O_1 = a) = \sum_{w_1} P(w_1, O_1 = a)P(W_2|w_1)$
$P(W_2 = 0, O_1 = a) = (0.27)(0.4) + (0.35)(0.8) = 0.388$
$P(W_2 = 1, O_1 = a) = (0.27)(0.6) + (0.35)(0.2) = 0.232$

(c) Using the previous calculation, compute $P(W_2, O_1 = a, O_2 = b)$.

$P(W_2, O_1 = a, O_2 = b) = P(W_2, O_1 = a)P(O_2 = b|W_2)$
$P(W_2 = 0, O_1 = a, O_2 = b) = (0.388)(0.1) = 0.0388$
$P(W_2 = 1, O_1 = a, O_2 = b) = (0.232)(0.5) = 0.116$

(d) Finally, compute $P(W_2|O_1 = a, O_2 = b)$.

Renormalizing the distribution above, we have
$P(W_2 = 0|O_1 = a, O_2 = b) = 0.0388/(0.0388 + 0.116) \approx 0.25$
$P(W_2 = 1|O_1 = a, O_2 = b) = 0.116/(0.0388 + 0.116) \approx 0.75$

Q2. HMMs

Consider a process where there are transitions among a finite set of states $s_1, \cdots, s_k$ over time steps $i = 1, \cdots, N$. Let the random variables $X_1, \cdots, X_N$ represent the state of the system at each time step and be generated as follows:

- Sample the initial state s from an initial distribution $P_1(X_1)$, and set $i = 1$
- Repeat the following:
  1. Sample a duration d from a duration distribution P_D over the integers $\{1, \cdots, M\}$, where M is the maximum duration.
  2. Remain in the current state s for the next d time steps, i.e., set
  $$x_i = x_{i+1} = \cdots = x_{i+d-1} = s \tag{1}$$
  3. Sample a successor state s' from a transition distribution $P_T(X_t|X_{t-1} = s)$ over the other states $s' \neq s$ (so there are no self transitions)
  4. Assign $i = i + d$ and $s = s'$.

This process continues indefinitely, but we only observe the first N time steps.

(a) Assuming that all three states s_1, s_2, s_3 are different, what is the probability of the sample sequence $s_1, s_1, s_2, s_2, s_2, s_3, s_3$? Write an algebraic expression. Assume $M \geq 3$.

$$p_1(s_1)p_D(2)p_T(s_2|s_1)p_D(3)p(s_3|s_2)(1 - p_D(1)) \tag{2}$$

At each time step i we observe a noisy version of the state X_i that we denote Y_i and is produced via a conditional distribution $P_E(Y_i|X_i)$.

(b) Only in this subquestion assume that $N > M$. Let $X_1, \cdots, X_N$ and $Y_1, \cdots, Y_N$ random variables defined as above. What is the maximum index $i \leq N - 1$ so that $X_1 \perp\!\!\!\perp X_N | X_i, X_{i+1}, \cdots, X_{N-1}$ is guaranteed?
$i = N - M$

(c) Only in this subquestion, assume the max duration $M = 2$, and P_D uniform over $\{1, 2\}$ and each x_i is in an alphabet $\{a, b\}$. For $(X_1, X_2, X_3, X_4, X_5, Y_1, Y_2, Y_3, Y_4, Y_5)$ draw a Bayes Net over these 10 random variables with the property that removing any of the edges would yield a Bayes net inconsistent with the given distribution.

(X1) at (0,0) X1; (X2) at (2,-2) X2; (X3) at (4,0) X3; (X4) at (6,-2) X4; (X5) at (8,0) X5; (Y1) at (0,-4)Y1; (Y2) at (2,-4)Y2; (Y3) at (4,-4)Y3; (Y4) at (6,-4)Y4; (Y5) at (8,-4)Y5; (X1) – (X2);(X2) – (X3);(X3) – (X4);(X4) – (X5);(X1) – (Y1);(X2) – (Y2);(X3) – (Y3);(X4) – (Y4);(X5) – (Y5);(X1) – (X3);(X2) – (X4);(X3) – (X5);

(d) In this part we will explore how to write the described process as an HMM with an extended state space. Write the states $z = (s, t)$ where s is a state of the original system and t represents the time elapsed in that state. For example, the state sequence $s_1, s_1, s_1, s_2, s_3, s_3$ would be represented as $(s_1, 1), (s_1, 2), (s_1, 3), (s_2, 1), (s_3, 1), (s_3, 2)$.

Answer all of the following in terms of the parameters $P_1(X_1), P_D(d), P_T(X_{j+1}|X_j), P_E(Y_i|X_i), k$ (total number of possible states), N and M (max duration).

