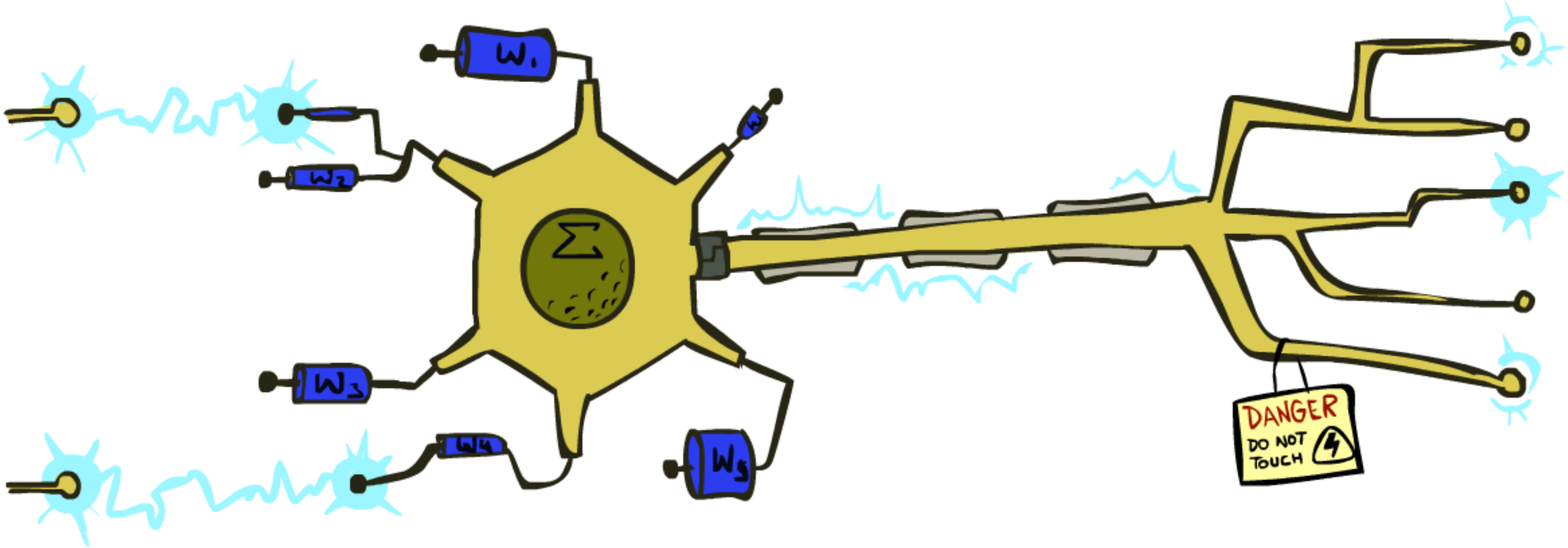
CS 188: Artificial Intelligence

Perceptrons and Logistic Regression



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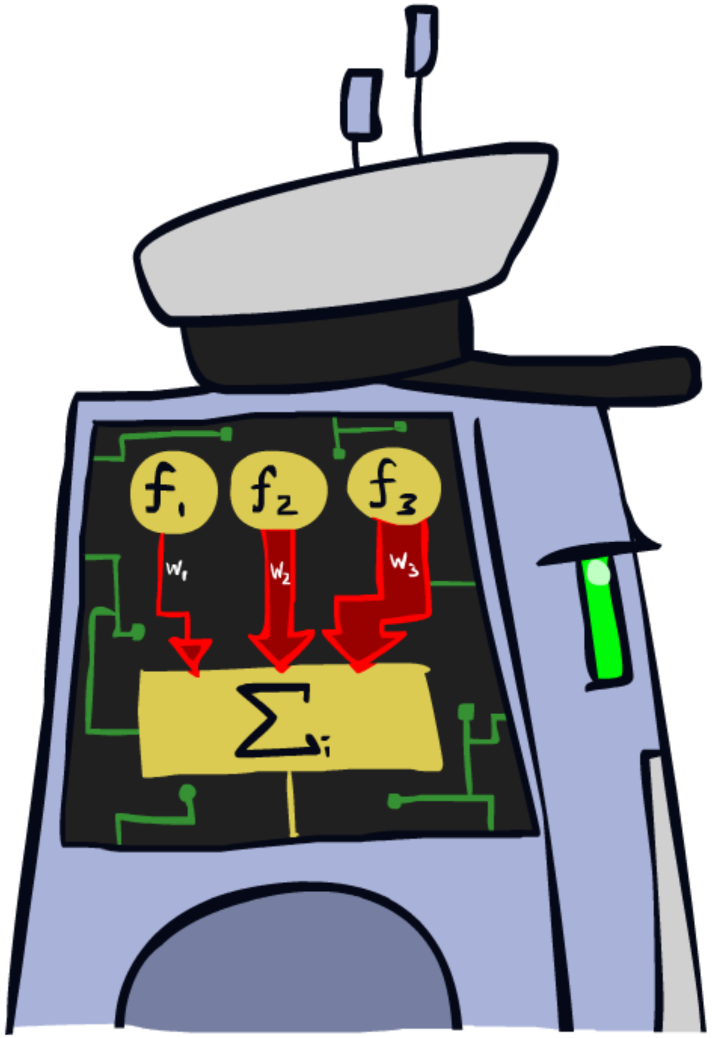
[These slides were created by Dan Klein and Pieter Abbeel for CS188 Intro to AI at UC Berkeley. All CS188 materials are available at http://ai.berkeley.edu.]

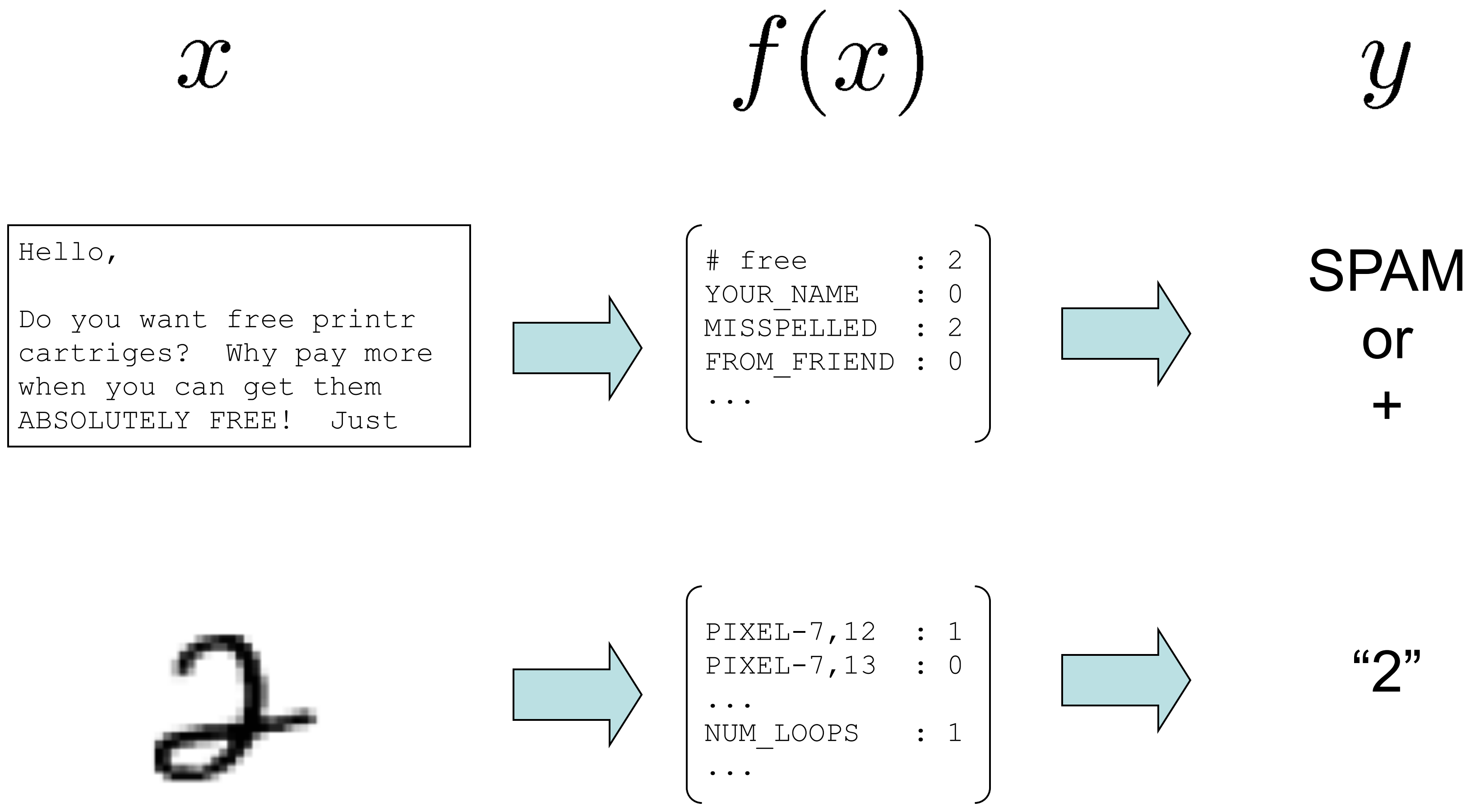
Announcements

- HW8 is due **Thursday, July 31**, 11:59 PM PT
- Project 4 is due **Friday, August 1**, 11:59 PM PT
- HW9 is due **Tuesday, August 5**, 11:59 PM PT
- HW10 is due **Thursday, August 7**, 11:59 PM PT
- Ignore assessment on HWs part B, but please show your work
- Final Exam is **Wednesday, August 13**, 7-10 PM PT
  - Accommodation requests by Wednesday, July 30, 11:59 PM PT

