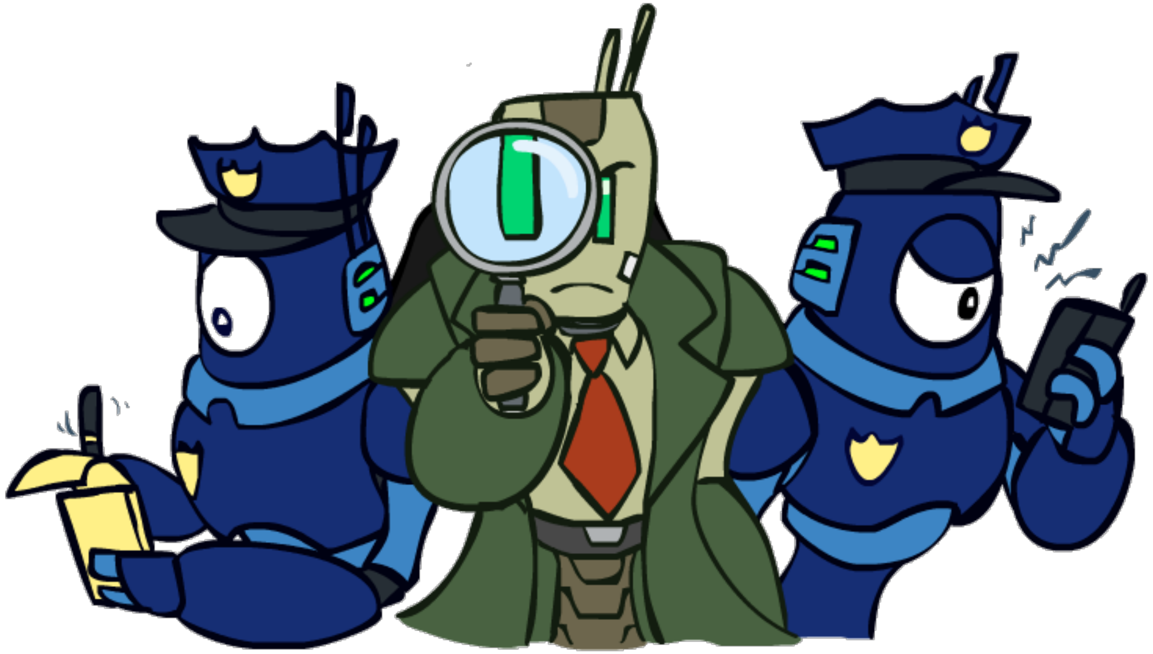
CS 188: Artificial Intelligence

Constraint Satisfaction Problems



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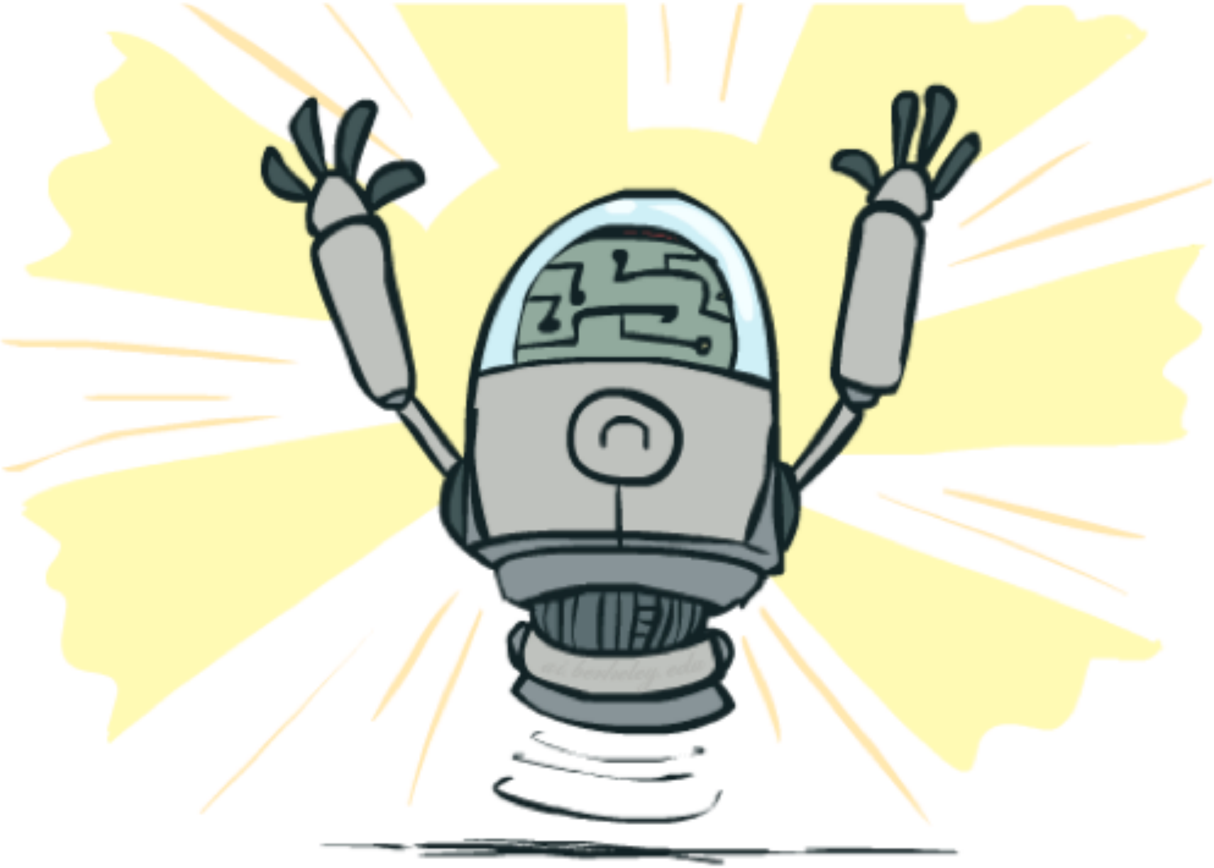
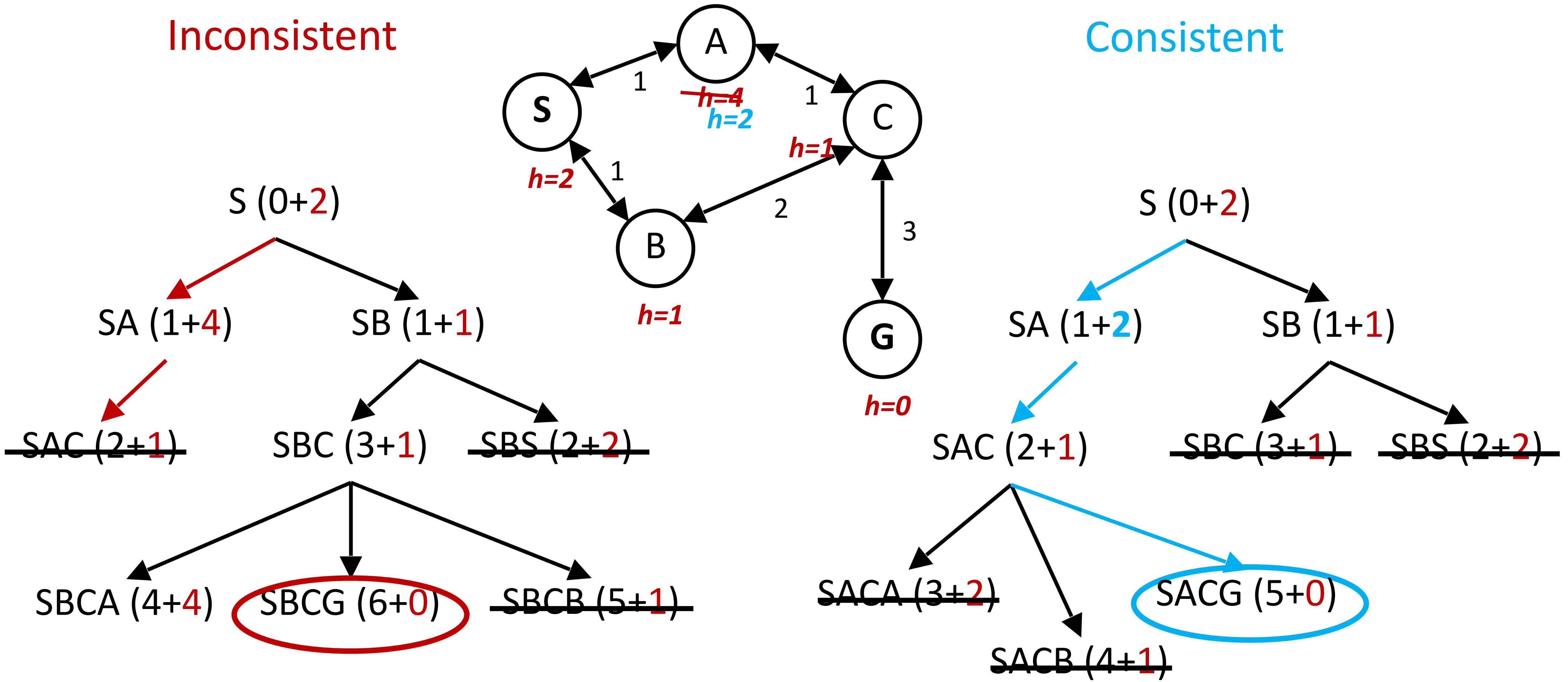
Announcements

- HW0 (optional) is due **Thursday, June 26**, 11:59 PM PT
- Project 0 (optional) is due **Friday, June 27**, 11:59 PM PT
- HW1 is due **Tuesday, July 1**, 11:59 PM PT
- HW2 is due **Thursday, July 3**, 11:59 PM PT
- Project 1 is due **Friday, July 4**, 11:59 PM PT

Optimality of A* Graph Search

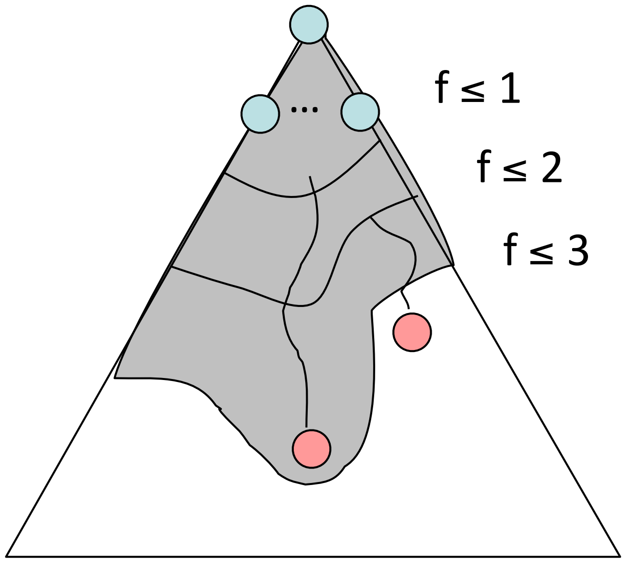


Consistency => non-decreasing f-score



Optimality of A* Graph Search

- Sketch: consider what A* does with a consistent heuristic:
  - Fact 1: In tree search, A* expands nodes in increasing total f value (f-contours)
  - Fact 2: For every state s, nodes that reach s optimally are expanded before nodes that reach s suboptimally
  - Result: A* graph search is optimal



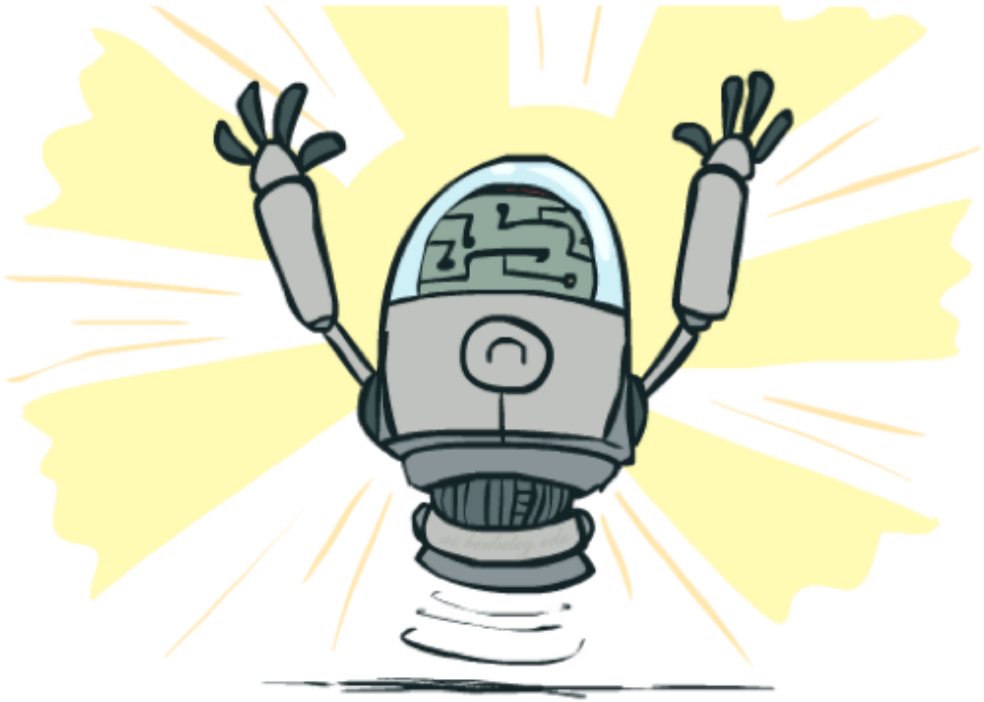
Optimality

- Tree search:
  - A* is optimal if heuristic is admissible
  - UCS is a special case (h = 0)

- Graph search:
  - A* optimal if heuristic is consistent
  - UCS optimal (h = 0 is consistent)

- Consistency implies admissibility

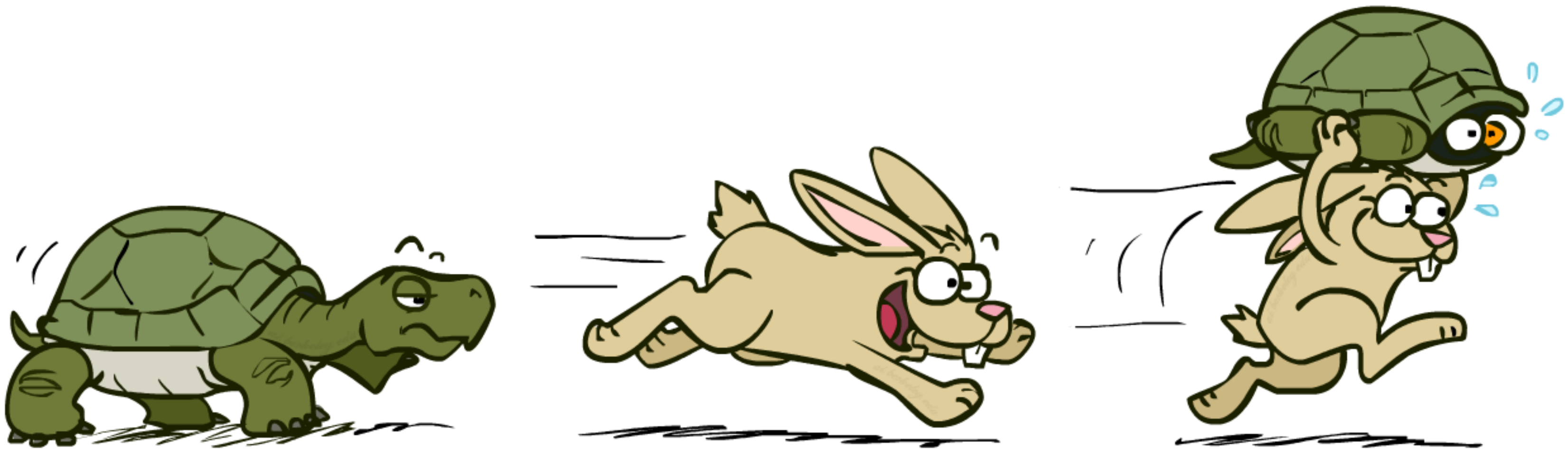
- In general, most natural admissible heuristics tend to be consistent, especially if from relaxed problems