(i) What is $P(Z_1)$?

$$P(x_1, t) = \begin{cases} P_1(x_1) & \text{if } t = 1 \\ 0 & \text{o.w.} \end{cases} \tag{3}$$

(ii) What is $P(Z_{i+1}|Z_i)$? Hint: You will need to break this into cases where the transition function will behave differently.

$$P(X_{i+1}, t_{i+1}|X_i, t_i) = \begin{cases} P_D(d \geq t_i + 1|d \geq t_i) & \text{when } X_{i+1} = X_i \text{ and } t_{i+1} = t_i + 1 \text{ and } t_{i+1} \leq M \\ P_T(X_{i+1}|X_i)P_D(d = t_i|d \geq t_i) & \text{when } X_{i+1} \neq X_i \text{ and } t_{i+1} = 1 \\ 0 & \text{o.w.} \end{cases}$$

Where $P_D(d \geq t_i + 1|d \geq t_i) = P_D(d \geq t_i + 1)/P_D(d \geq t_i)$.
Being in X_i, t_i, we know that d was drawn $d \geq t_i$. Conditioning on this fact, we have two choices, if $d > t_i$ then the next state is $X_{i+1} = X_i$, and if $d = t_i$ then $X_{i+1} \neq X_i$ drawn from the transition distribution and $t_{i+1} = 1$.
(4)

(iii) What is $P(Y_i|Z_i)$?
$p(Y_i|X_i, t_i) = P_E(Y_i|X_i)$

(e) In this question we explore how to write an algorithm to compute $P(X_N|y_1, \cdots, y_N)$ using the particular structure of this process.

Write $P(X_t|y_1, \cdots, y_{t-1})$ in terms of other factors. Construct an answer by checking the correct boxes below:

$$P(X_t|y_1, \cdots, y_{t-1}) = \underline{\quad\textbf{(i)}\quad} \; \underline{\quad\textbf{(ii)}\quad} \; \underline{\quad\textbf{(iii)}\quad}$$

(i)
- ● $\sum_{i=1}^{k} \sum_{d=1}^{M} \sum_{d'=1}^{M}$
- ○ $\sum_{i=1}^{k}$
- ○ $\sum_{i=1}^{k} \sum_{d=1}^{M}$
- ○ $\sum_{d=1}^{M}$

(ii)
- ○ $P(Z_t = (X_t, d)|Z_{t-1} = (s_i, d))$
- ○ $P(X_t|X_{t-1} = s_d)$
- ○ $P(X_t|X_{t-1} = s_i)$
- ● $P(Z_t = (X_t, d')|Z_{t-1} = (s_i, d))$

(iii)
- ○ $P(Z_{t-1} = (s_d, i)|y_1, \cdots, y_{t-1})$
- ● $P(Z_{t-1} = (s_i, d)|y_1, \cdots, y_{t-1})$
- ○ $P(X_{t-1} = s_d|y_1, \cdots, y_{t-1})$
- ○ $P(X_{t-1} = s_i|y_1, \cdots, y_{t-1})$

(iv) Now we would like to include the evidence y_t in the picture. What would be the running time of each update of the **whole table** $P(X_t|y_1, \cdots, y_t)$?. Assume tables corresponding to any factors used in **(i)**, **(ii)**, **(iii)** have already been computed.

- ○ $O(k^2)$
- ● $O(k^2M^2)$
- ○ $O(k^2M)$
- ○ $O(kM)$

Note: Computing $P(X_N|y_1, \cdots, y_N)$ will take time $N\times$ your answer in **(iv)**.

Just the running time for filtering when the state space is the space of pairs (x_i, t_i),

Given $B_{t-1}(z)$, the step $p(z_t|y_1, \cdots, y_{t-1})$ can be done in time kM. (size of the statespace for z).

The computation to include the y_t evidence can be done in $O(1)$ per z_t.

Therefore each update to the table per evidence point will take $(Mk)^2$. So it is $O((Mk)^2)$.

Using N steps, the whole algorithm will take $O(Nk^2M^2)$ to compute $P(X_N|Y_1, \cdots, Y_N)$.

(v) Describe an update rule to compute $P(X_t|y_1, \cdots, y_{t-1})$ that is faster than the one you discovered in parts **(i)**, **(ii)**, **(iii)**. **Specify its running time**. Hint: Use the structure of the transitions $Z_{t-1} \rightarrow Z_t$.

Answer is $O(k^2M + kM)$.

The answer from the previous section is:

$$P(X_t|y_1, \cdots, y_{t-1}) = \sum_{i=1}^{k} \sum_{d=1}^{M} \sum_{d'=1}^{M} P(Z_t = (X_t, d')|Z_{t-1} = (s_i, d)) P(Z_{t-1} = (s_i, d)|y_1, \cdots, y_{t-1}) \quad (5)$$

To compute this value we only really need to loop through those transitions $P(Z_t = (X_t, d')|Z_{t-1} = (s_i, d))$ that can happen with nonzero probability.

For all $X_t = s$ we need to sum over all factors of the form $P(Z_t = (s, d')|Z_{t-1} = (s_i, d))P(X_{t-1} = s_i|y_i, \cdots, y_{t-1})$. For a fixed s the factor $P(Z_t = (X_t, d')|Z_{t-1} = (s_i, d))$ can be nonzero only when $s_i = s$ and $d' = d + 1$ (M tuples). And when $s_i \neq s$ and $d' = 1$ and $d = 1, \cdots, M$ (kM tuples).

Since this needs to be performed for all k possible values of s, the answer to update the whole table is $O(k^2M + kM)$.