Linear Classifiers

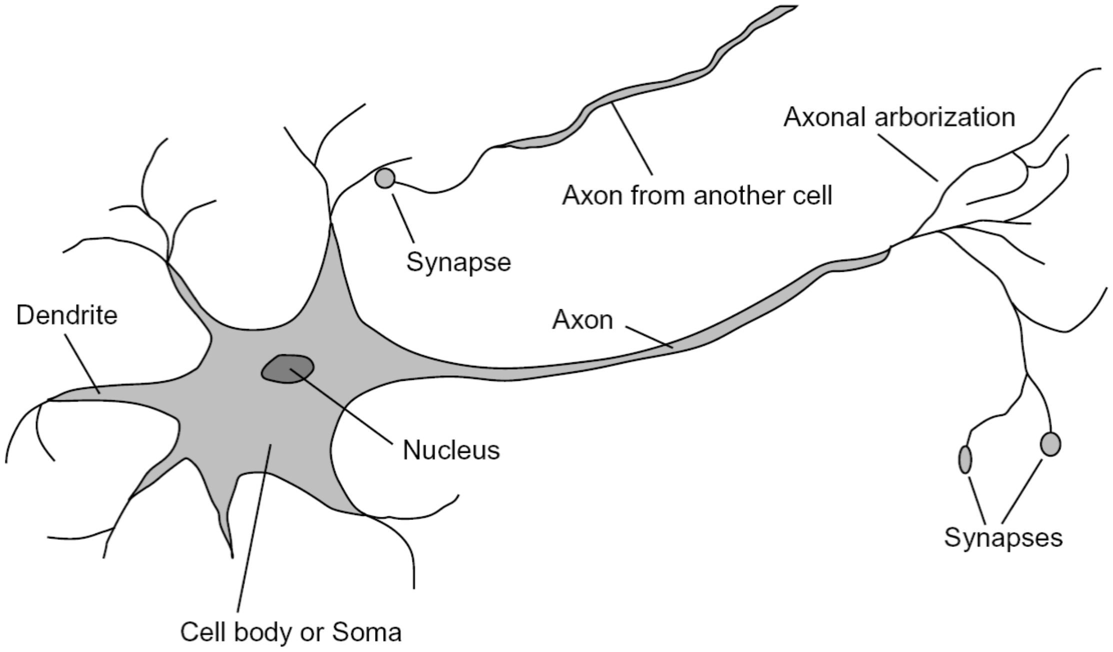


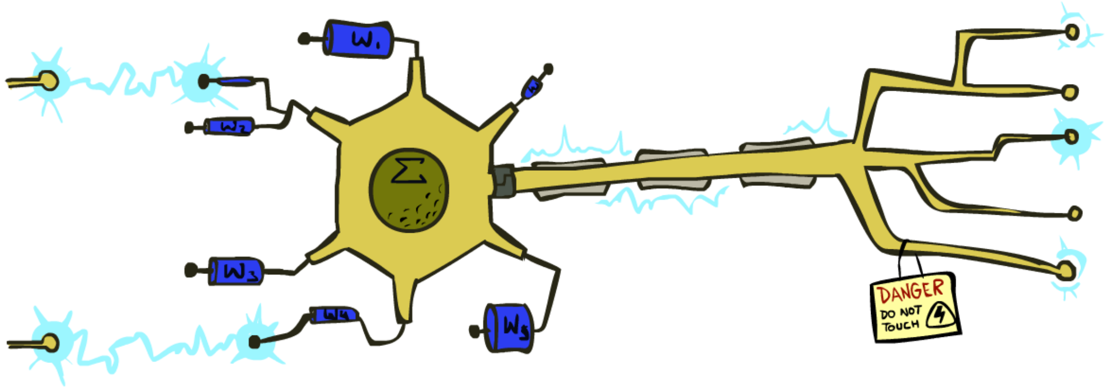
Feature Vectors



Some (Simplified) Biology

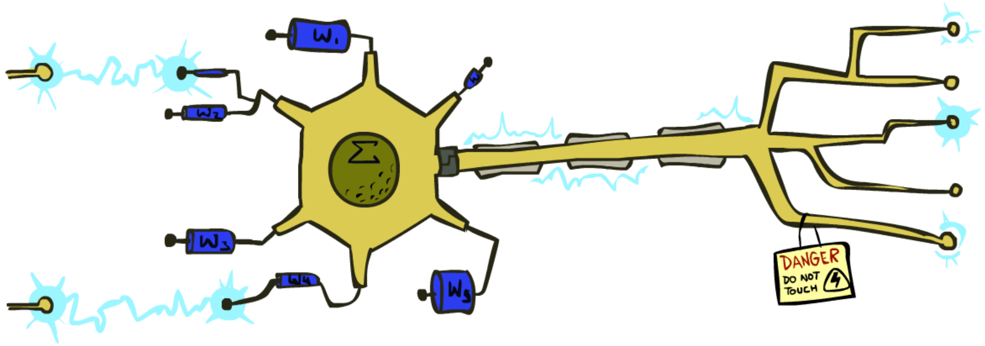
- Very loose inspiration: human neurons





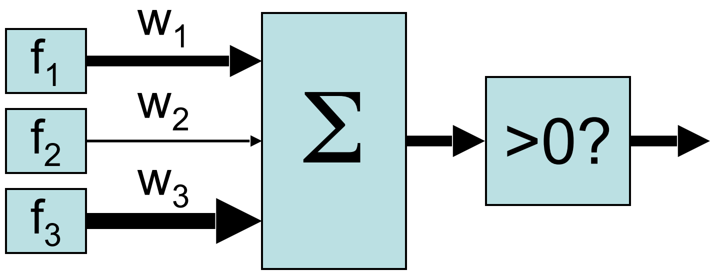
Linear Classifiers

- Inputs are feature values
- Each feature has a weight
- Sum is the activation



$$\text{activation}_w(x) = \sum_i w_i \cdot f_i(x) = w \cdot f(x)$$

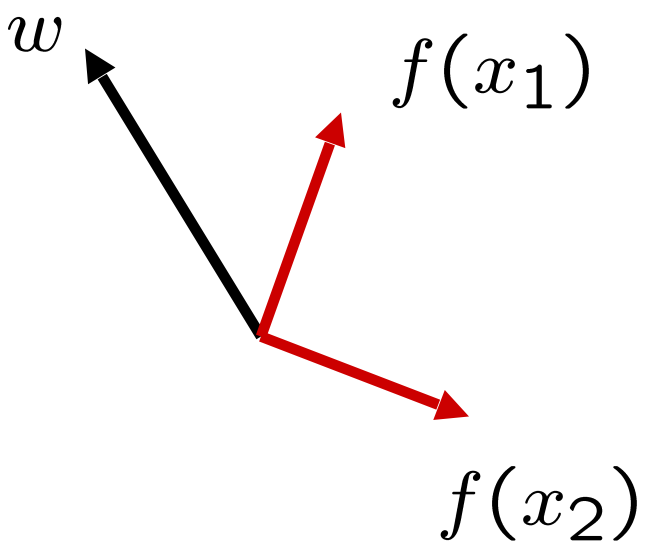
- If the activation is:
  - Positive, output +1
  - Negative, output -1



Weights

- Binary case: compare features to a weight vector
- Learning: figure out the weight vector from examples

```
# free      : 4
YOUR_NAME   :-1
MISSPELLED  : 1
FROM_FRIEND :-3
...
```