A*: Summary

A*: Summary

- A* uses both backward costs and (estimates of) forward costs
- A* is optimal with admissible / consistent heuristics
- Heuristic design is key: often use relaxed problems



Appendix: Search Pseudo-Code

Tree Search Pseudo-Code

```
function TREE-SEARCH(problem, fringe) return a solution, or failure
    fringe ← INSERT(MAKE-NODE(INITIAL-STATE[problem]), fringe)
    loop do
        if fringe is empty then return failure
        node ← REMOVE-FRONT(fringe)
        if GOAL-TEST(problem, STATE[node]) then return node
        for child-node in EXPAND(STATE[node], problem) do
            fringe ← INSERT(child-node, fringe)
        end
    end
```

Graph Search Pseudo-Code

```
function GRAPH-SEARCH(problem, fringe) return a solution, or failure
    closed ← an empty set
    fringe ← INSERT(MAKE-NODE(INITIAL-STATE[problem]), fringe)
    loop do
        if fringe is empty then return failure
        node ← REMOVE-FRONT(fringe)
        if GOAL-TEST(problem, STATE[node]) then return node
        if STATE[node] is not in closed then
            add STATE[node] to closed
            for child-node in EXPAND(STATE[node], problem) do
                fringe ← INSERT(child-node, fringe)
            end
    end
```

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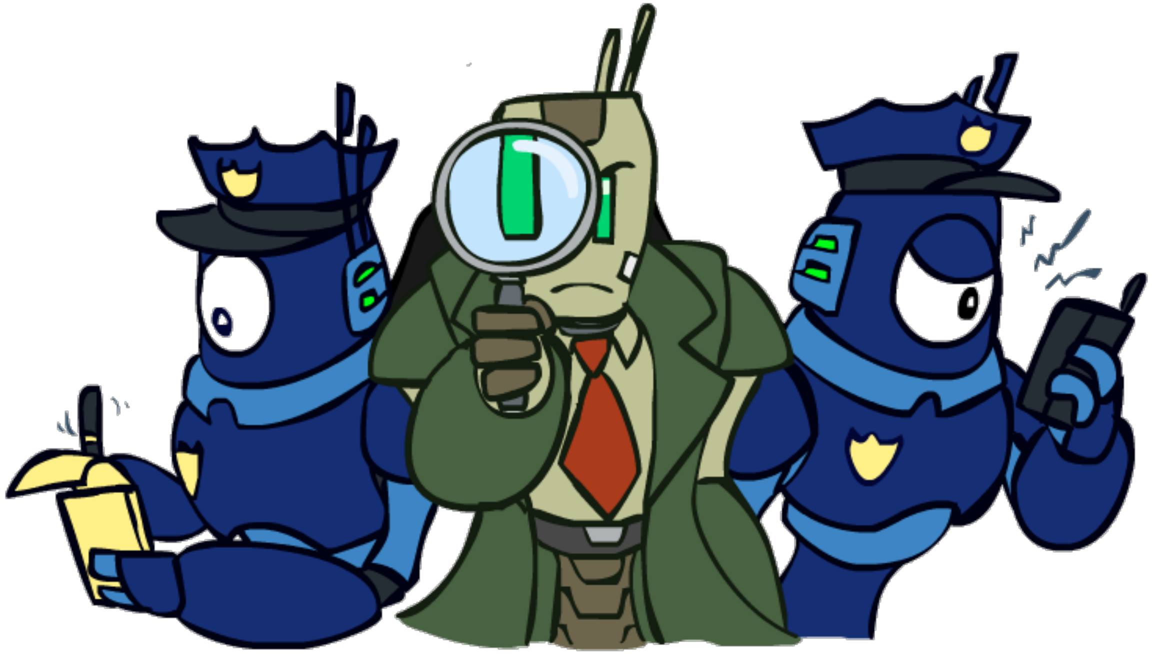
Constraint Satisfaction Problems



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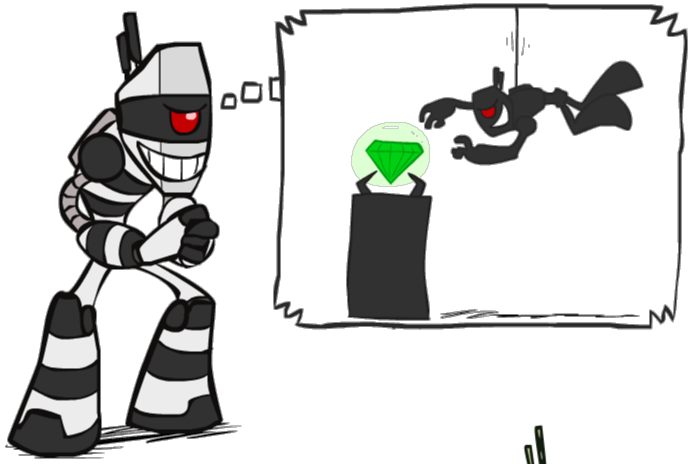
What is Search For?

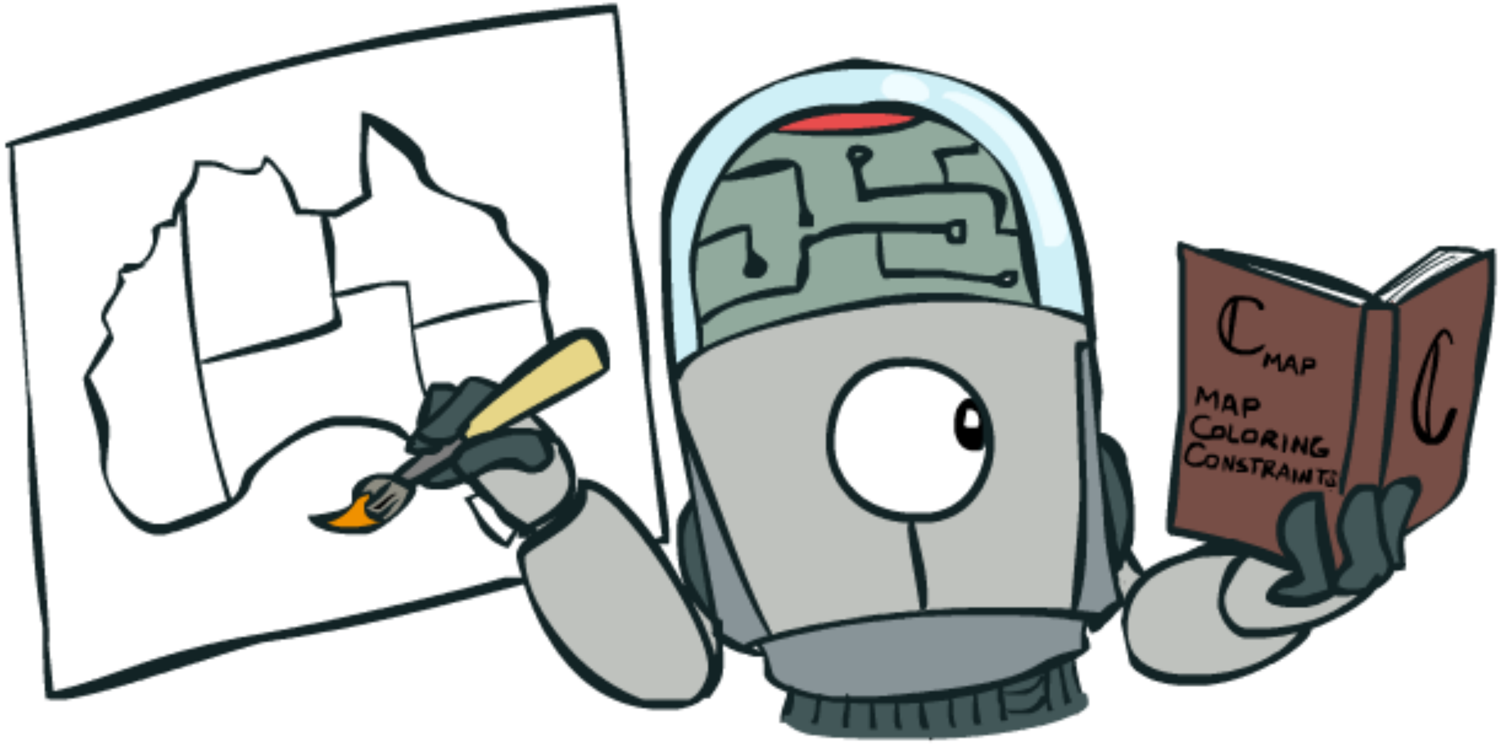
- Assumptions about the world: a single agent, deterministic actions, fully observed state, discrete state space



- Planning: sequences of actions
  - The path to the goal is the important thing
  - Paths have various costs, depths
  - Heuristics give problem-specific guidance

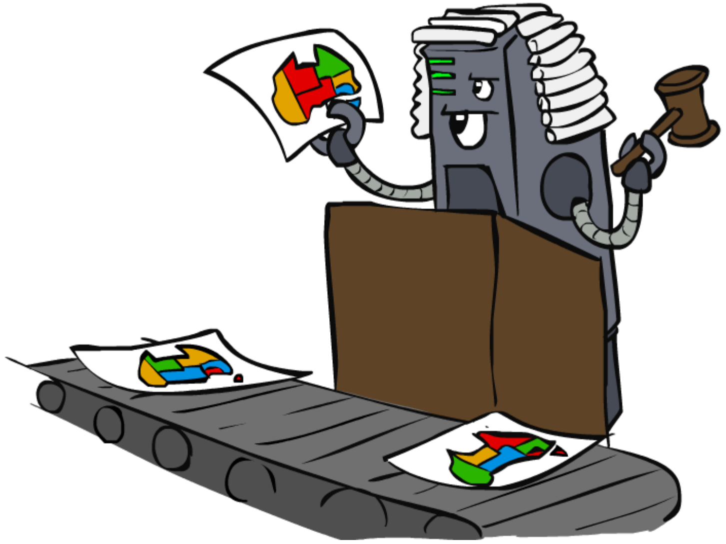
- Identification: assignments to variables
  - The goal itself is important, not the path
  - All paths at the same depth (for some formulations)
  - CSPs are a specialized class of identification problems

Constraint Satisfaction Problems

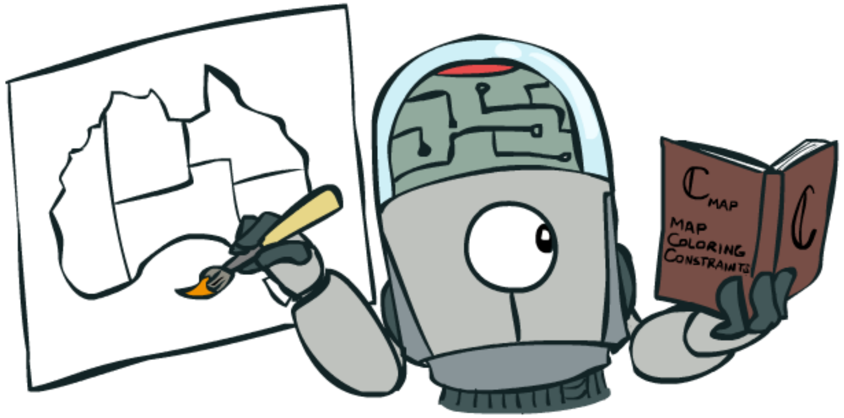


Constraint Satisfaction Problems

- Standard search problems:
  - State is a "black box": arbitrary data structure
  - Goal test can be any function over states
  - Successor function can also be anything



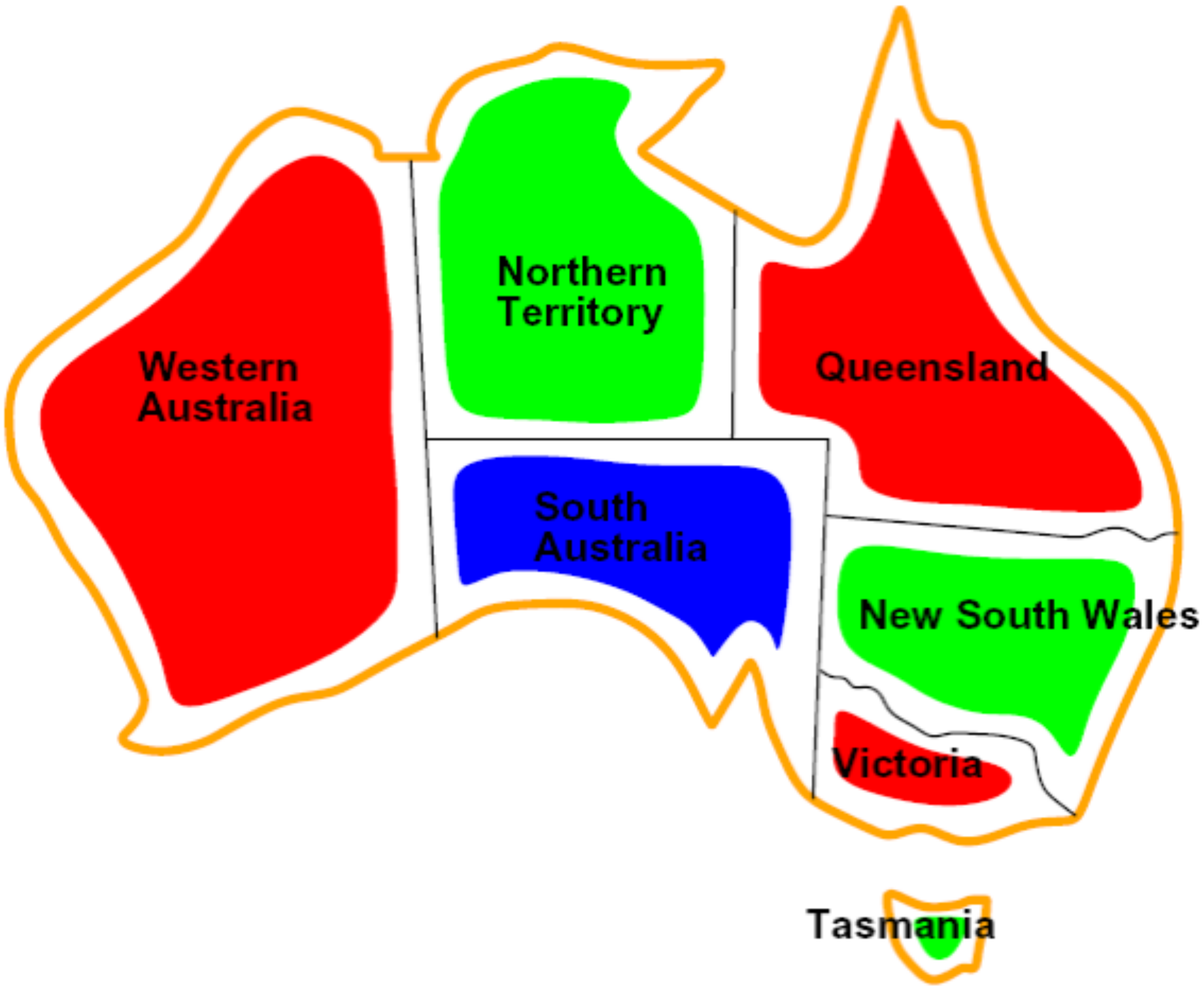
- Constraint satisfaction problems (CSPs):
  - A special subset of search problems
  - State is defined by variables X_i with values from a domain D (sometimes D depends on i)
  - Goal test is a set of constraints specifying allowable combinations of values for subsets of variables



