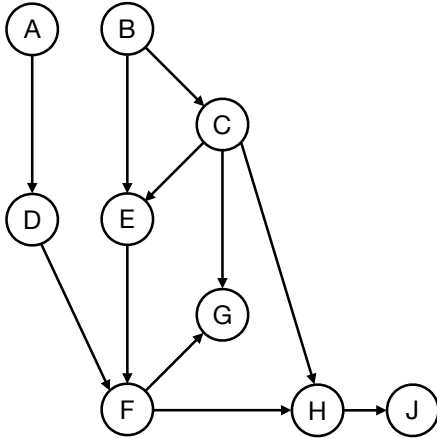## Q1. Bayes Nets: Elimination

(a) Consider running variable elimination on the Bayes Net shown below.


First, we eliminate D to create a factor f_1
Next, we eliminate E to create a factor f_2
Next, we eliminate H to create a factor f_3

From the list below, select all factors that remain after D, E and H have been eliminated.

- [ ] f_1
- [ ] f_2
- [ ] f_3

- [ ] $P(A)$
- [ ] $P(A|F)$
- [ ] $P(B)$
- [ ] $P(B|A)$
- [ ] $P(B|C)$
- [ ] $P(C)$
- [ ] $P(C|B)$

- [ ] $P(D|A)$
- [ ] $P(D|A, F)$
- [ ] $P(E|B)$
- [ ] $P(E|C)$
- [ ] $P(E|B, C)$
- [ ] $P(F)$
- [ ] $P(F|A, B)$
- [ ] $P(F|G)$

- [ ] $P(G|C)$
- [ ] $P(G|C, F)$
- [ ] $P(H|F)$
- [ ] $P(H|C, F)$
- [ ] $P(H|J)$
- [ ] $P(J|C)$
- [ ] $P(J|F)$
- [ ] $P(J|H)$

(b) Consider the Bayes Net shown below. Each variable in the Bayes Net can take on two possible values.
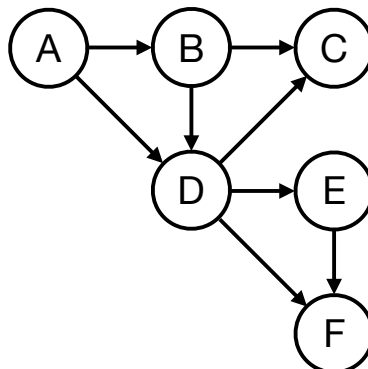
You are given the query $P(C|F)$, which you would like to answer using variable elimination. Please find a variable elimination ordering where the largest intermediate factor created during variable elimination is as small as possible.
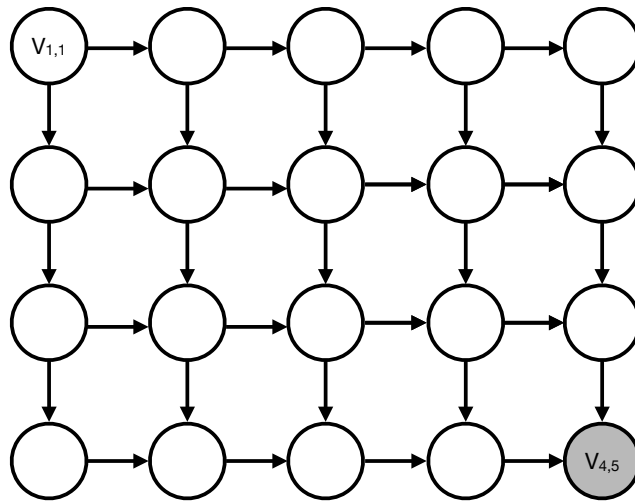
Elimination ordering: _____