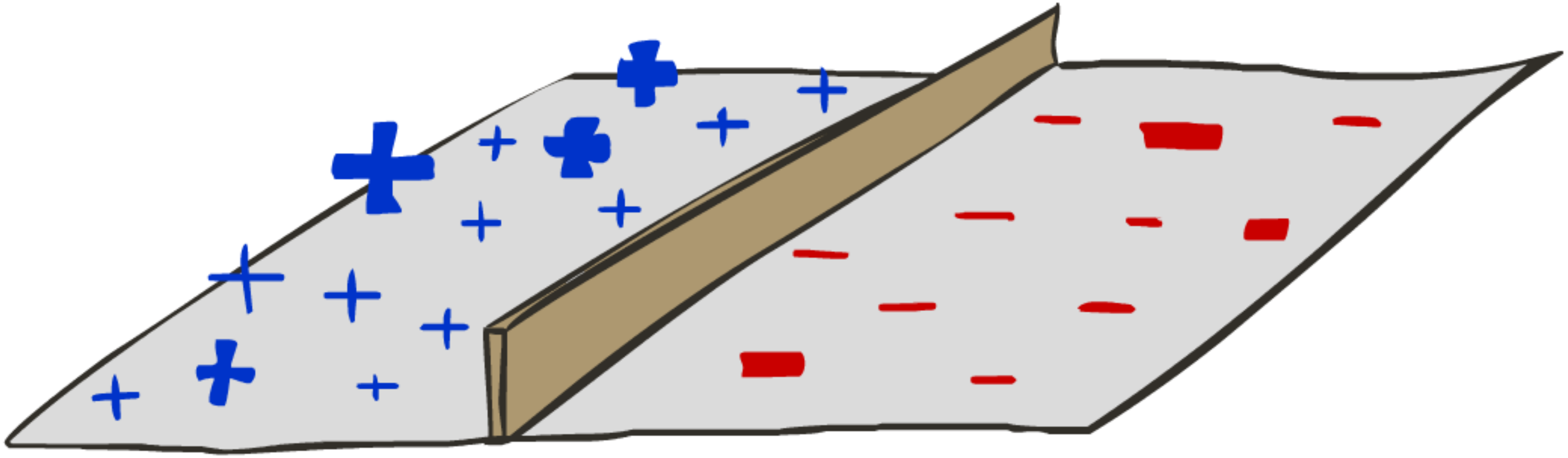



```
# free      : 2
YOUR_NAME   : 0
MISSPELLED  : 2
FROM_FRIEND : 0
...
```

```
# free      : 0
YOUR_NAME   : 1
MISSPELLED  : 1
FROM_FRIEND : 1
...
```

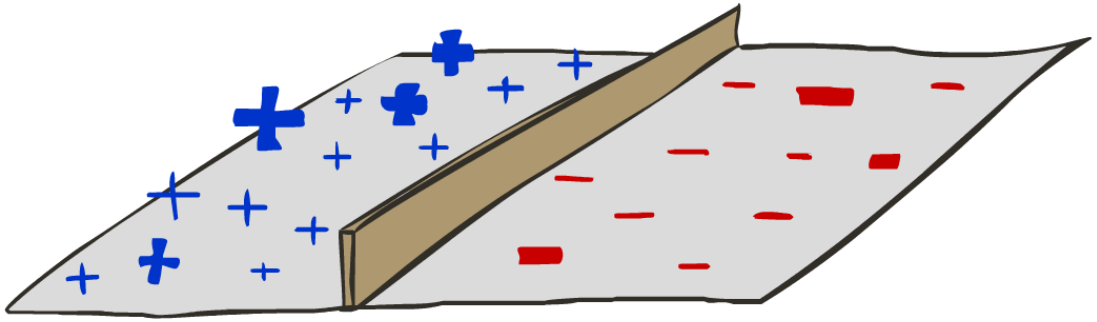
Dot product $w \cdot f$ positive means the positive class

Decision Rules



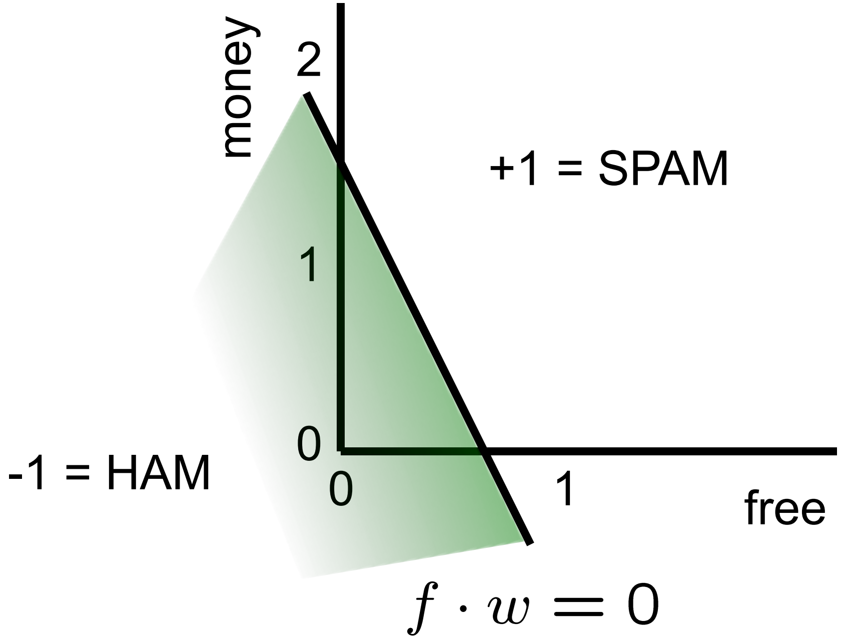
Binary Decision Rule

- In the space of feature vectors
  - Examples are points
  - Any weight vector is a hyperplane
  - One side corresponds to Y=+1
  - Other corresponds to Y=-1



$$w$$

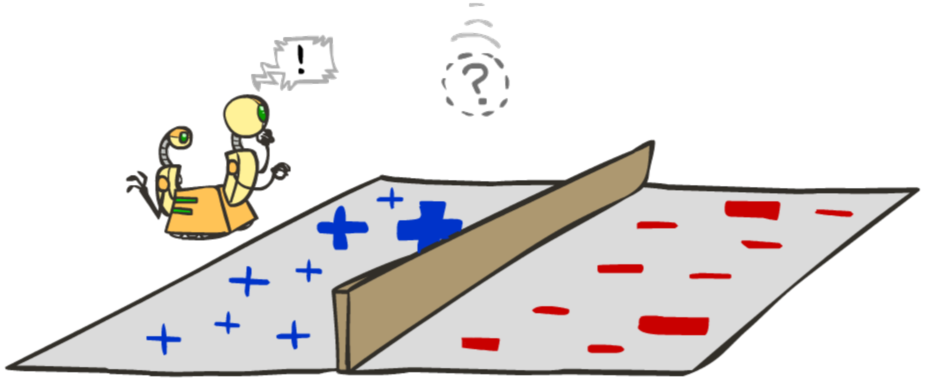
```
BIAS  : -3
free  :  4
money :  2
...
```



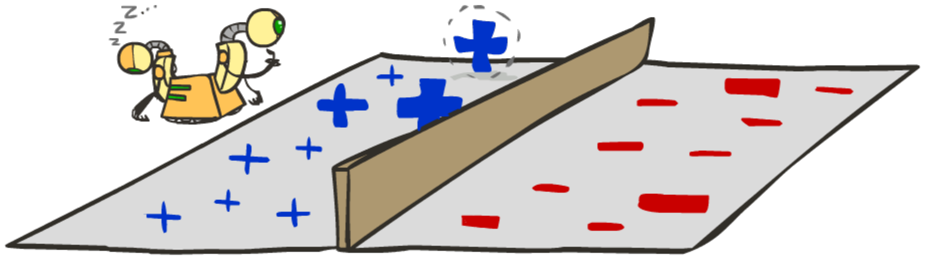
Weight Updates

Learning: Binary Perceptron

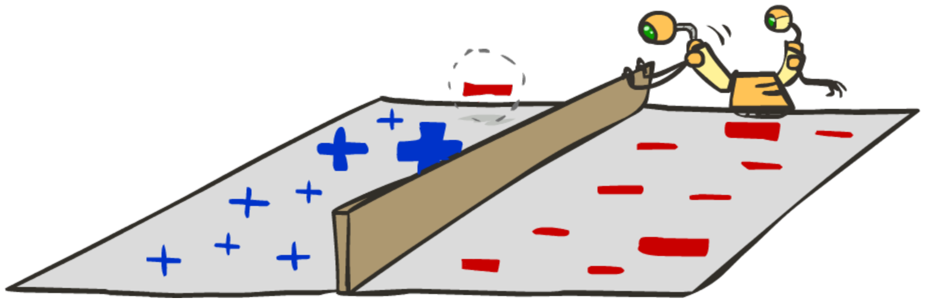
- Start with weights = 0
- For each training instance:
  - Classify with current weights



  - If correct (i.e., y=y*), no change!