- Simple example of a *formal representation language*

- Allows useful general-purpose algorithms with more power than standard search algorithms

CSP Examples

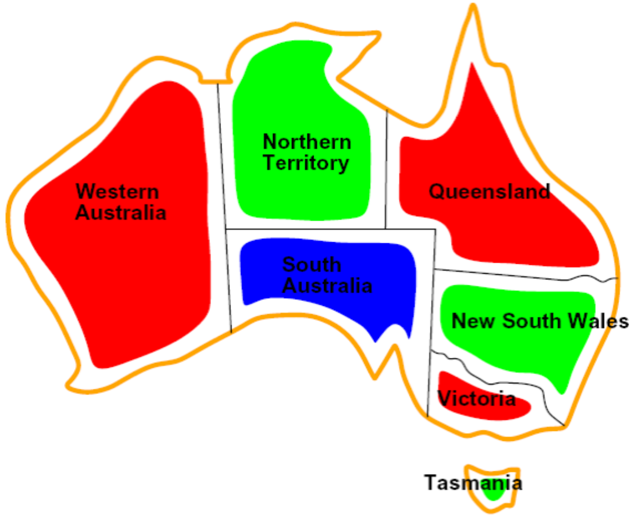


Example: Map Coloring

- Variables: $\mathrm{WA},\ \mathrm{NT},\ \mathrm{Q},\ \mathrm{NSW},\ \mathrm{V},\ \mathrm{SA},\ \mathrm{T}$

- Domains: $D = \{\mathrm{red}, \mathrm{green}, \mathrm{blue}\}$

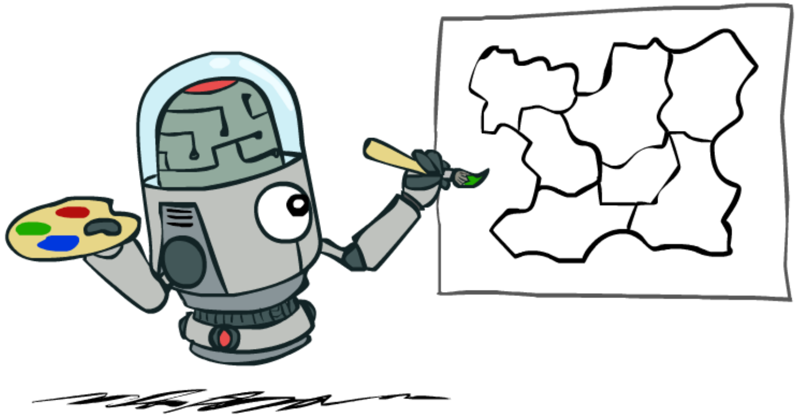
- Constraints: adjacent regions must have different colors



Implicit: $\mathrm{WA} \neq \mathrm{NT}$

Explicit: $(\mathrm{WA}, \mathrm{NT}) \in \{(\mathrm{red}, \mathrm{green}), (\mathrm{red}, \mathrm{blue}), \ldots\}$

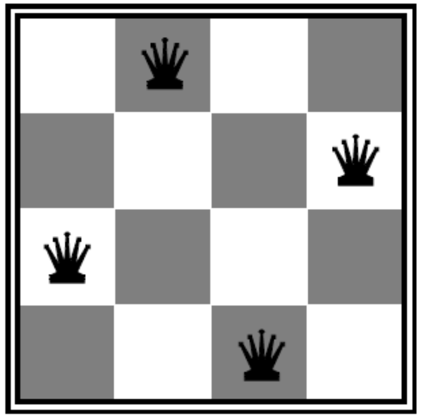
- Solutions are assignments satisfying all constraints, e.g.:

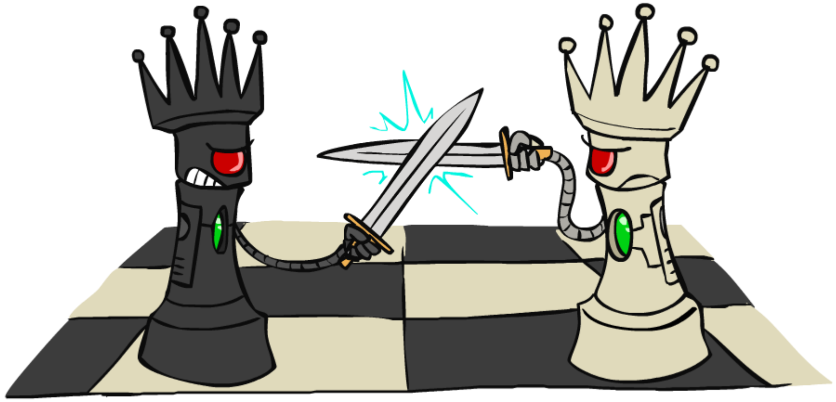


$\{\mathrm{WA}{=}\mathrm{red},\ \mathrm{NT}{=}\mathrm{green},\ \mathrm{Q}{=}\mathrm{red},\ \mathrm{NSW}{=}\mathrm{green},$
$\mathrm{V}{=}\mathrm{red},\ \mathrm{SA}{=}\mathrm{blue},\ \mathrm{T}{=}\mathrm{green}\}$

Example: N-Queens

- Formulation 1:
  - Variables: X_{ij}
  - Domains: $\{0, 1\}$
  - Constraints



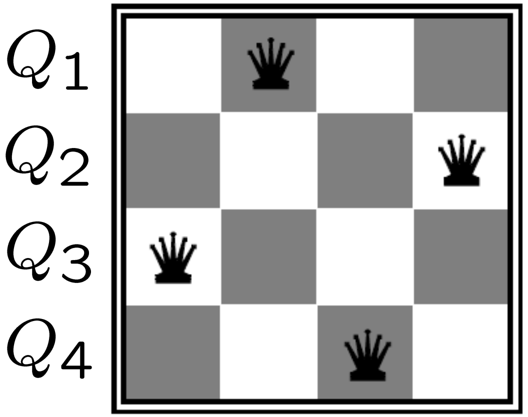


$$\forall i, j, k \;\; (X_{ij}, X_{ik}) \in \{(0,0), (0,1), (1,0)\}$$
$$\forall i, j, k \;\; (X_{ij}, X_{kj}) \in \{(0,0), (0,1), (1,0)\}$$
$$\forall i, j, k \;\; (X_{ij}, X_{i+k,j+k}) \in \{(0,0), (0,1), (1,0)\}$$
$$\forall i, j, k \;\; (X_{ij}, X_{i+k,j-k}) \in \{(0,0), (0,1), (1,0)\}$$

$$\sum_{i,j} X_{ij} = N$$

Example: N-Queens

- Formulation 2:
  - Variables: Q_k
  - Domains: $\{1, 2, 3, \ldots N\}$
  - Constraints:

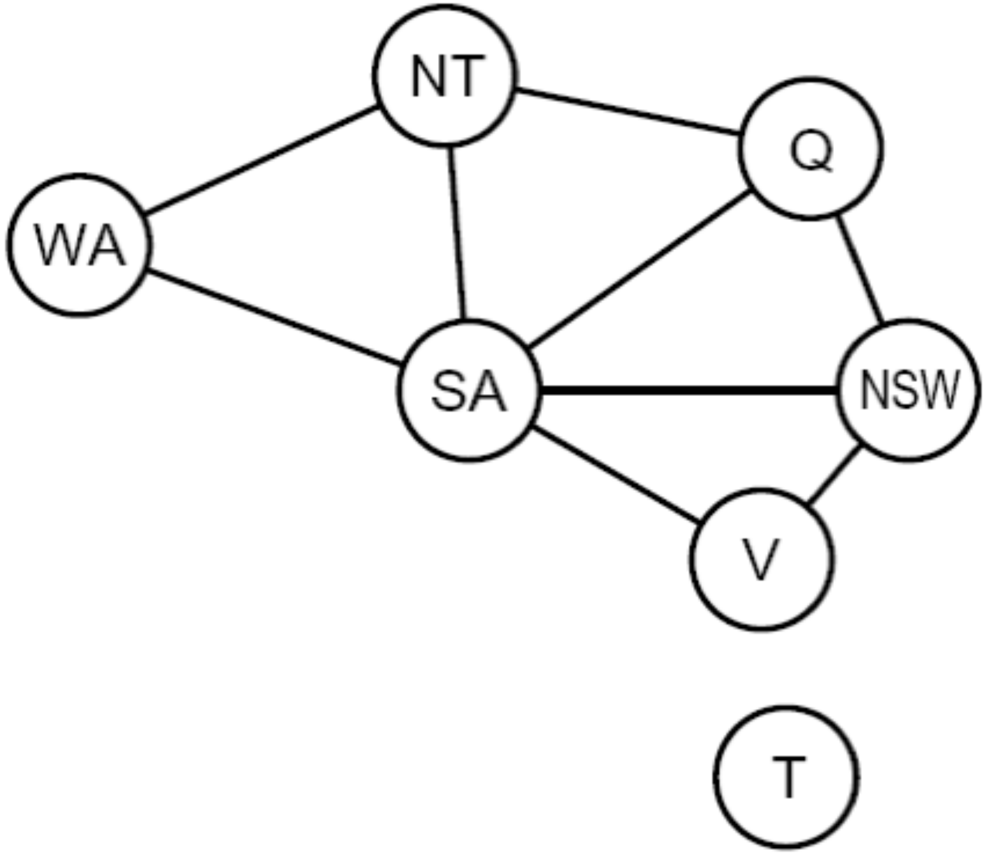


Implicit: $\forall i, j \;\; \text{non-threatening}(Q_i, Q_j)$

Explicit: $(Q_1, Q_2) \in \{(1, 3), (1, 4), \ldots\}$

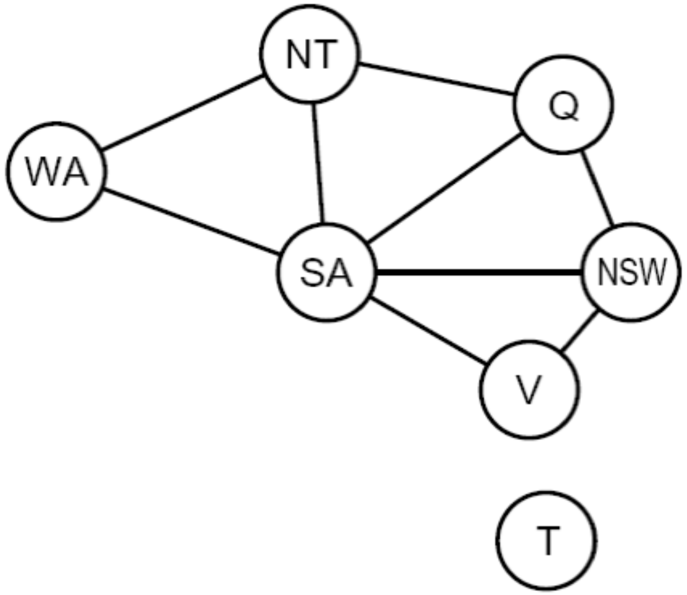
$\cdots$

Constraint Graphs



Constraint Graphs

- Binary CSP: each constraint relates (at most) two variables
- Binary constraint graph: nodes are variables, arcs show constraints
- General-purpose CSP algorithms use the graph structure to speed up search. E.g., Tasmania is an independent subproblem!



[Demo: CSP applet (made available by aispace.org) -- n-queens]

Example: Cryptarithmetic

- Variables:

  $F \ T \ U \ W \ R \ O \ X_1 \ X_2 \ X_3$

- Domains:

  $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$

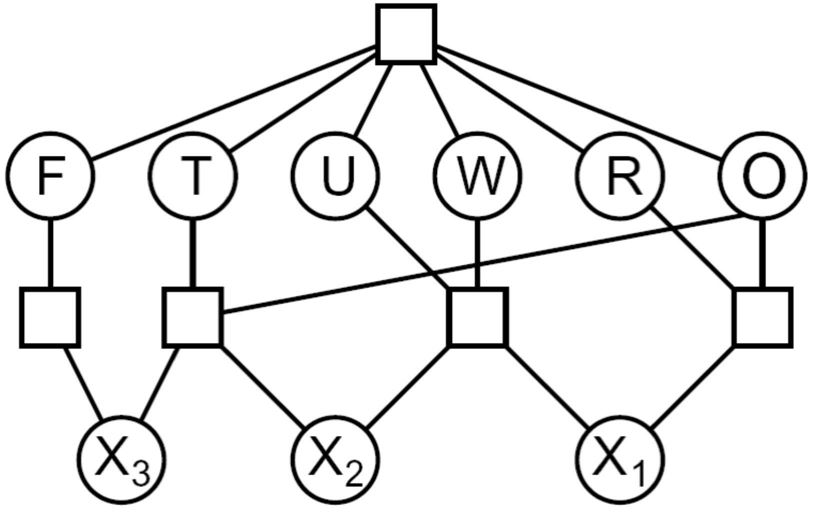
- Constraints:

  $\text{alldiff}(F, T, U, W, R, O)$

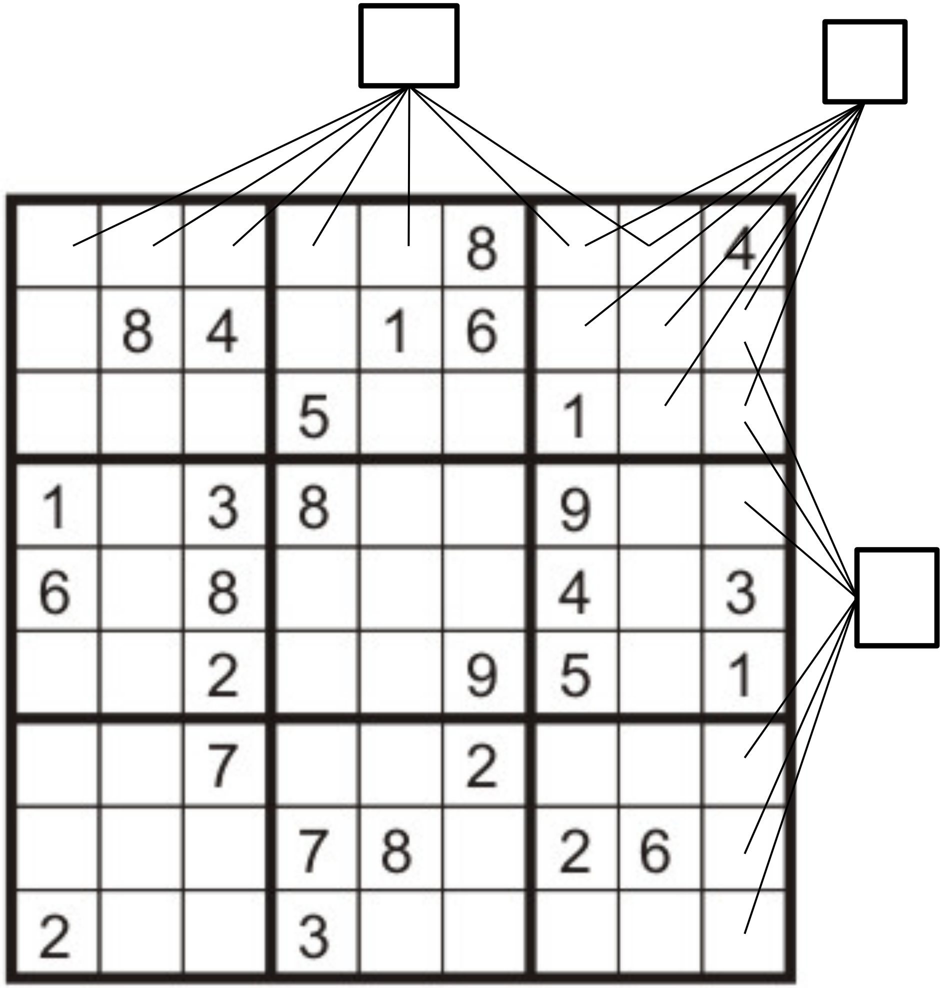
  $O + O = R + 10 \cdot X_1$

  $\cdots$

```
    T W O
  + T W O
  -------
  F O U R
```



Example: Sudoku



- Variables:
  - Each (open) square
- Domains:
  - {1,2,…,9}
- Constraints:

  9-way alldiff for each column

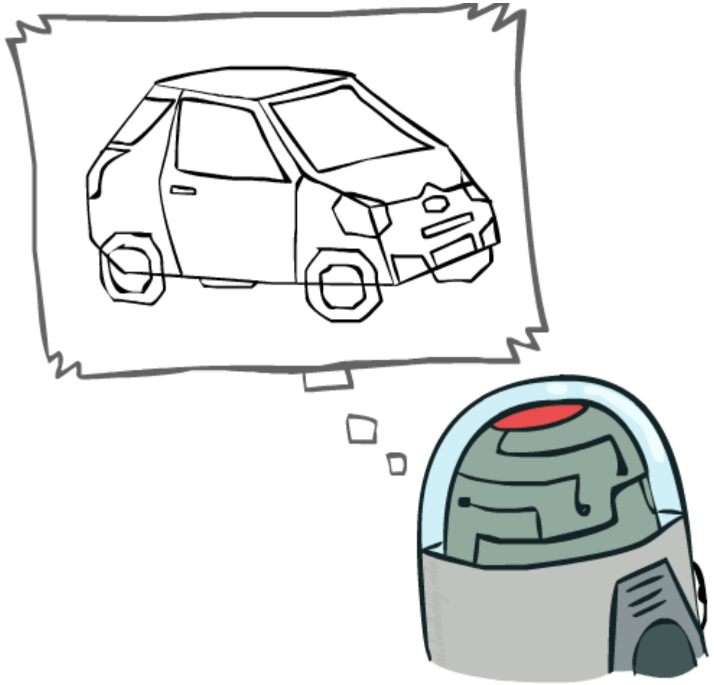
  9-way alldiff for each row

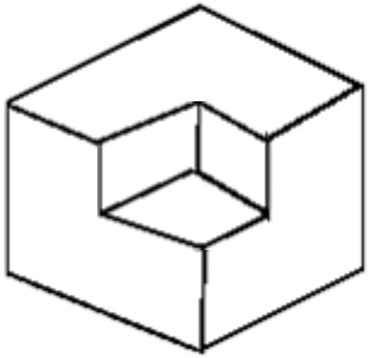
  9-way alldiff for each region

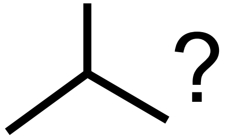
  (or can have a bunch of pairwise inequality constraints)

Example: The Waltz Algorithm

- The Waltz algorithm is for interpreting line drawings of solid polyhedra as 3D objects
- An early example of an AI computation posed as a CSP

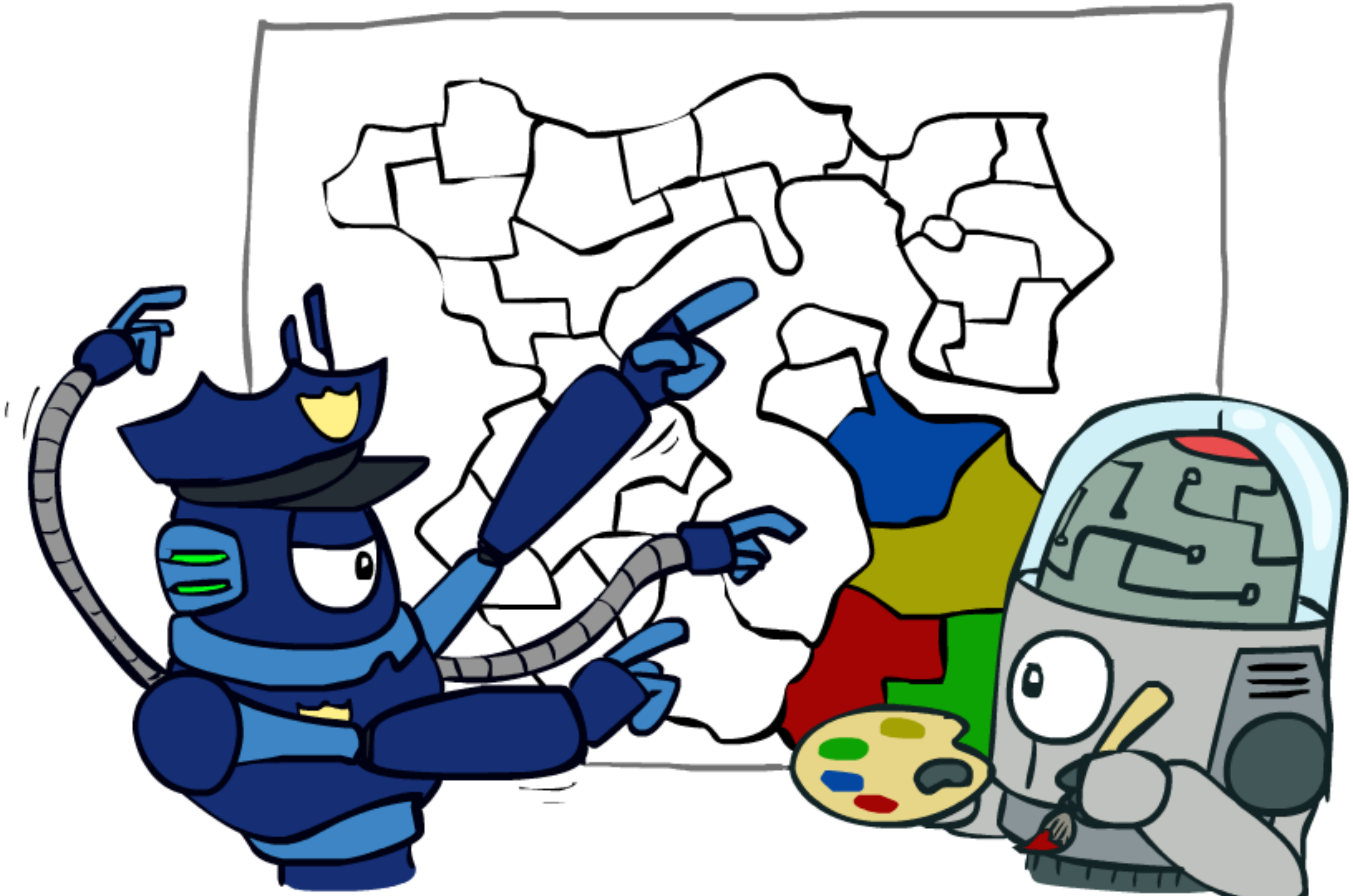






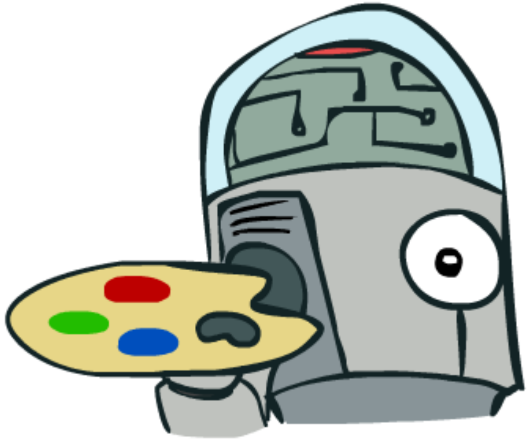
- Approach:
  - Each intersection is a variable
  - Adjacent intersections impose constraints on each other
  - Solutions are physically realizable 3D interpretations

Varieties of CSPs and Constraints



Varieties of CSPs

- Discrete Variables
  - Finite domains
    - Size d means $O(d^n)$ complete assignments
    - E.g., Boolean CSPs, including Boolean satisfiability (NP-complete)
  - Infinite domains (integers, strings, etc.)
    - E.g., job scheduling, variables are start/end times for each job
    - Linear constraints solvable, nonlinear undecidable



- Continuous variables
  - E.g., start/end times for Hubble Telescope observations
  - Linear constraints solvable in polynomial time by Linear Programming methods (see cs170 for a bit of this theory)

Varieties of Constraints

- Varieties of Constraints
  - Unary constraints involve a single variable (equivalent to reducing domains), e.g.:

$$SA \neq green$$

  - Binary constraints involve pairs of variables, e.g.:

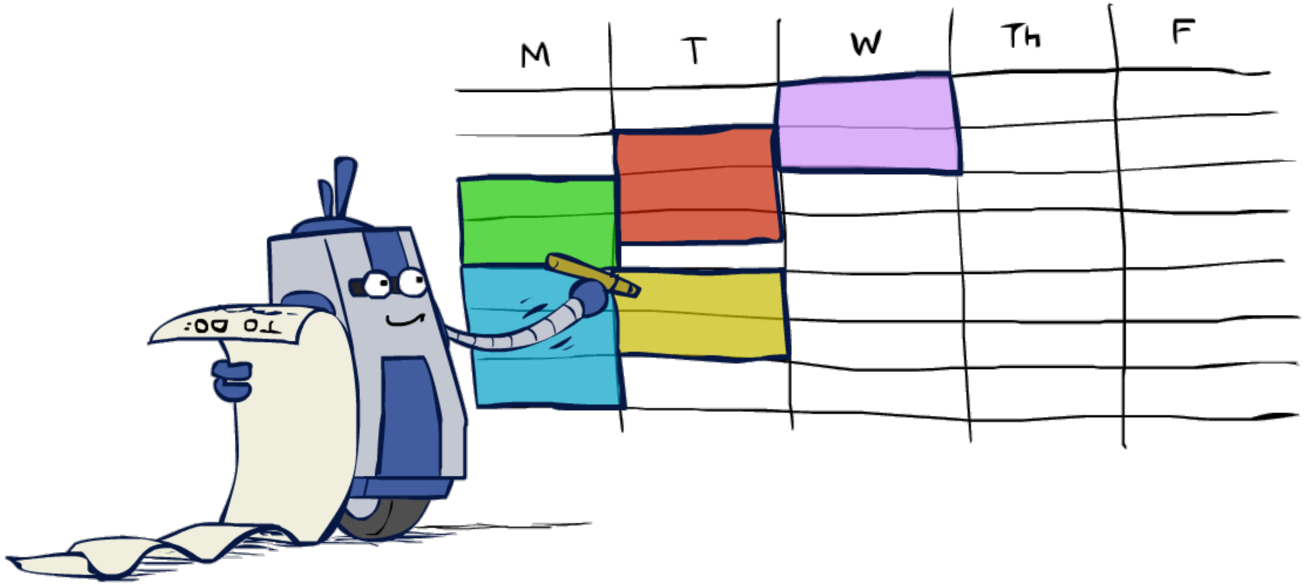
$$SA \neq WA$$

  - Higher-order constraints involve 3 or more variables:
    e.g., cryptarithmetic column constraints

- Preferences (soft constraints):
  - E.g., red is better than green
  - Often representable by a cost for each variable assignment
  - Gives constrained optimization problems
  - (We'll ignore these until we get to Bayes' nets)

Real-World CSPs

- Scheduling problems: e.g., when can we all meet?
- Timetabling problems: e.g., which class is offered when and where?
- Assignment problems: e.g., who teaches what class
- Hardware configuration
- Transportation scheduling
- Factory scheduling
- Circuit layout
- Fault diagnosis
- … lots more!