(c) Consider doing inference in an m x n lattice Bayes Net, as shown below. The network consists of mn binary variables $V_{i,j}$, and you have observed that $V_{m,n} = +v_{m,n}$.
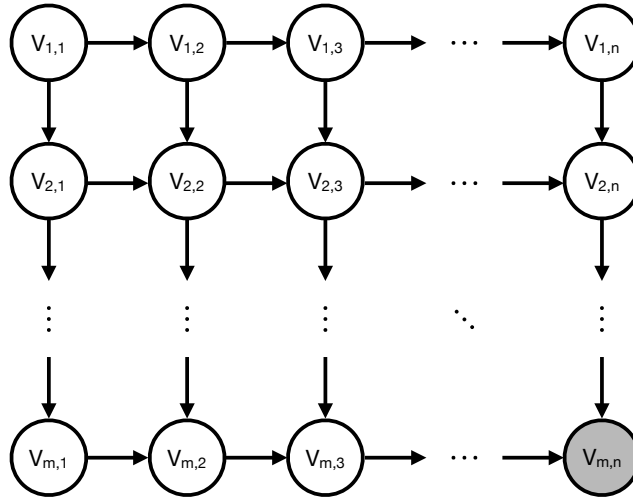
You wish to calculate $P(V_{1,1}| + v_{m,n})$ using variable elimination. To maximize computational efficiency, you wish to use a variable elimination ordering for which the size of the largest generated factor is as small as possible.

(i) First consider the special case where $m = 4$ and $n = 5$. A reproduction of the lattice is shown below, with variable names for non-query variables omitted. Please provide your optimal elimination ordering for this example by numbering the nodes below in the order they will be eliminated (i.e. write a number such as $1, 2, 3, \ldots$ inside every node that will be eliminated.)


(ii) Now consider the general case (assume $m > 2$ and $n > 2$). What is the size of the largest factor generated under the most efficient elimination ordering? Your answer should be the number of rows in the factor's table, expressed in terms of m and n.

Size (number of rows) of the largest factor: _____

Q2. Bayesian Networks

Consider the below Bayesian Network. Assume each event A, B, C, D, E, F is an indicator random variable.
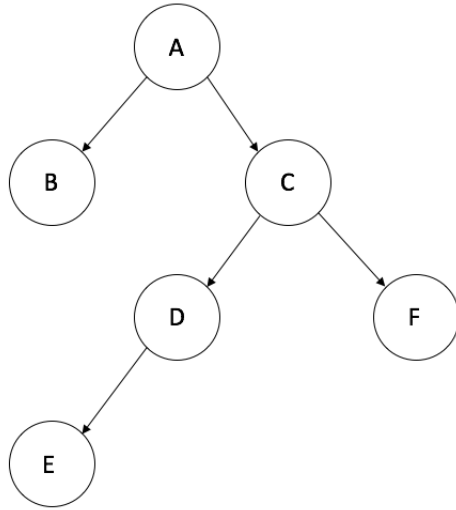
(a) **Independence and Variable Elimination**

    (i) [*true* or *false*] $F \perp\!\!\!\perp D$

    (ii) [*true* or *false*] $F \perp\!\!\!\perp E|D$

    (iii) You are asked to calculate the following query: $P(F| + e)$. You decide to use variable elimination to solve this problem. You eliminate D first and then C. The resulting factor you obtain is $f(+e, F|A, B$. Write $f(+e, F|A, B)$ in terms of the relevant conditional probabilities encoded in the Bayesian network.

———————————————

    (iv) We just calculated $f(+e, F|A, B)$ and now the next step is to eliminate B followed by then eliminating A. The resulting factor from these elimination steps is $f(+e, F)$. Write $f(+e, F)$ in terms of $f(+e, F|A, B)$ and relevant conditional probabilities in the Bayesian Network after performing these two steps of variable elimination.

———————————————

    (v) The last step in variable elimination is normalizing our final factor $f(+e, F)$ to obtain $P(F| + e)$. Write $P(F| + e)$ in terms of $f(+e, F)$ (perform the normalization step).

———————————————

4

Q3. Bayes Nets: Sampling

Consider the following Bayes Net, where we have observed that $B = +b$ and $D = +d$.