  - If wrong: adjust the weight vector



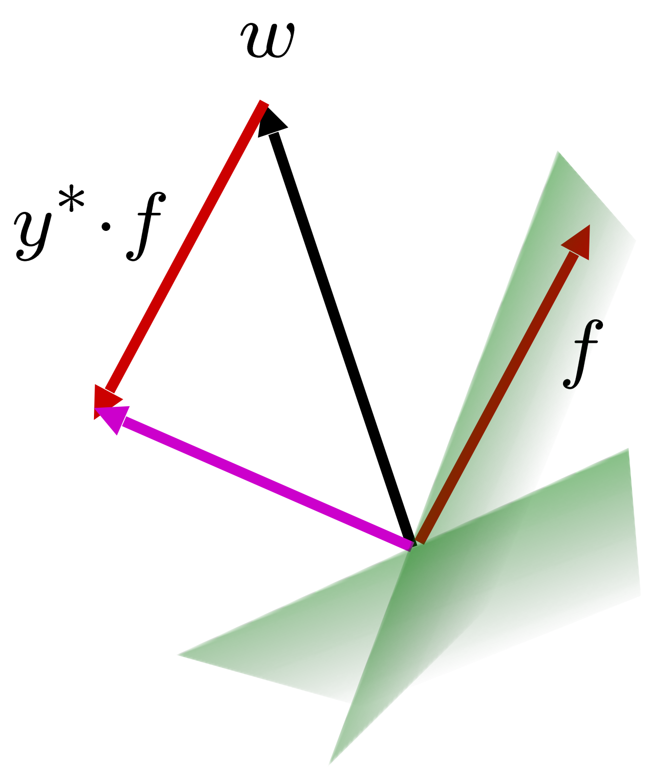
Learning: Binary Perceptron

- Start with weights = 0
- For each training instance:
  - Classify with current weights

$$y = \begin{cases} +1 & \text{if } \; w \cdot f(x) \geq 0 \\ -1 & \text{if } \; w \cdot f(x) < 0 \end{cases}$$

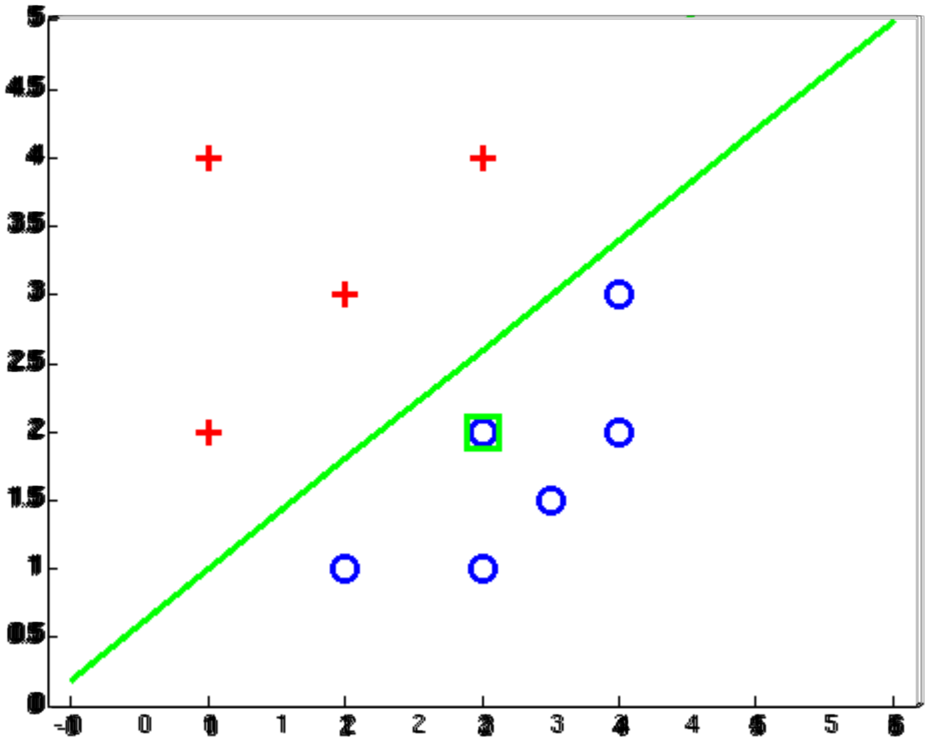
  - If correct (i.e., y=y*), no change!
  - If wrong: adjust the weight vector by adding or subtracting the feature vector. Subtract if y* is -1.

$$w = w + y^* \cdot f$$



Examples: Perceptron

- Separable Case



Multiclass Decision Rule

- If we have multiple classes:
  - A weight vector for each class:

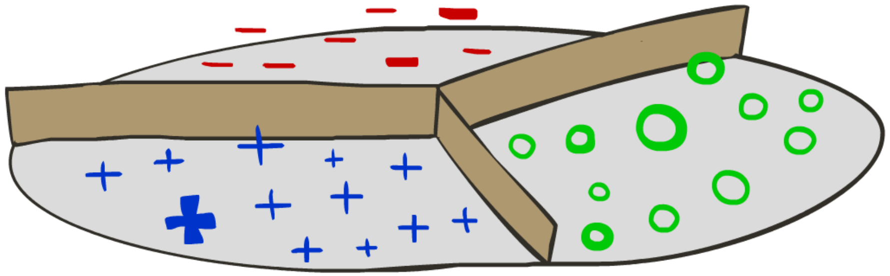
$$w_y$$

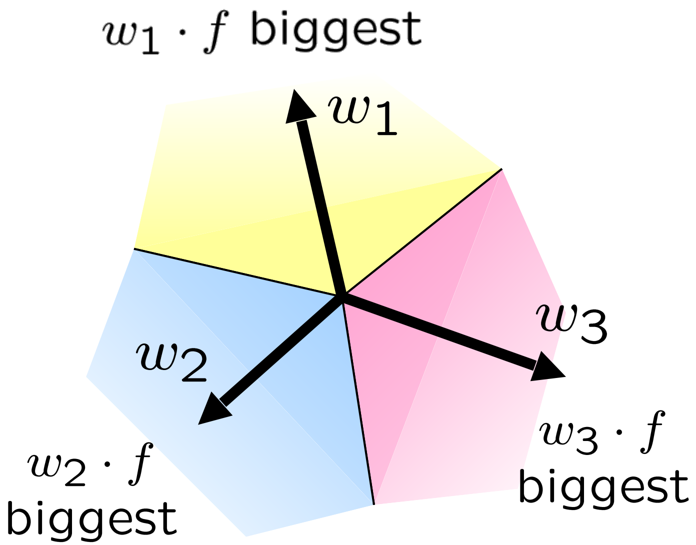
  - Score (activation) of a class y:

$$w_y \cdot f(x)$$

  - Prediction highest score wins

$$y = \arg\max_y \; w_y \cdot f(x)$$





Binary = multiclass where the negative class has weight zero

Learning: Multiclass Perceptron

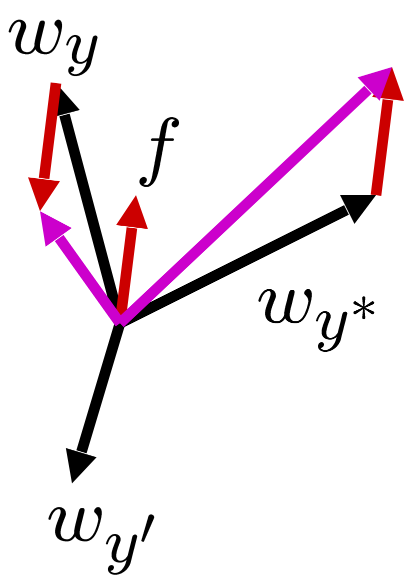
- Start with all weights = 0
- Pick up training examples one by one
- Predict with current weights

$$y = \arg\max_y \; w_y \cdot f(x)$$

- If correct, no change!
- If wrong: lower score of wrong answer, raise score of right answer

$$w_y = w_y - f(x)$$

$$w_{y^*} = w_{y^*} + f(x)$$



Example: Multiclass Perceptron

"win the vote"

"win the election"

"win the game"

```
Sample A
BIAS : 1
win  : 1
game : 0
vote : 1
the  : 1
```

wS . fA = 1; wP . fA = 0; wT . fA = 0

```
Sample B
BIAS : 1
win  : 1
game : 0
vote : 0
the  : 1
```

wS . fB = -2; wP . fB = 3; wT . fB = 0

```
Sample C
BIAS : 1
win  : 1
game : 1
vote : 0
the  : 1
```

wS . fC = -2; wP . fC = 3; wT . fC = 0

w_{SPORTS}

			fA		wS		fC		wS
BIAS	1		1		0		1		1
win	0		1		-1		1		0
game	0	-	0	=	0	+	1	=	1
vote	0		1		-1		0		-1
the	0		1		-1		1		0
...					...				...