- Many real-world problems involve real-valued variables…

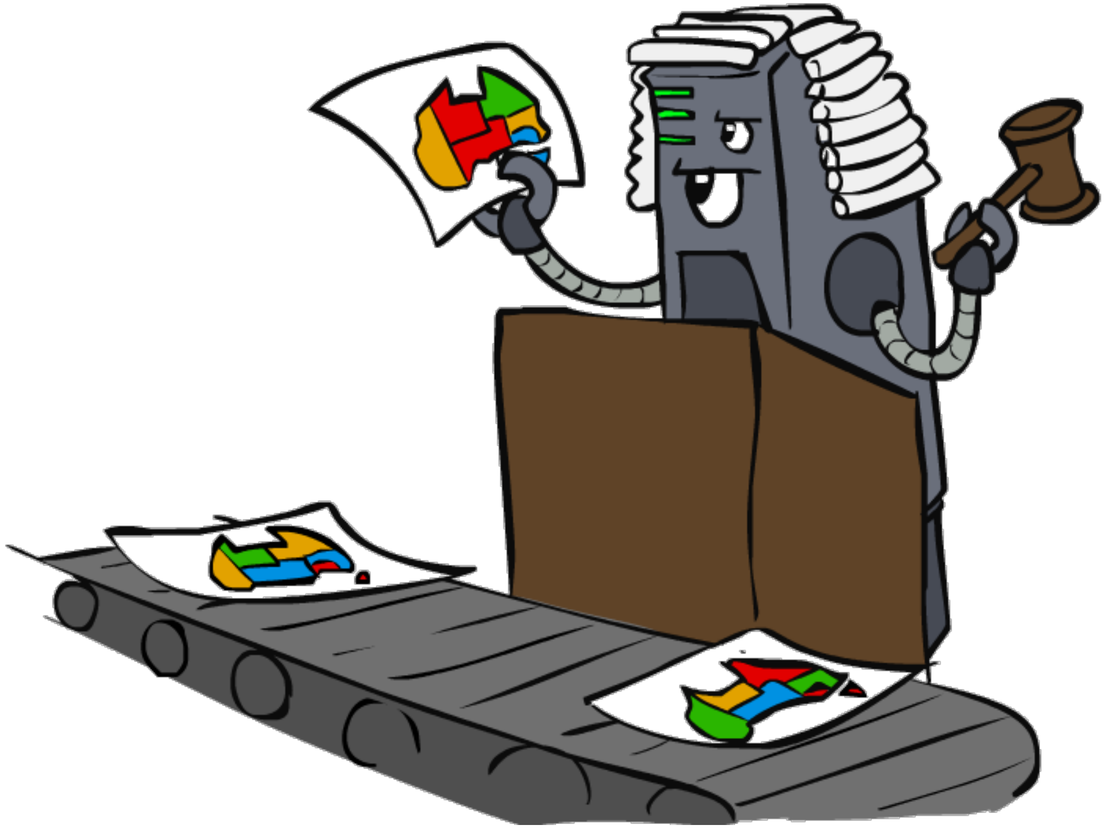
Solving CSPs

Standard Search Formulation

- Standard search formulation of CSPs

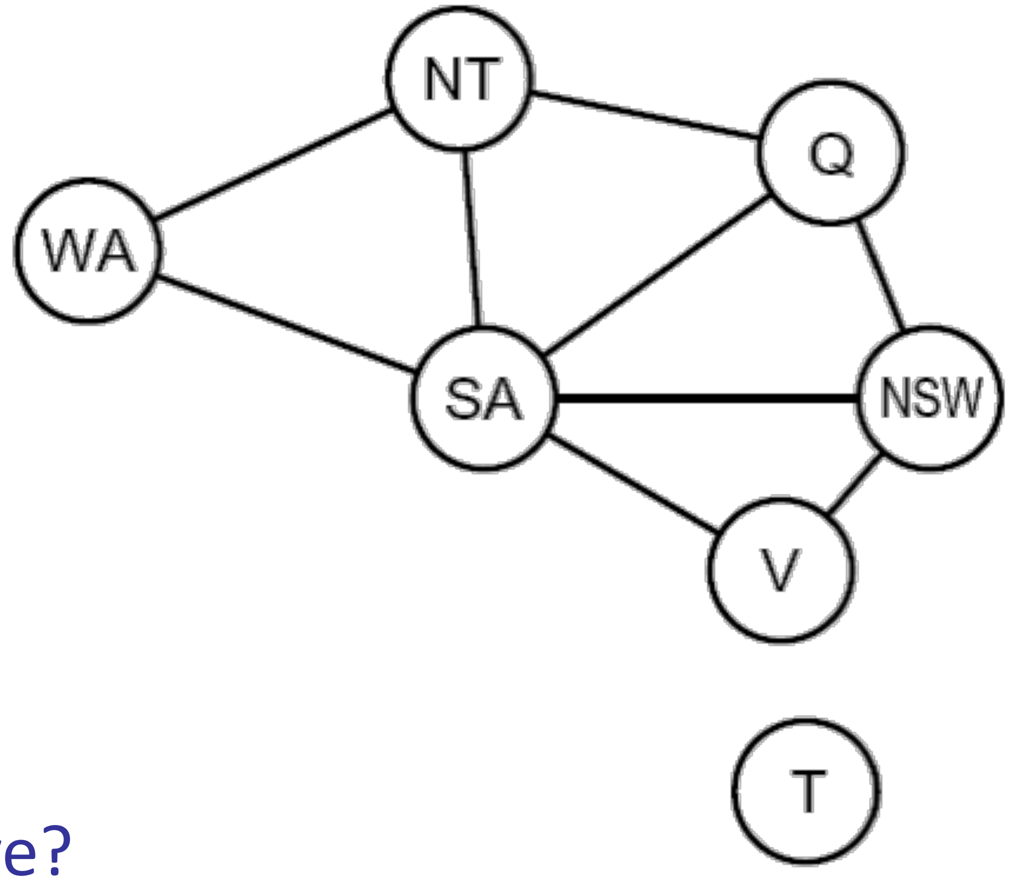
- States defined by the values assigned so far (partial assignments)
  - Initial state: the empty assignment, {}
  - Successor function: assign a value to an unassigned variable
  - Goal test: the current assignment is complete and satisfies all constraints

- We'll start with the straightforward, naïve approach, then improve it



Search Methods

- What would BFS do?

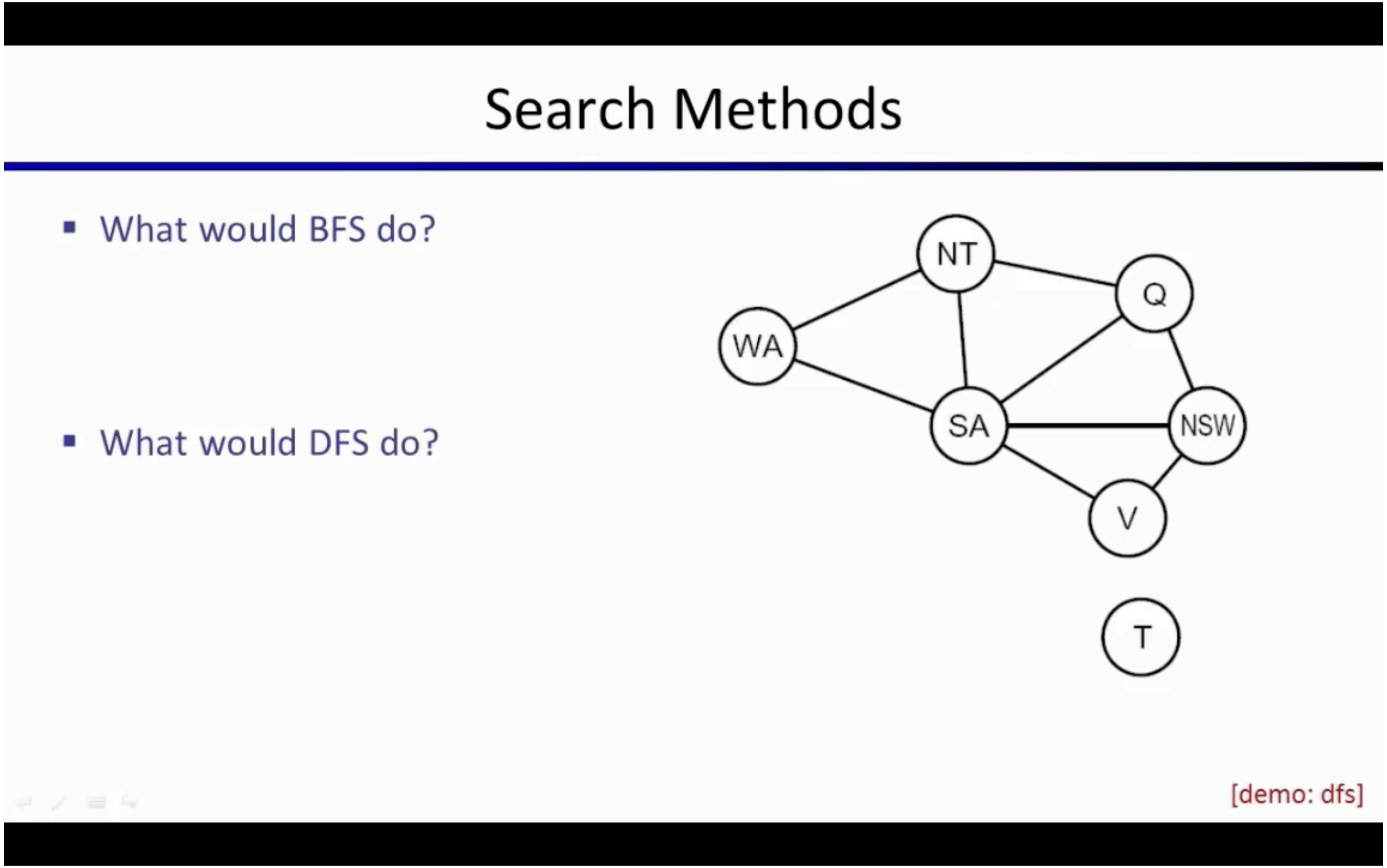


- What would DFS do?

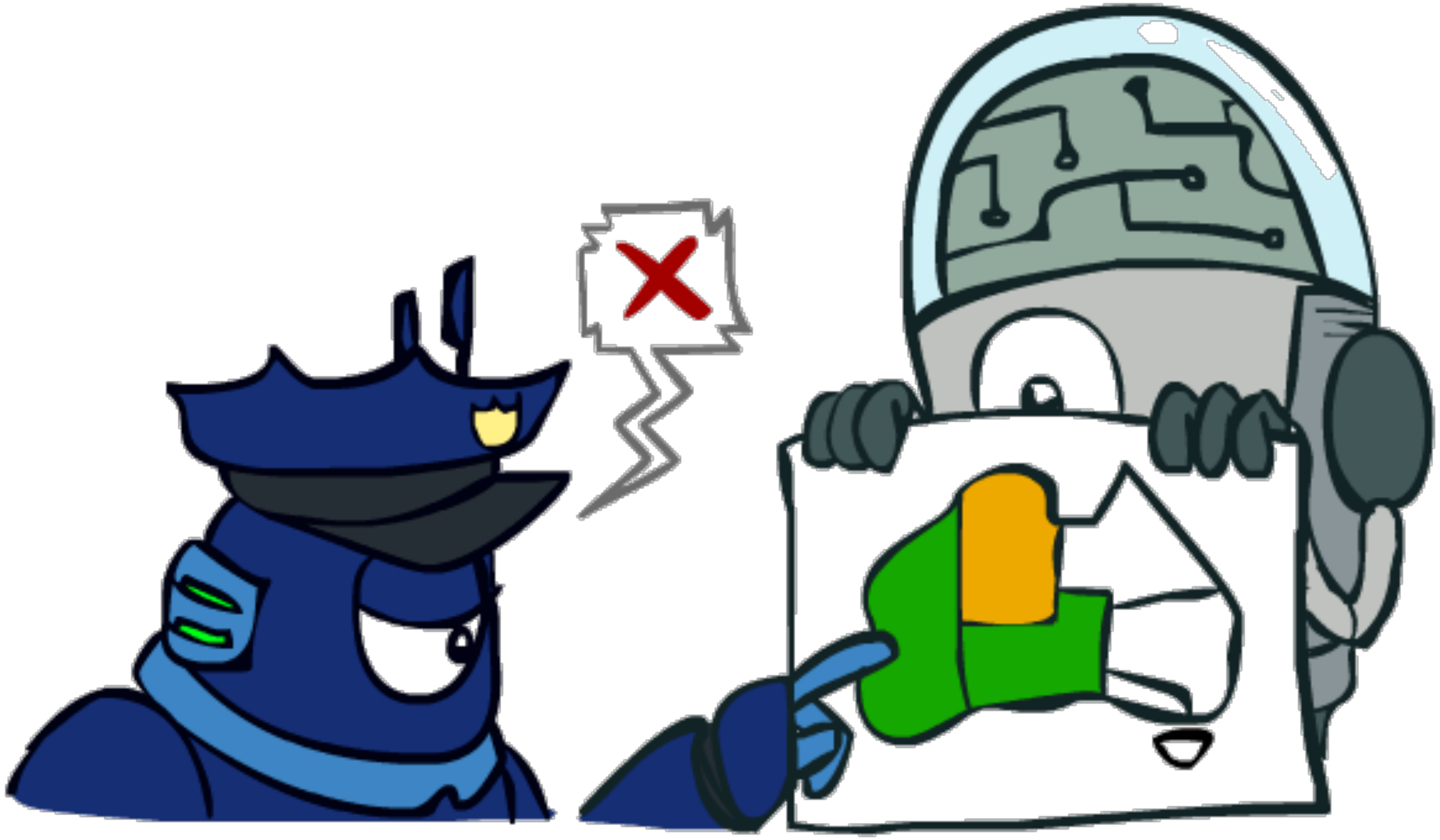
- What problems does naïve search have?

[Demo: coloring -- dfs]