$P(D|A,C)$			
$+a$	$+c$	$+d$	0.6
$+a$	$+c$	$-d$	0.4
$+a$	$-c$	$+d$	0.1
$+a$	$-c$	$-d$	0.9
$-a$	$+c$	$+d$	0.2
$-a$	$+c$	$-d$	0.8
$-a$	$-c$	$+d$	0.5
$-a$	$-c$	$-d$	0.5

$P(B|A)$		
$+a$	$+b$	0.8
$+a$	$-b$	0.2
$-a$	$+b$	0.4
$-a$	$-b$	0.6

$P(C|B)$		
$+b$	$+c$	0.1
$+b$	$-c$	0.9
$-b$	$+c$	0.7
$-b$	$-c$	0.3

$P(A)$	
$+a$	0.5
$-a$	0.5
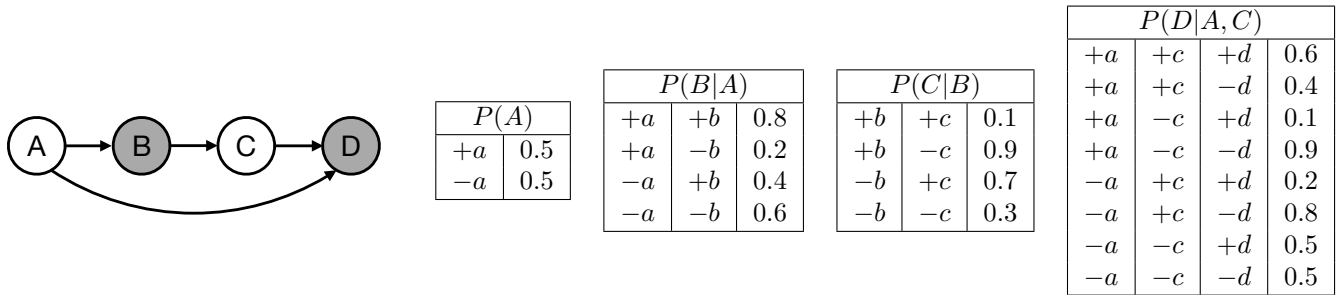
(a) Consider doing Gibbs sampling for this example. Assume that we have initialized all variables to the values $+a, +b, +c, +d$. We then un-assign the variable C, such that we have $A = +a$, $B = +b$, $C = ?$, $D = +d$. Calculate the probabilities for new values of C at this stage of the Gibbs sampling procedure.

$P(C = +c \text{ at next step}) = $ _____ $\qquad\qquad$ $P(C = -c \text{ at next step}) = $ _____

(b) Consider a sampling scheme that is a hybrid of rejection sampling and likelihood-weighted sampling. Under this scheme, we first perform rejection sampling for the variables A and B. We then take the sampled values for A and B and extend the sample to include values for variables C and D, using likelihood-weighted sampling.

   (i) Below is a list of candidate samples. Mark the samples that would be rejected by the rejection sampling portion of the hybrid scheme.

   ☐ $(-a, -b)$ $\qquad\qquad$ ☐ $(+a, +b)$ $\qquad\qquad$ ☐ $(+a, -b)$ $\qquad\qquad$ ☐ $(-a, +b)$

   (ii) To decouple from part (i), you now receive a *new* set of samples shown below. Fill in the weights for these samples under our hybrid scheme.

   (iii) Use the weighted samples to calculate an estimate for $P(+a| + b, +d)$.

$-a$	$+b$	$-c$	$+d$	_____
$+a$	$+b$	$-c$	$+d$	_____
$+a$	$+b$	$-c$	$+d$	_____
$-a$	$+b$	$+c$	$+d$	_____
$+a$	$+b$	$+c$	$+d$	_____

   scheme.

(c) We now attempt to design an alternative hybrid sampling scheme that combines elements of likelihood-weighted and rejection sampling. For each proposed scheme, indicate whether it is valid, i.e. whether the weighted samples it produces correctly approximate the distribution $P(A, C| + b, +d)$.

   (i) *First collect a likelihood-weighted sample for the variables A and B. Then switch to rejection sampling for the variables C and D. In case of rejection, the values of A and B and the sample weight are **thrown away**. Sampling then restarts from node **A**.*
   ○ Valid ○ Invalid

   (ii) *First collect a likelihood-weighted sample for the variables A and B. Then switch to rejection sampling for the variables C and D. In case of rejection, the values of A and B and the sample weight are **retained**. Sampling then restarts from node **C**.*
   ○ Valid ○ Invalid