$w_{POLITICS}$

			fA		wP		fC		wP
BIAS	0		1		1		1		0
win	0		1		1		1		0
game	0	+	0	=	0	-	1	=	-1
vote	0		1		1		0		1
the	0		1		1		1		0
...					...				...

w_{TECH}

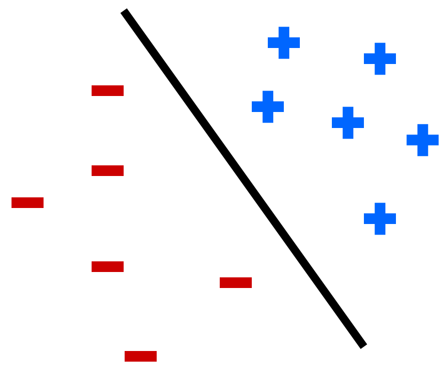
BIAS	0
win	0
game	0
vote	0
the	0
...	

Properties of Perceptrons

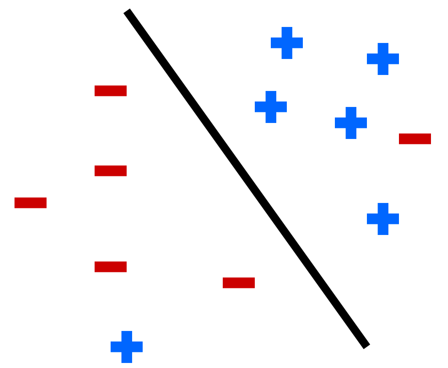
- Separability: true if some parameters get the training set perfectly correct
- Convergence: if the training is separable, perceptron will eventually converge (binary case)
- Mistake Bound: the maximum number of mistakes (binary case) related to the *margin* or degree of separability

$$\text{mistakes} < \frac{k}{\delta^2}$$

Separable

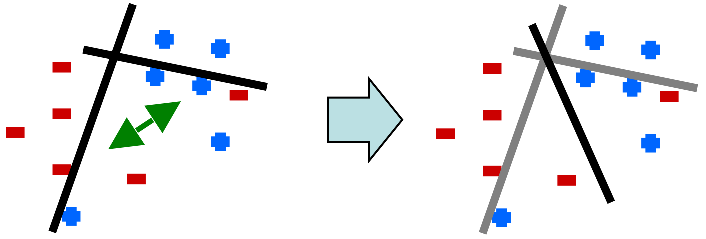


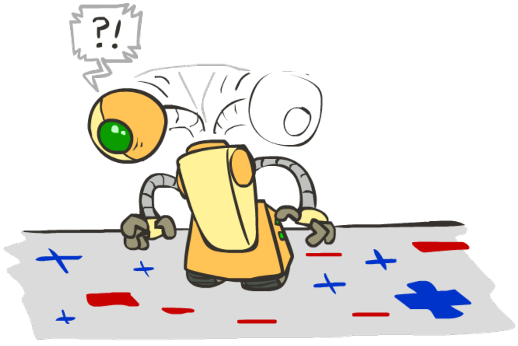
Non-Separable



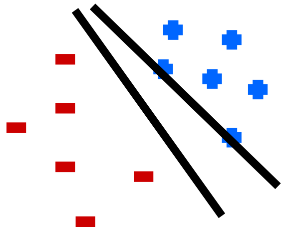
Problems with the Perceptron

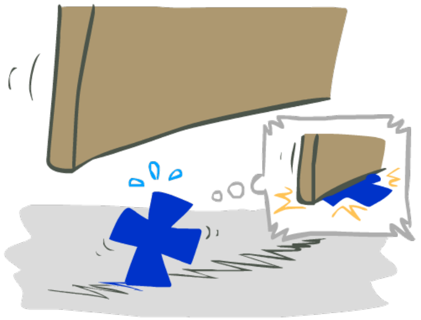
- Noise: if the data isn't separable, weights might thrash
  - Averaging weight vectors over time can help (averaged perceptron)



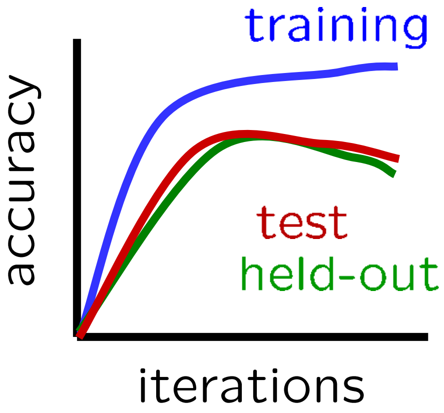


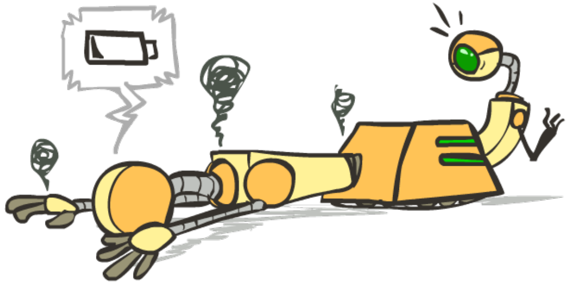
- Mediocre generalization: finds a "barely" separating solution



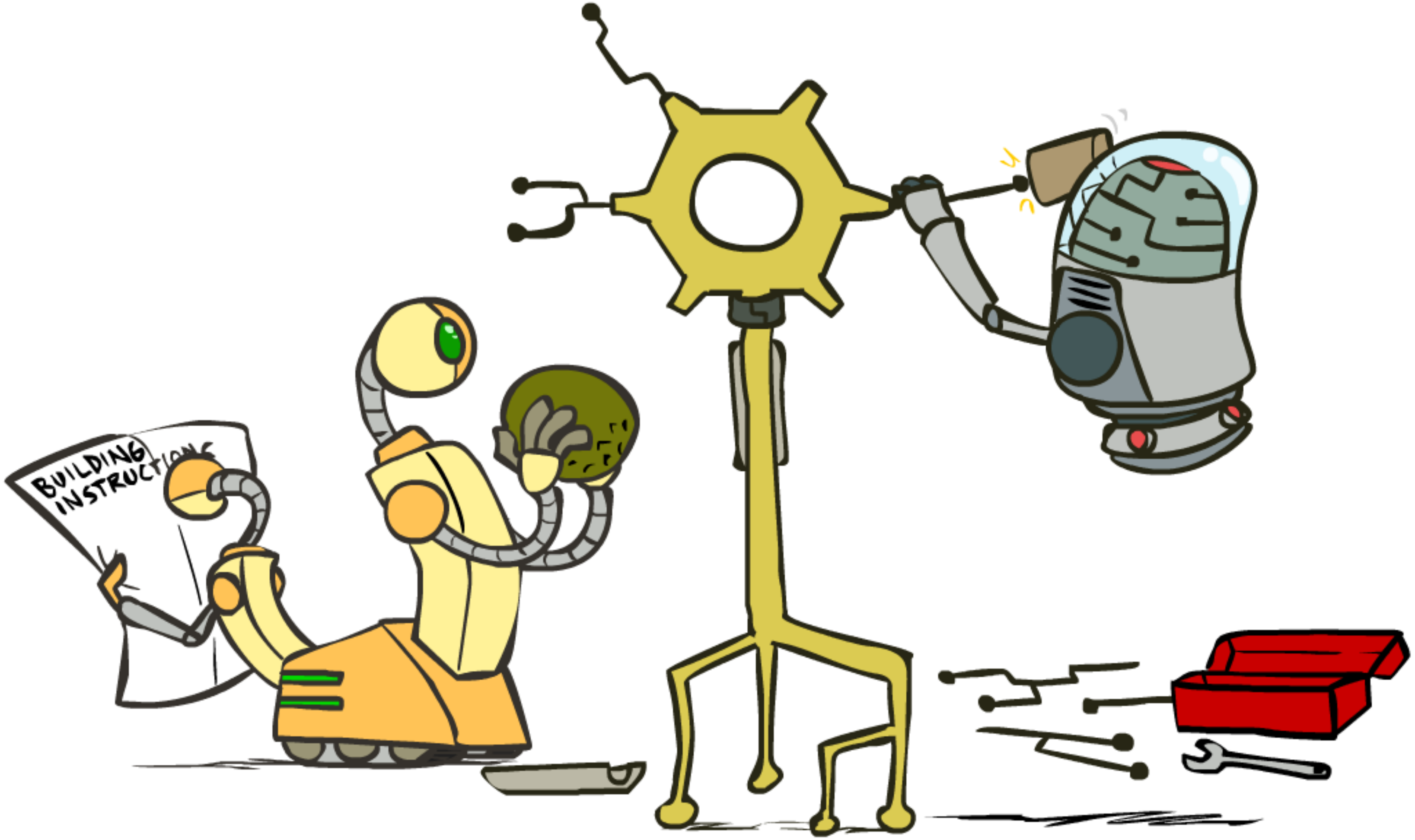


- Overtraining: test / held-out accuracy usually rises, then falls
  - Overtraining is a kind of overfitting