Video of Demo Coloring -- DFS

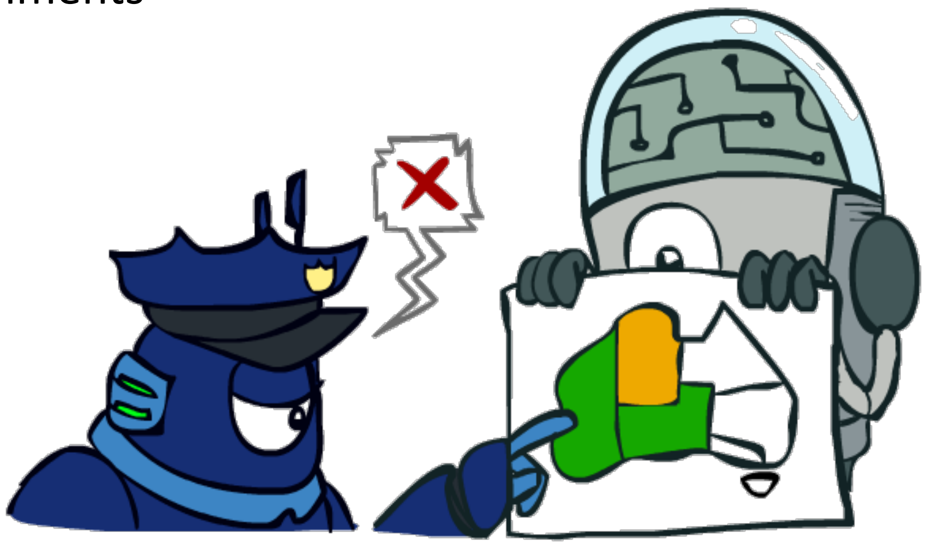


Backtracking Search

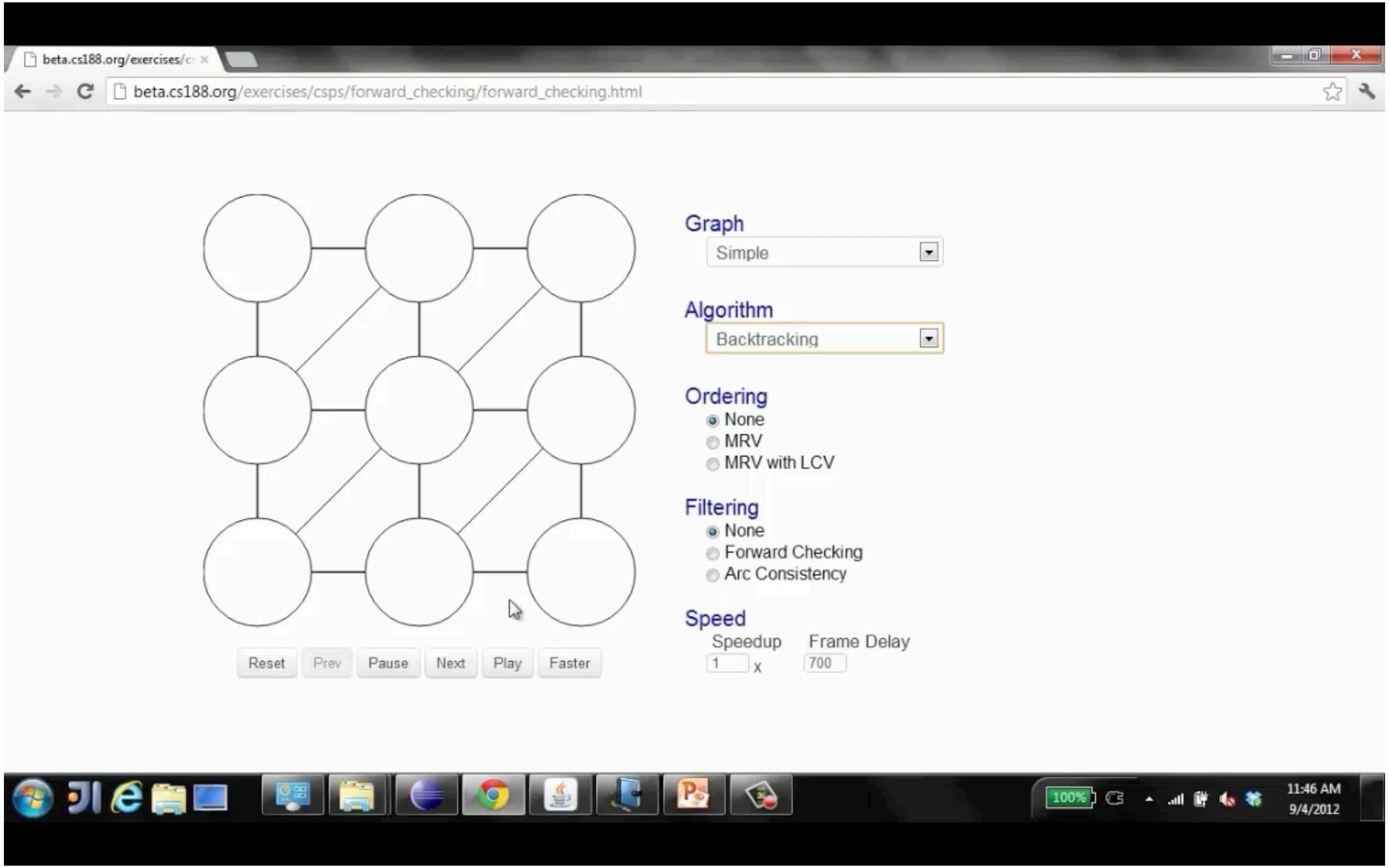


Backtracking Search

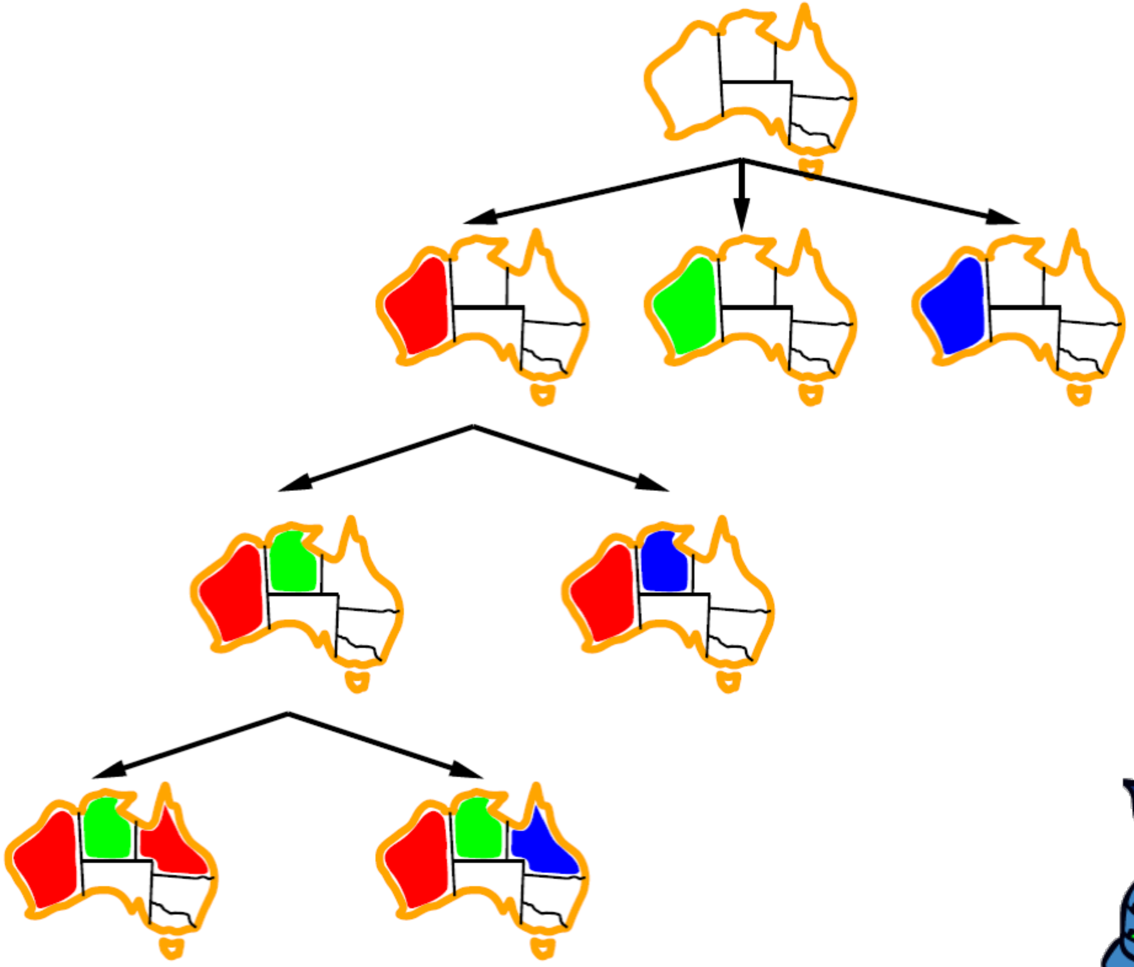
- Backtracking search is the basic uninformed algorithm for solving CSPs
- Idea 1: One variable at a time
  - Variable assignments are commutative, so fix ordering
  - I.e., [WA = red then NT = green] same as [NT = green then WA = red]
  - Only need to consider assignments to a single variable at each step
- Idea 2: Check constraints as you go
  - I.e. consider only values which do not conflict with previous assignments
  - Might have to do some computation to check the constraints
  - “Incremental goal test”
- Depth-first search with these two improvements is called *backtracking search* (not the best name)
- Can solve n-queens for n $\approx$ 25

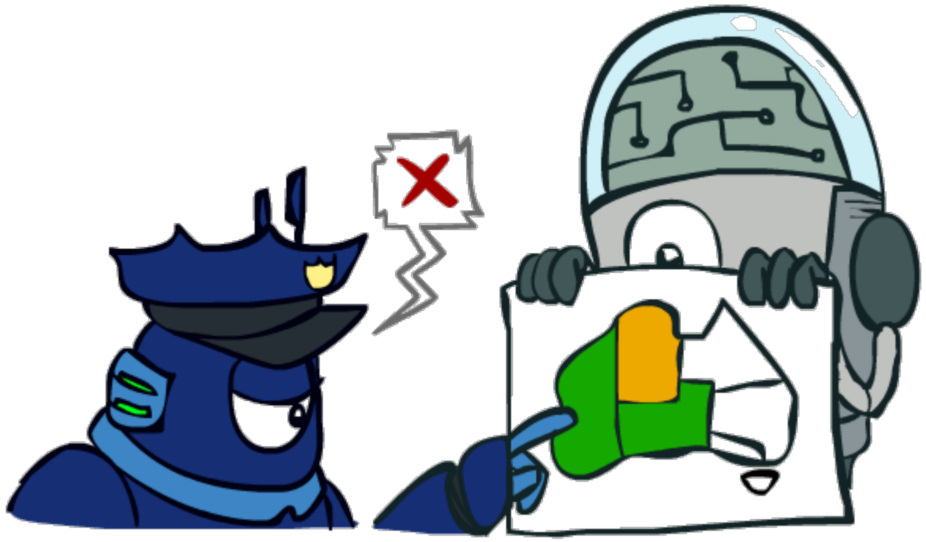


Video of Demo Coloring – Backtracking



Backtracking Example





Backtracking Search

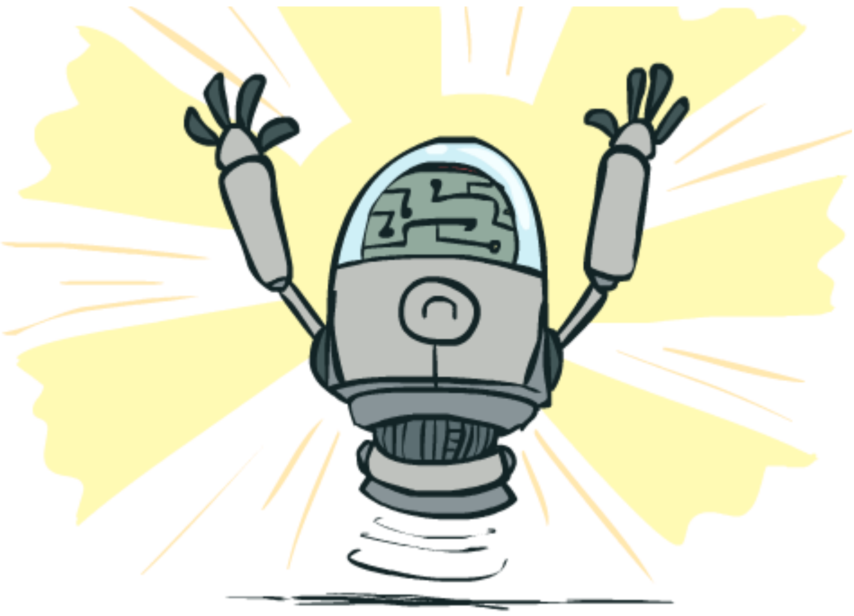
```
function BACKTRACKING-SEARCH(csp) returns solution/failure
   return RECURSIVE-BACKTRACKING({ }, csp)

function RECURSIVE-BACKTRACKING(assignment, csp) returns soln/failure
   if assignment is complete then return assignment
   var ← SELECT-UNASSIGNED-VARIABLE(VARIABLES[csp], assignment, csp)
   for each value in ORDER-DOMAIN-VALUES(var, assignment, csp) do
      if value is consistent with assignment given CONSTRAINTS[csp] then
         add {var = value} to assignment
         result ← RECURSIVE-BACKTRACKING(assignment, csp)
         if result ≠ failure then return result
         remove {var = value} from assignment
   return failure
```

- Backtracking = DFS + variable-ordering + fail-on-violation
- What are the choice points?

[Demo: coloring -- backtracking]

Improving Backtracking

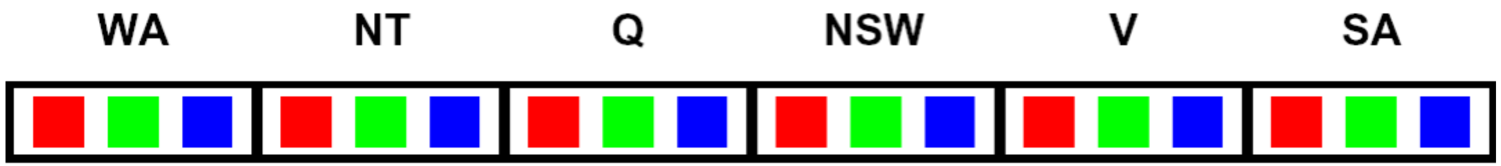
- General-purpose ideas give huge gains in speed
- Ordering:
  - Which variable should be assigned next?
  - In what order should its values be tried?
- Filtering: Can we detect inevitable failure early?
- Structure: Can we exploit the problem structure?



Filtering

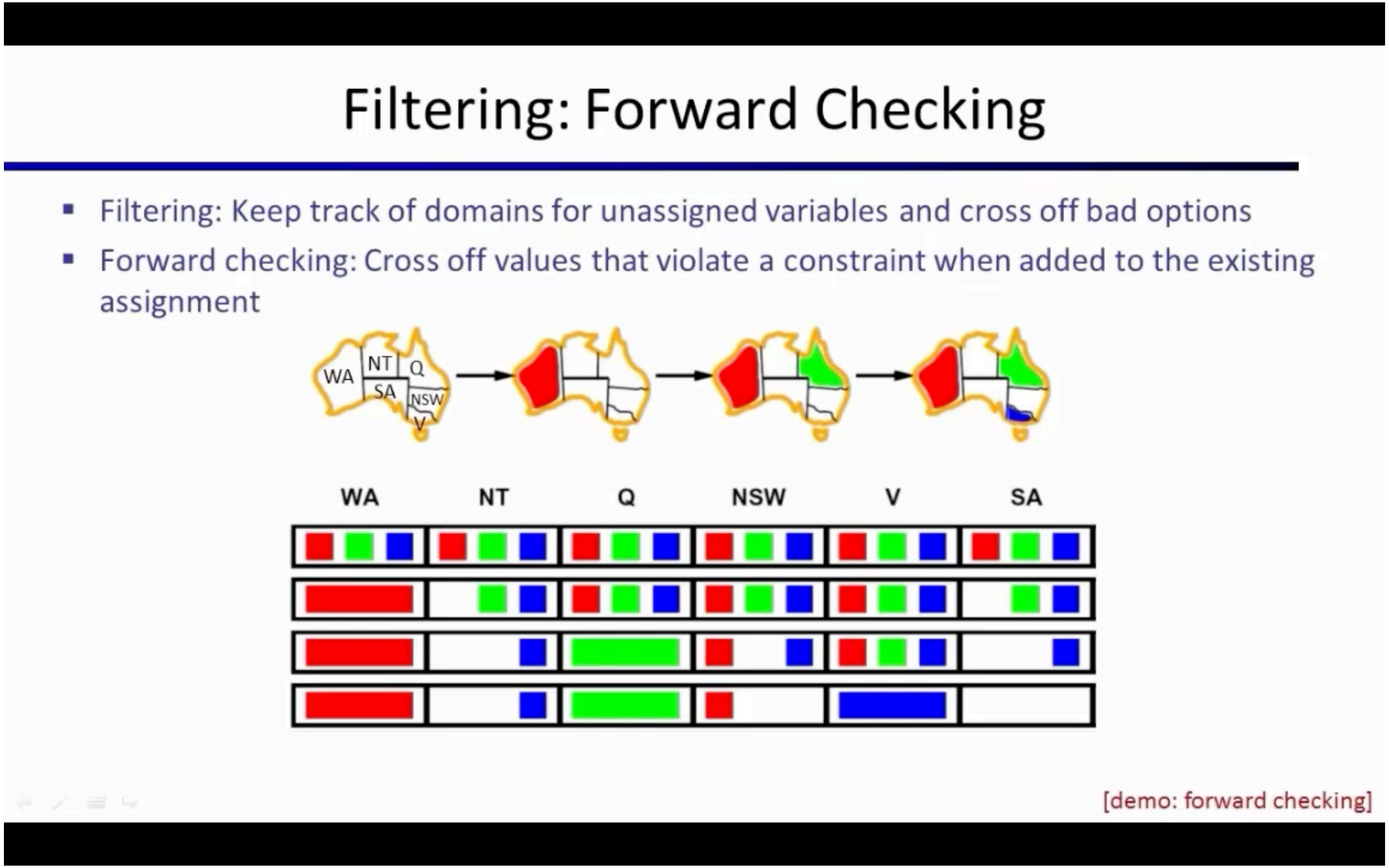
Filtering: Forward Checking

- Filtering: Keep track of domains for unassigned variables and cross off bad options
- Forward checking: Cross off values that violate a constraint when added to the existing assignment