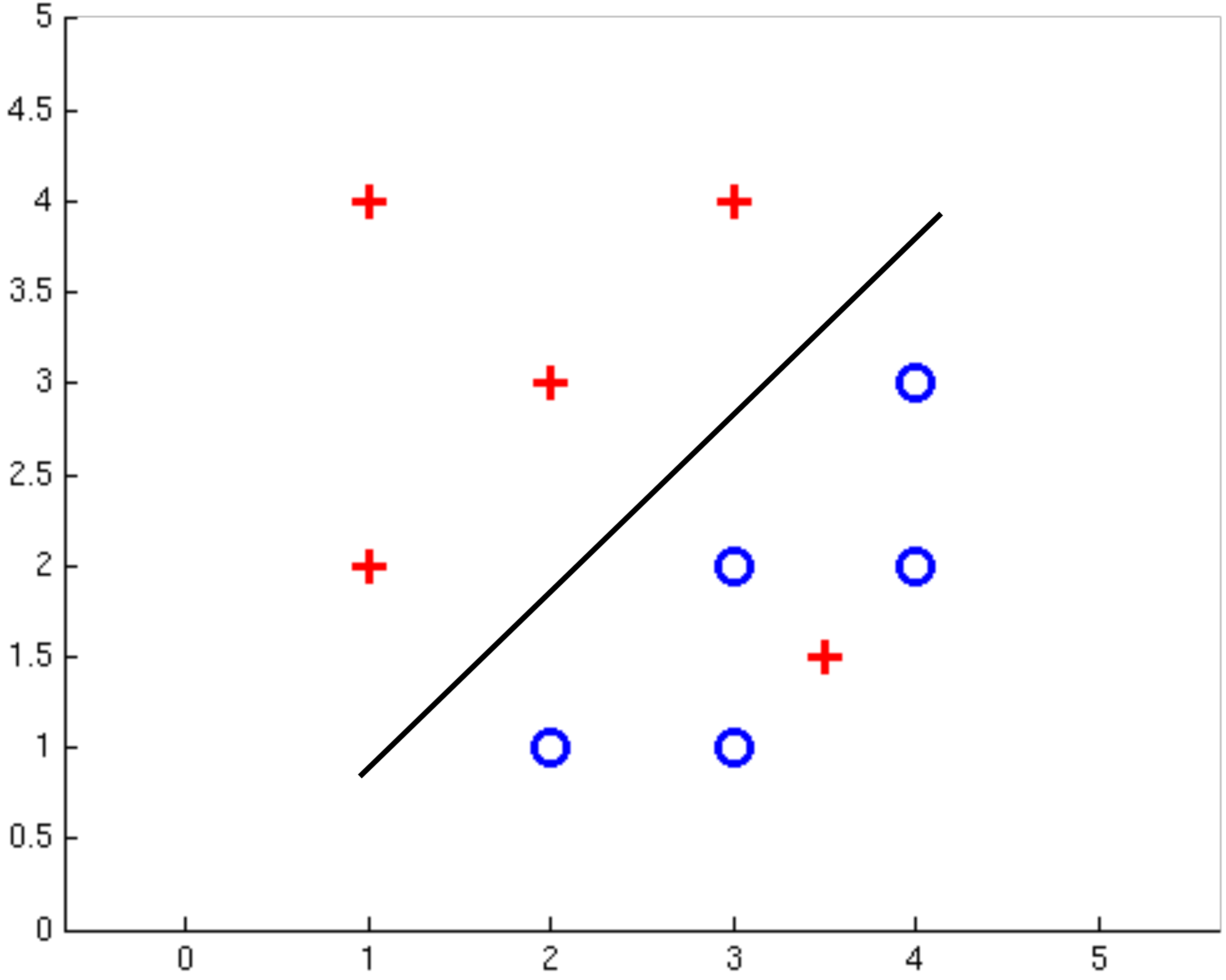


Improving the Perceptron

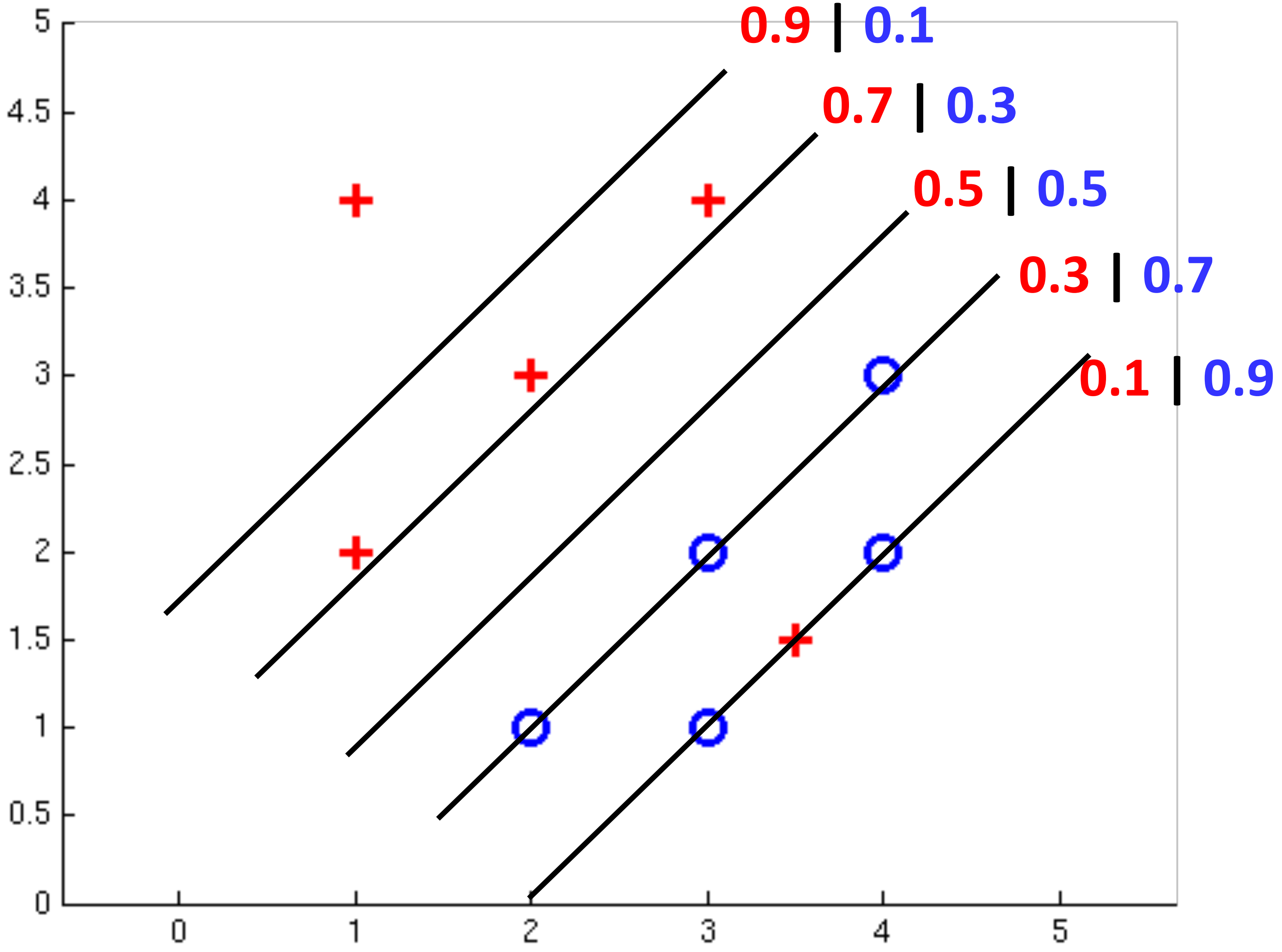


Non-Separable Case: Deterministic Decision

Even the best linear boundary makes at least one mistake



Non-Separable Case: Probabilistic Decision

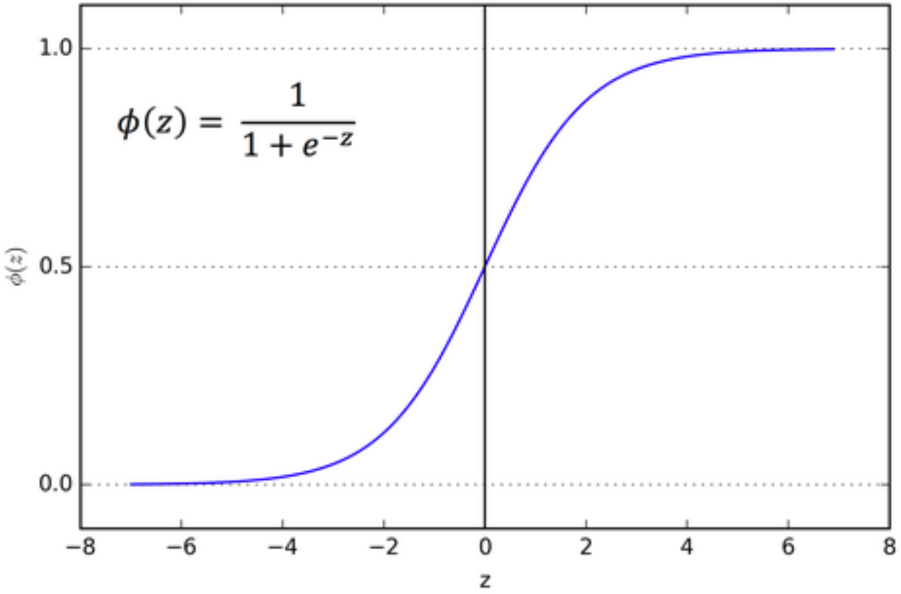


How to get probabilistic decisions?

- Perceptron scoring: $z = w \cdot f(x)$
- If $z = w \cdot f(x)$ very positive → want probability going to 1
- If $z = w \cdot f(x)$ very negative → want probability going to 0

- Sigmoid function

$$\phi(z) = \frac{1}{1 + e^{-z}}$$



Best w?

- Maximum likelihood estimation:

$$\max_w \quad ll(w) = \max_w \quad \sum_i \log P(y^{(i)}|x^{(i)}; w)$$

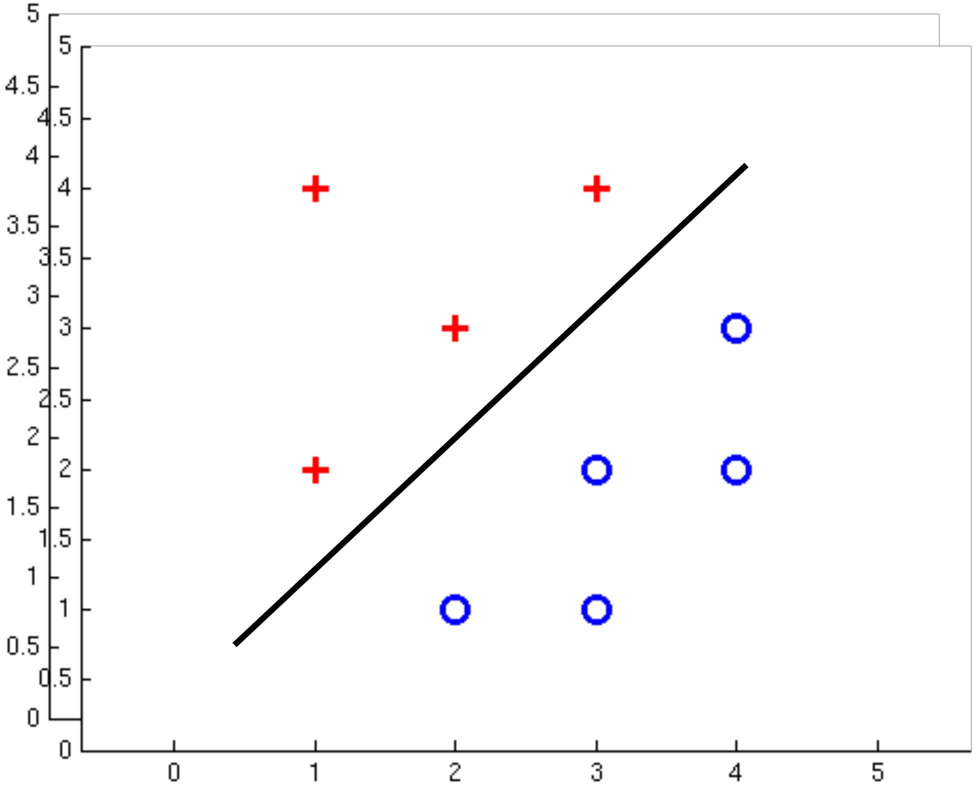
with:

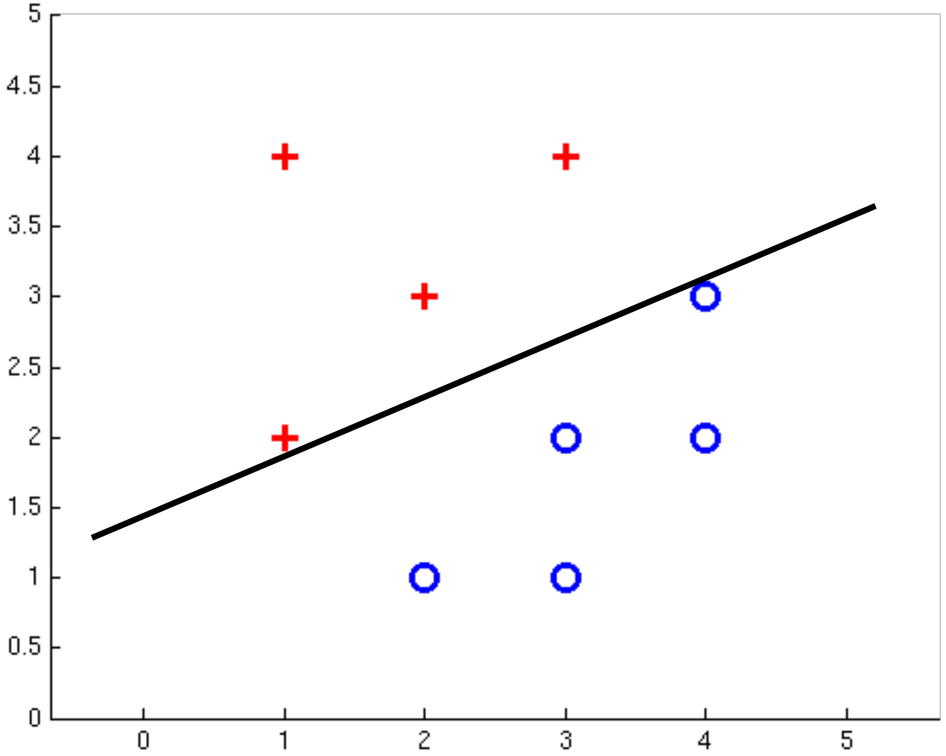
$$P(y^{(i)} = +1|x^{(i)}; w) = \frac{1}{1 + e^{-w \cdot f(x^{(i)})}}$$