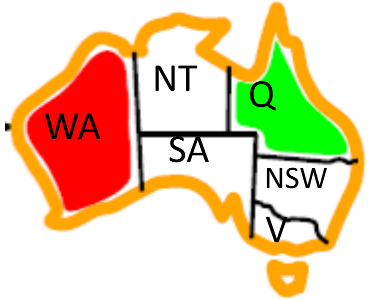
[Demo: coloring -- forward checking]

Video of Demo Coloring – Backtracking with Forward Checking



Filtering: Constraint Propagation

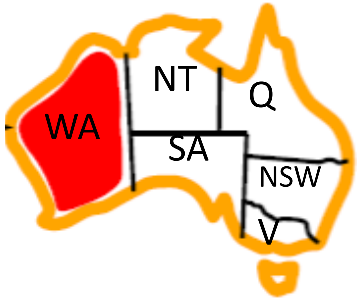
- Forward checking propagates information from assigned to unassigned variables, but doesn't provide early detection for all failures:

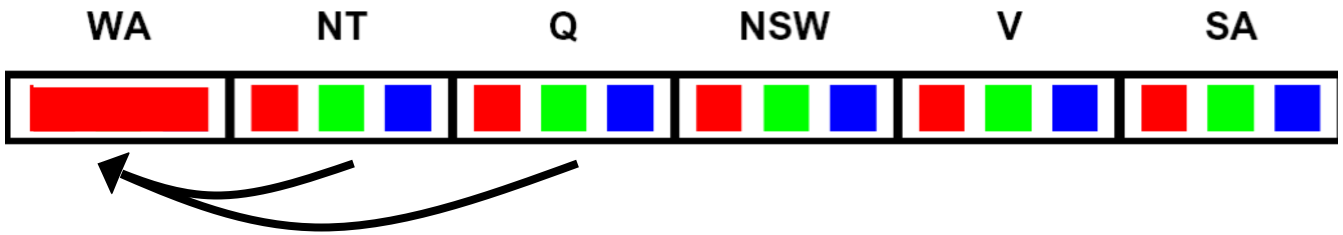


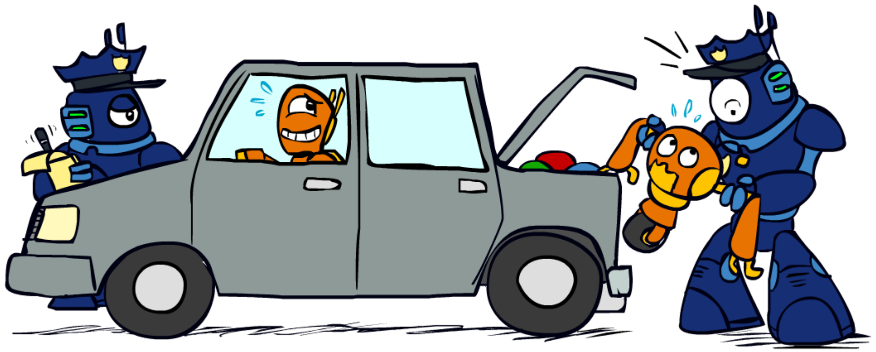
- NT and SA cannot both be blue!
- Why didn't we detect this yet?
- *Constraint propagation:* reason from constraint to constraint

Consistency of A Single Arc

- An arc X → Y is consistent iff for *every* x in the tail there is *some* y in the head which could be assigned without violating a constraint





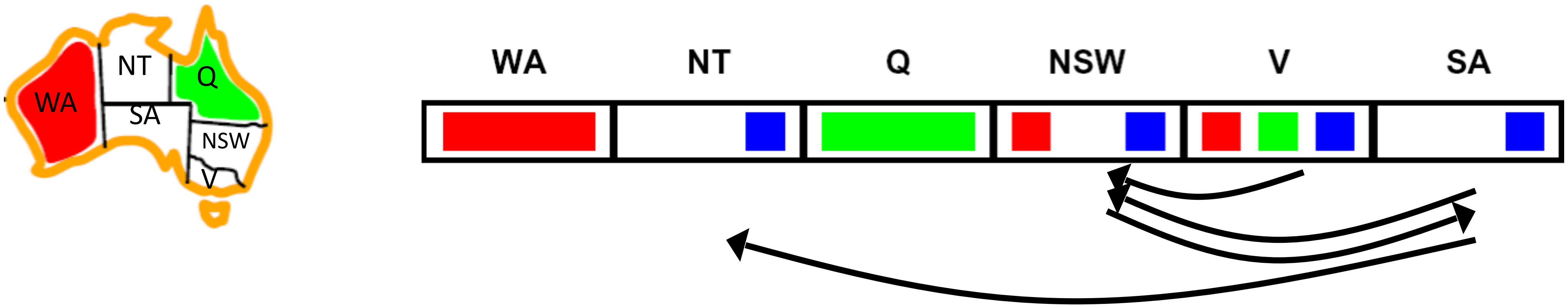


Delete from the tail!

- Forward checking: Enforcing consistency of arcs pointing to each new assignment

Arc Consistency of an Entire CSP

- A simple form of propagation makes sure all arcs are consistent:



- Important: If X loses a value, neighbors of X need to be rechecked!
- Arc consistency detects failure earlier than forward checking
- Can be run as a preprocessor or after each assignment
- What's the downside of enforcing arc consistency?

Remember: Delete from the tail!

Enforcing Arc Consistency in a CSP

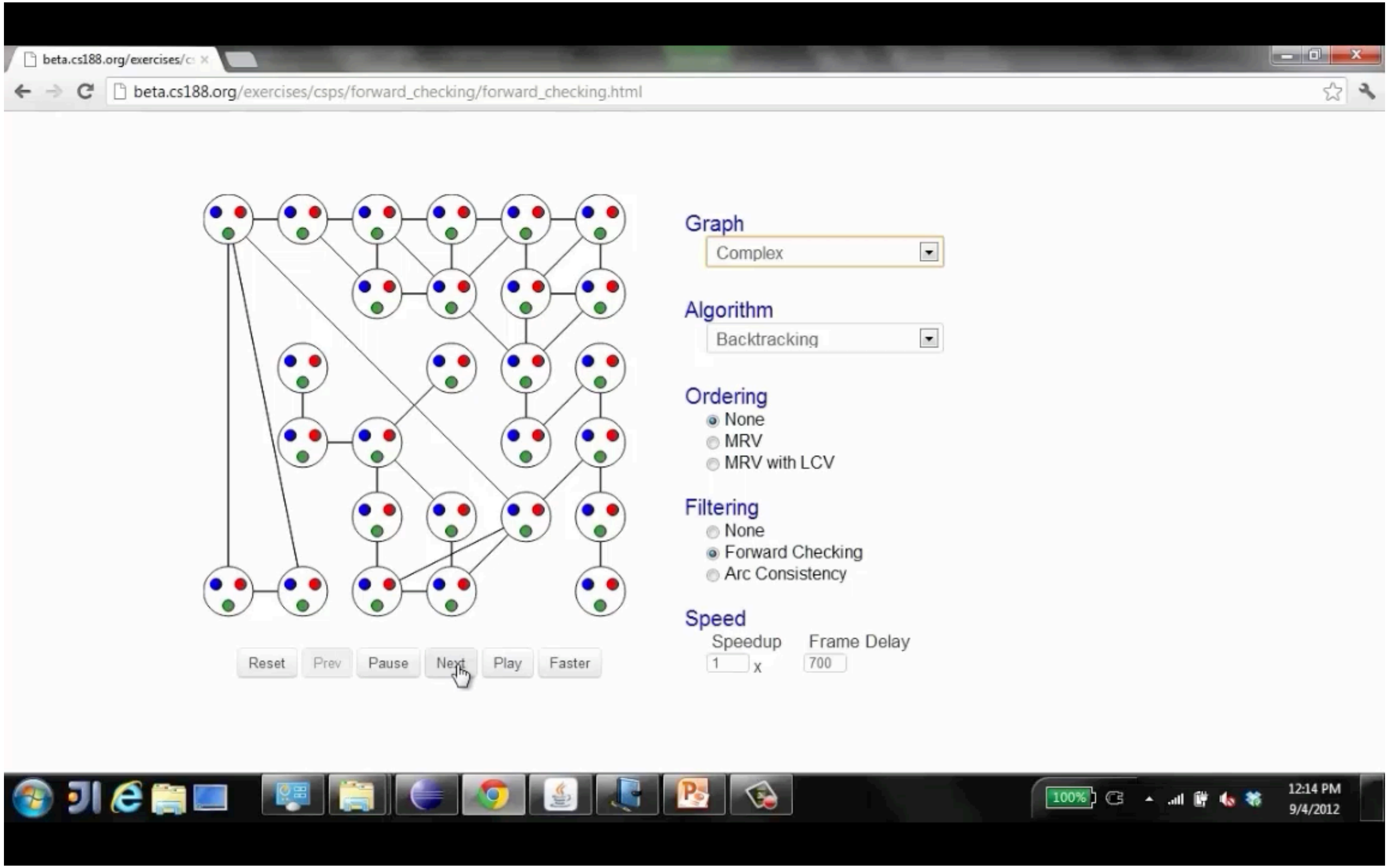
```
function AC-3( csp) returns the CSP, possibly with reduced domains
   inputs: csp, a binary CSP with variables {X1, X2, ..., Xn}
   local variables: queue, a queue of arcs, initially all the arcs in csp

   while queue is not empty do
      (Xi, Xj) ← REMOVE-FIRST(queue)
      if REMOVE-INCONSISTENT-VALUES(Xi, Xj) then
         for each Xk in NEIGHBORS[Xi] do
            add (Xk, Xi) to queue
```

```
function REMOVE-INCONSISTENT-VALUES( Xi, Xj) returns true iff succeeds
   removed ← false
   for each x in DOMAIN[Xi] do
      if no value y in DOMAIN[Xj] allows (x,y) to satisfy the constraint Xi ↔ Xj
         then delete x from DOMAIN[Xi]; removed ← true
   return removed
```

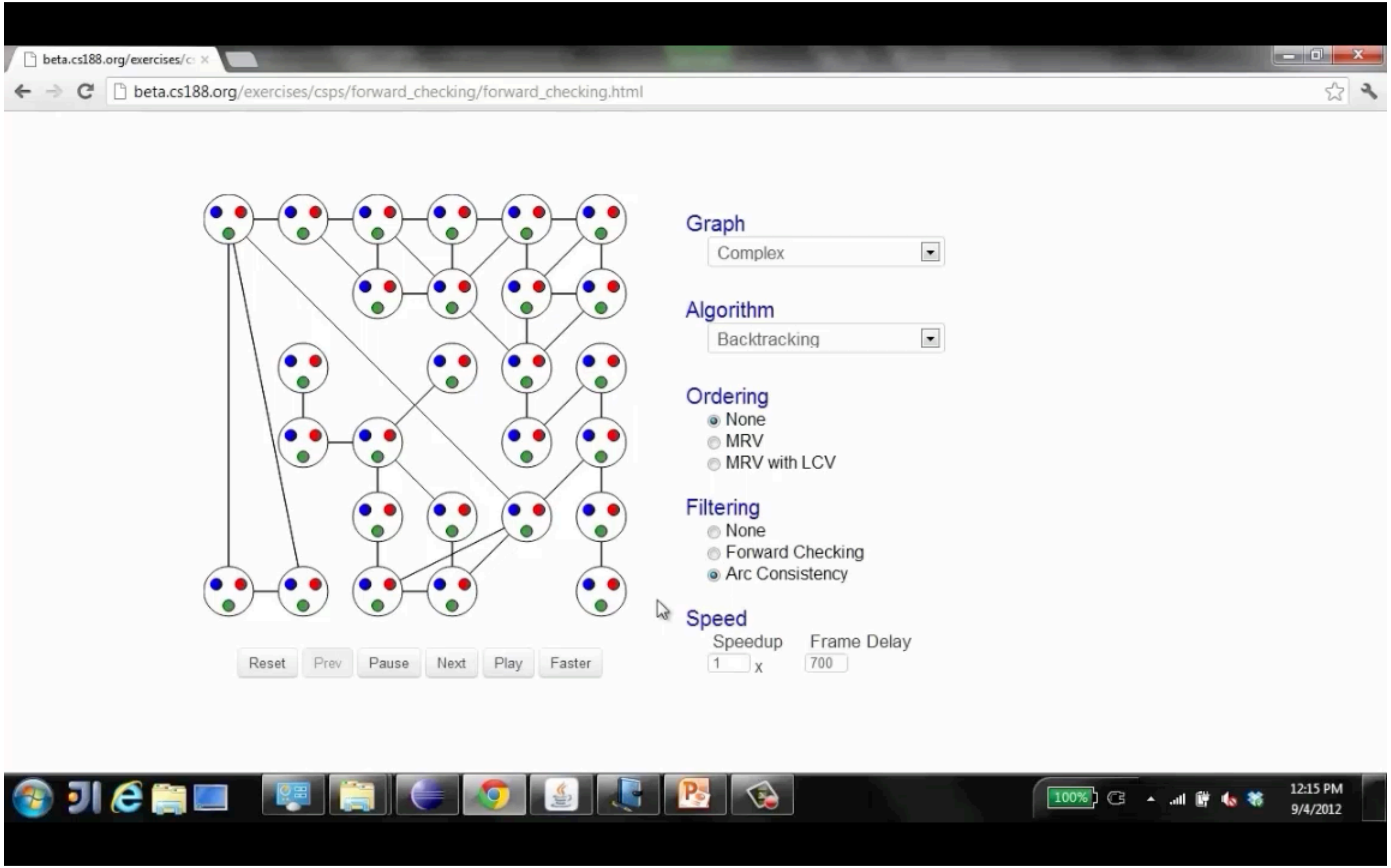
- Runtime: $O(n^2d^3)$, can be reduced to $O(n^2d^2)$
- … but detecting all possible future problems is NP-hard – why?

[Demo: CSP applet (made available by aispace.org) -- n-queens]

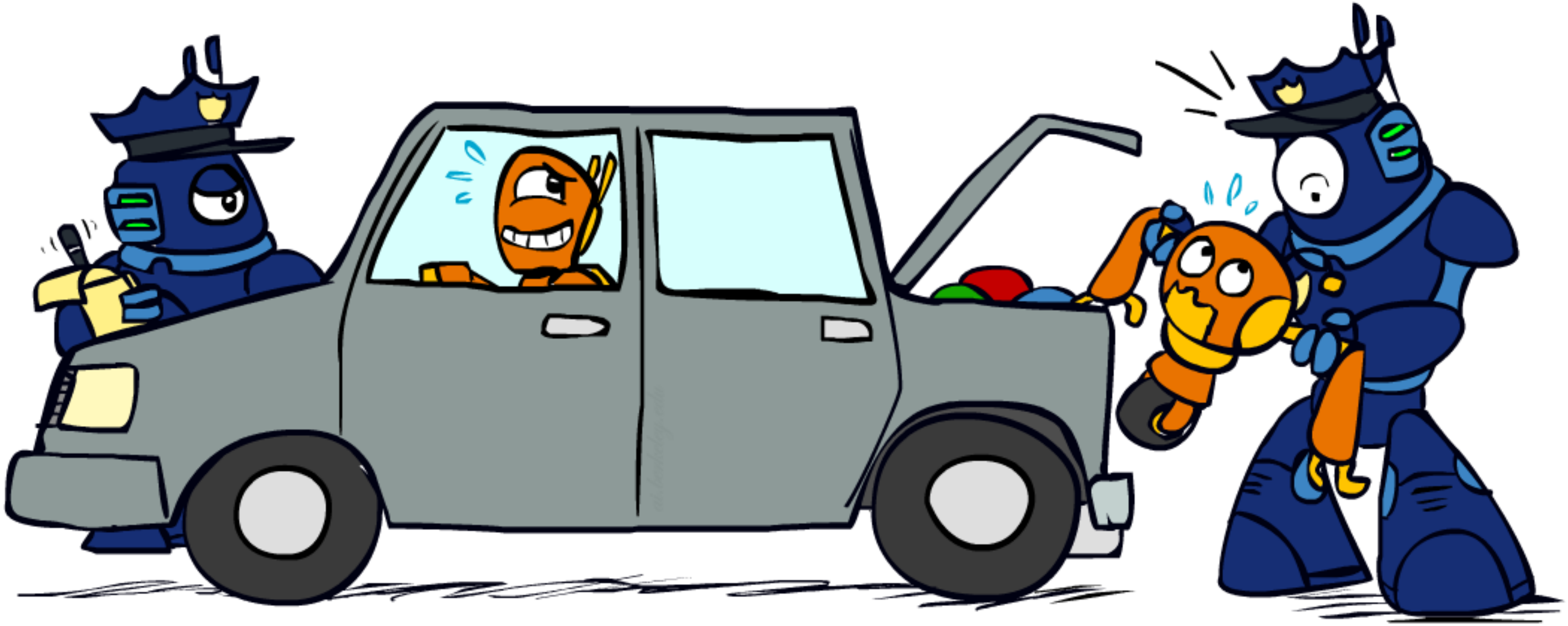
Video of Demo Coloring – Backtracking with Forward Checking – Complex Graph



Video of Demo Coloring – Backtracking with Arc Consistency – Complex Graph



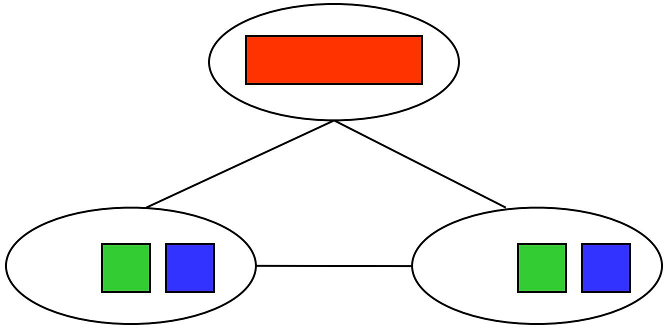
Arc Consistency and Beyond

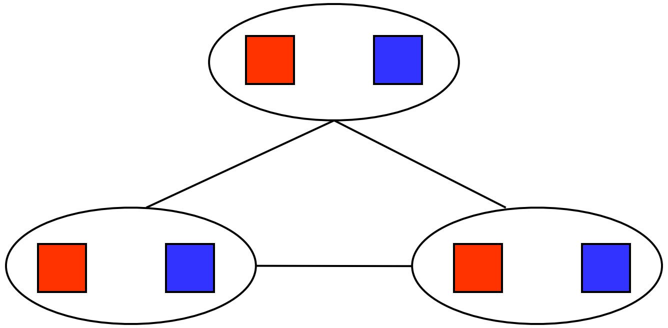


Limitations of Arc Consistency

- After enforcing arc consistency:
  - Can have one solution left
  - Can have multiple solutions left
  - Can have no solutions left (and not know it)

- Arc consistency still runs inside a backtracking search!

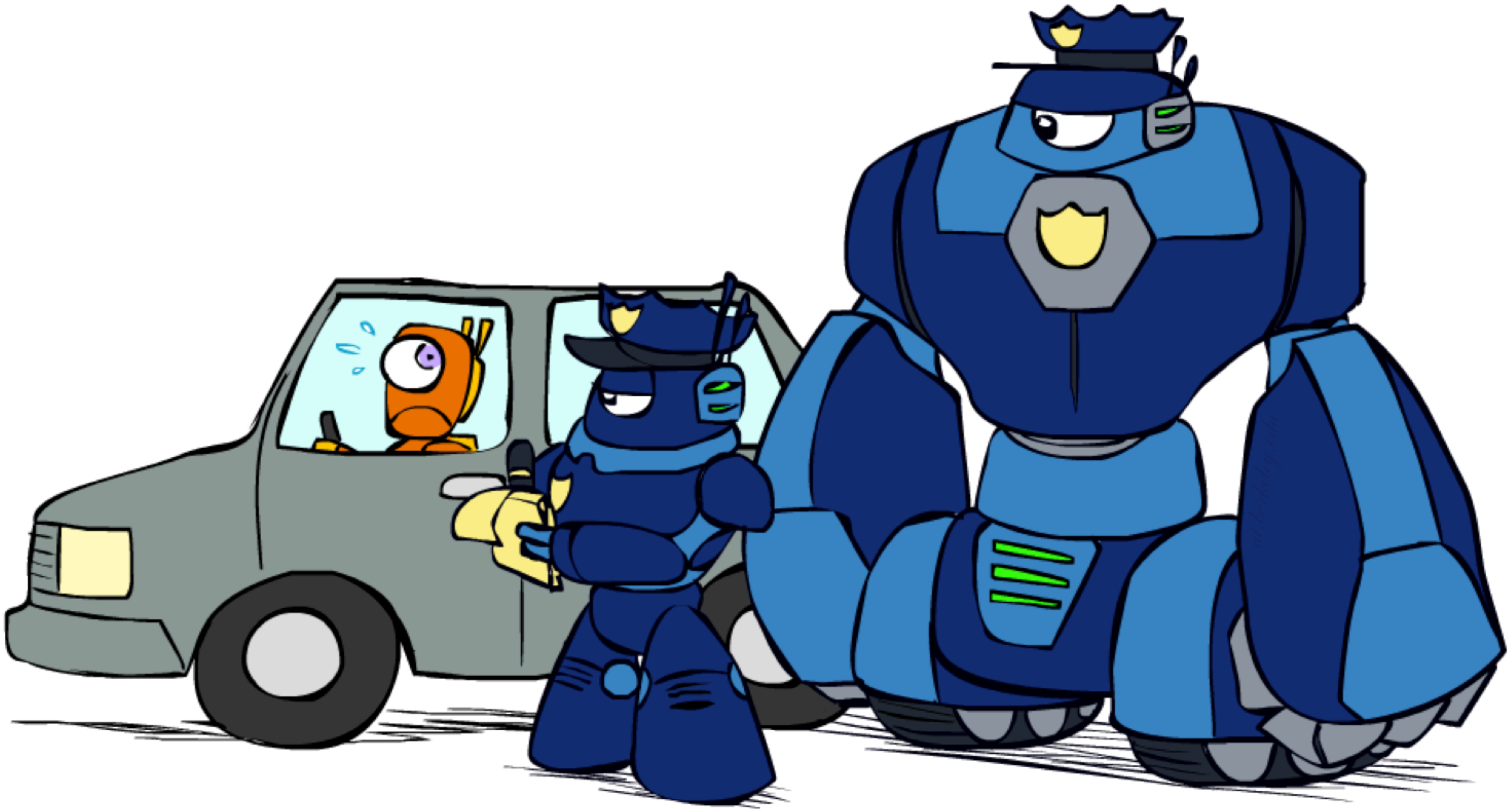




What went wrong here?

[Demo: coloring -- forward checking]
[Demo: coloring -- arc consistency]

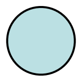
K-Consistency



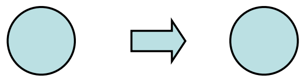
K-Consistency

Increasing degrees of consistency

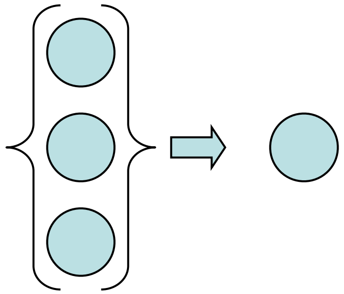
1-Consistency (Node Consistency): Each single node's domain has a value which meets that node's unary constraints



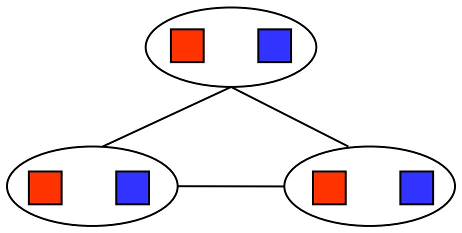
2-Consistency (Arc Consistency): For each pair of nodes, any consistent assignment to one can be extended to the other



K-Consistency: For each k nodes, any consistent assignment to k-1 can be extended to the k^{th} node.



Higher k more expensive to compute



(You need to know the k=2 case: arc consistency)

Strong K-Consistency

Strong k-consistency: also k-1, k-2, … 1 consistent

Claim: strong n-consistency means we can solve without backtracking!

Why?

- Choose any assignment to any variable
- Choose a new variable
- By 2-consistency, there is a choice consistent with the first
- Choose a new variable
- By 3-consistency, there is a choice consistent with the first 2
- …

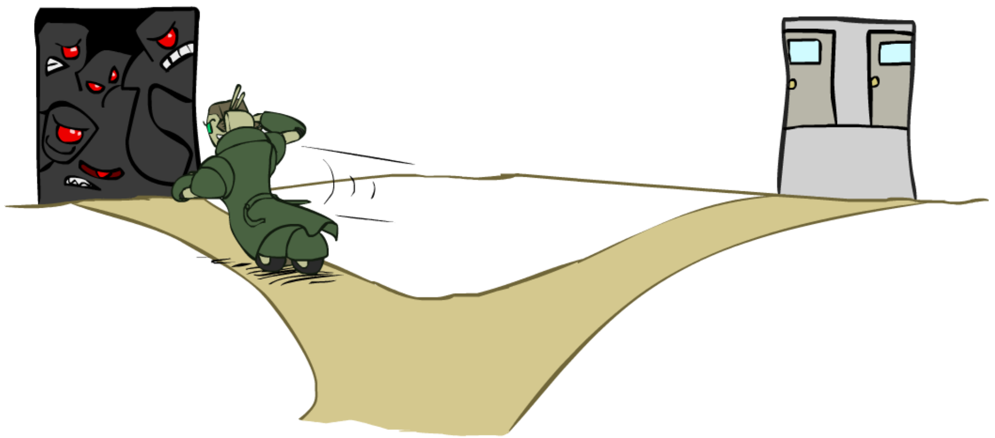
Lots of middle ground between arc consistency and n-consistency! (e.g. k=3, called path consistency)

Ordering

Ordering: Minimum Remaining Values

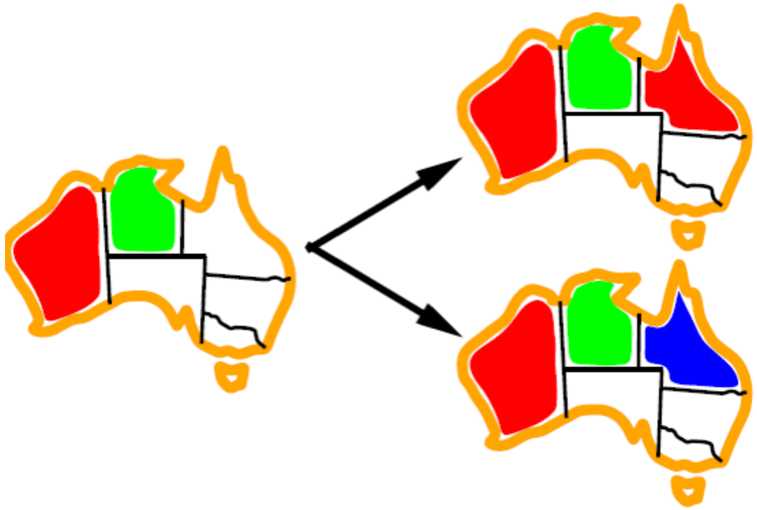
- Variable Ordering: Minimum remaining values (MRV):
  - Choose the variable with the fewest legal left values in its domain

- Why min rather than max?
- Also called “most constrained variable”
- “Fail-fast” ordering



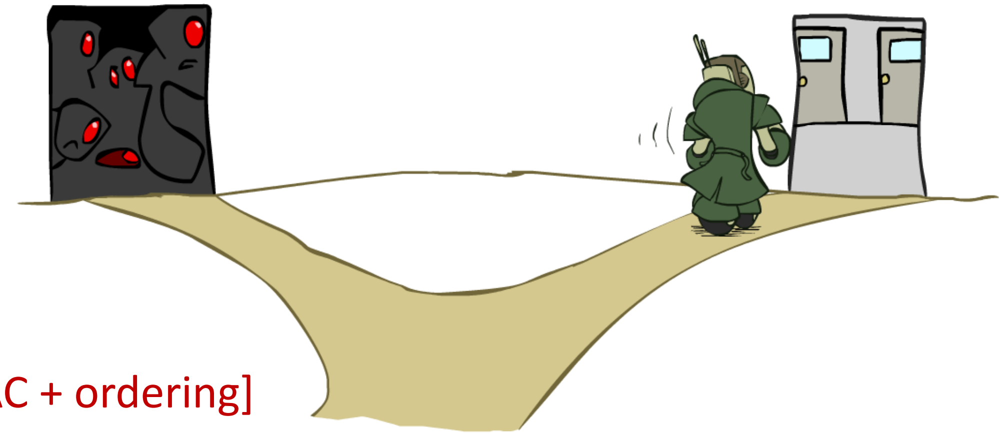
Ordering: Least Constraining Value

- Value Ordering: Least Constraining Value
  - Given a choice of variable, choose the *least constraining value*
  - I.e., the one that rules out the fewest values in the remaining variables
  - Note that it may take some computation to determine this! (E.g., rerunning filtering)



- Why least rather than most?

- Combining these ordering ideas makes 1000 queens feasible



[Demo: coloring – backtracking + AC + ordering]