$$P(y^{(i)} = -1|x^{(i)}; w) = 1 - \frac{1}{1 + e^{-w \cdot f(x^{(i)})}}$$

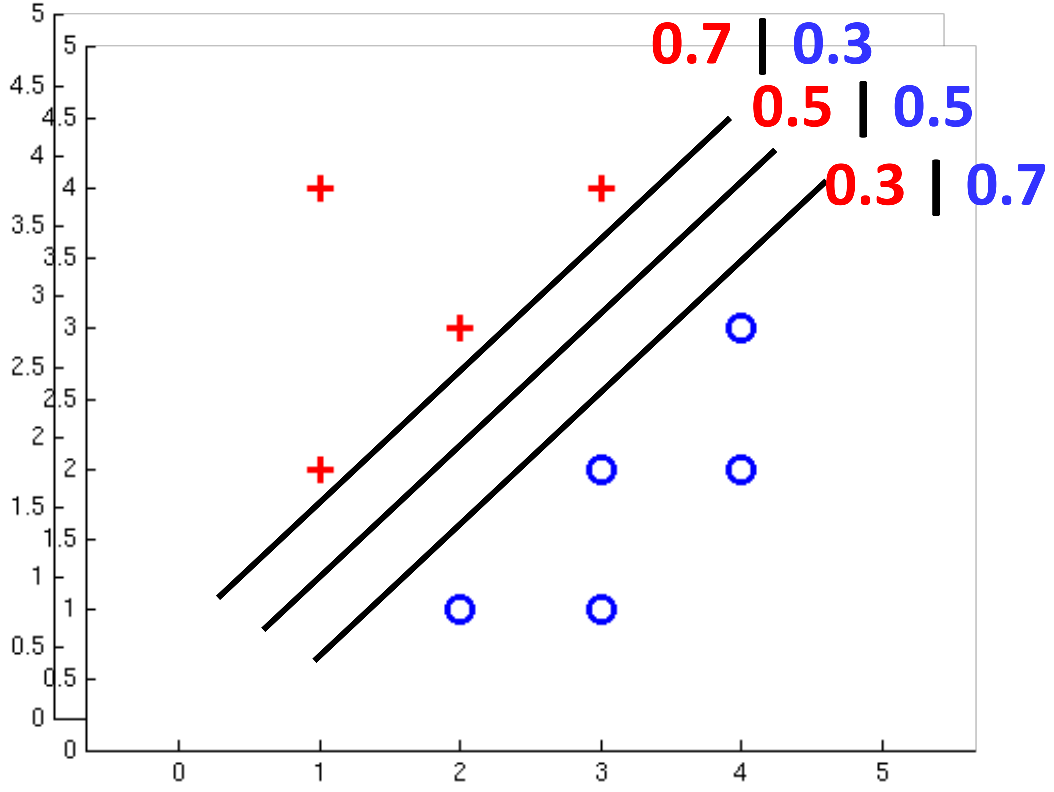
= Logistic Regression

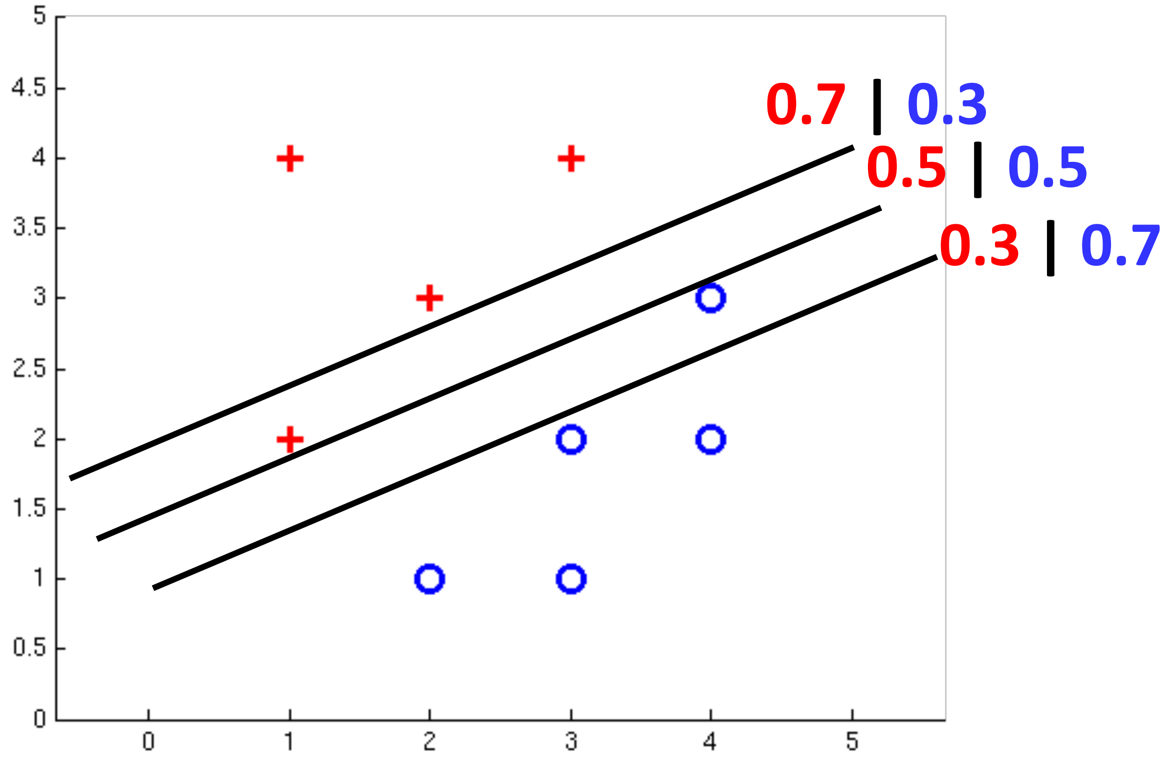
Separable Case: Deterministic Decision – Many Options





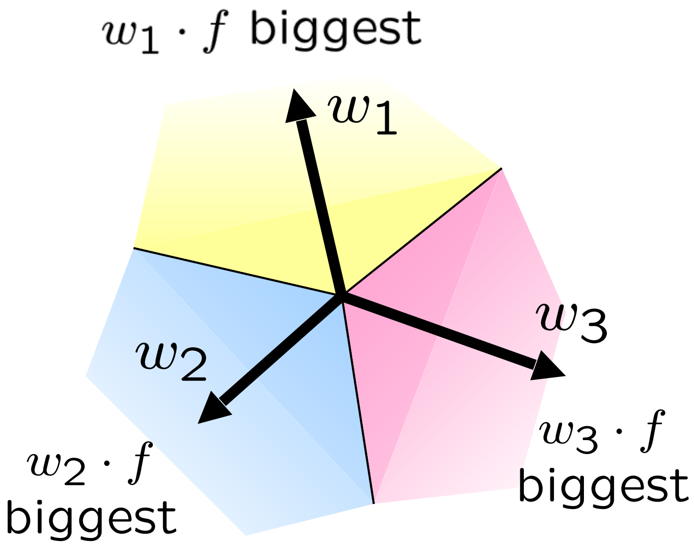
Separable Case: Probabilistic Decision – Clear Preference





Multiclass Logistic Regression

- Recall Perceptron:
  - A weight vector for each class: w_y
  - Score (activation) of a class y: $w_y \cdot f(x)$
  - Prediction highest score wins $y = \arg\max_y \; w_y \cdot f(x)$



- How to make the scores into probabilities?

$$\underbrace{z_1, z_2, z_3}_{\text{original activations}} \rightarrow \underbrace{\frac{e^{z_1}}{e^{z_1} + e^{z_2} + e^{z_3}}, \frac{e^{z_2}}{e^{z_1} + e^{z_2} + e^{z_3}}, \frac{e^{z_3}}{e^{z_1} + e^{z_2} + e^{z_3}}}_{\text{softmax activations}}$$

Best w?

- Maximum likelihood estimation:

$$\max_{w} \quad ll(w) = \max_{w} \quad \sum_{i} \log P(y^{(i)}|x^{(i)}; w)$$

with:

$$P(y^{(i)}|x^{(i)}; w) = \frac{e^{w_{y^{(i)}} \cdot f(x^{(i)})}}{\sum_{y} e^{w_{y} \cdot f(x^{(i)})}}$$

= Multi-Class Logistic Regression

Choosing weights

- Optimization
  - i.e., how do we solve:

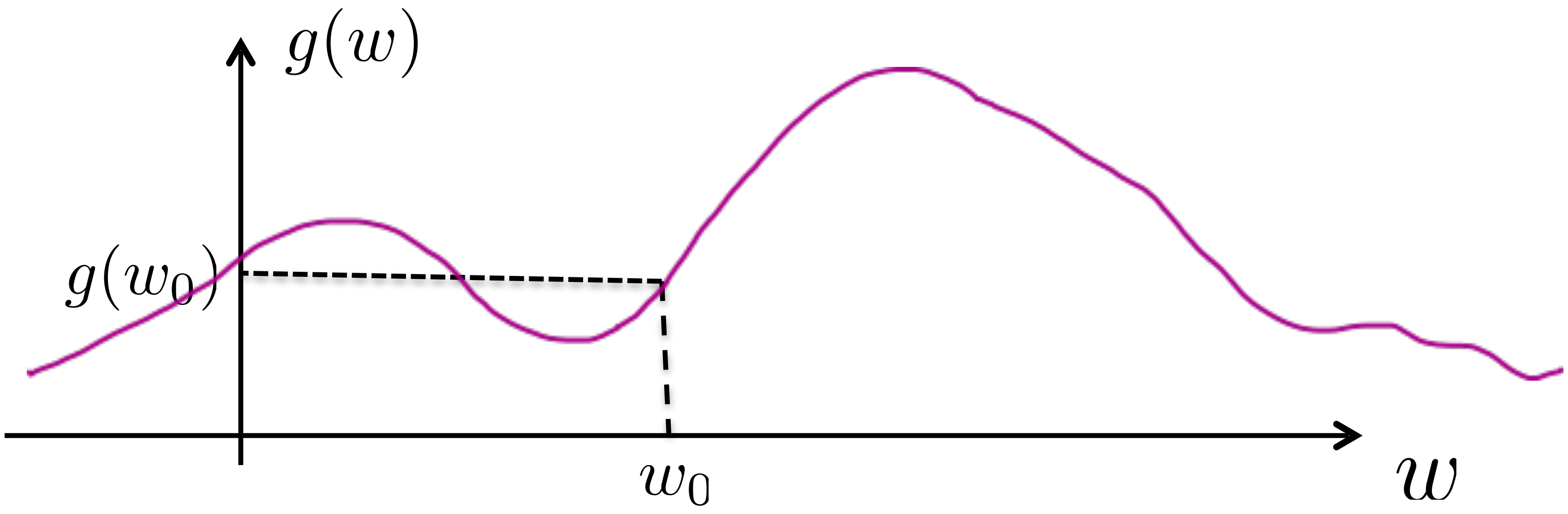
$$\max_w \quad ll(w) = \max_w \quad \sum_i \log P(y^{(i)}|x^{(i)}; w)$$

Hill Climbing

- Recall from CSPs lecture: simple, general idea
  - Start wherever
  - Repeat: move to the best neighboring state
  - If no neighbors better than current, quit

- What's particularly tricky when hill-climbing for multiclass logistic regression?
  - Optimization over a continuous space
    - Infinitely many neighbors!
    - How to do this efficiently?

1-D Optimization

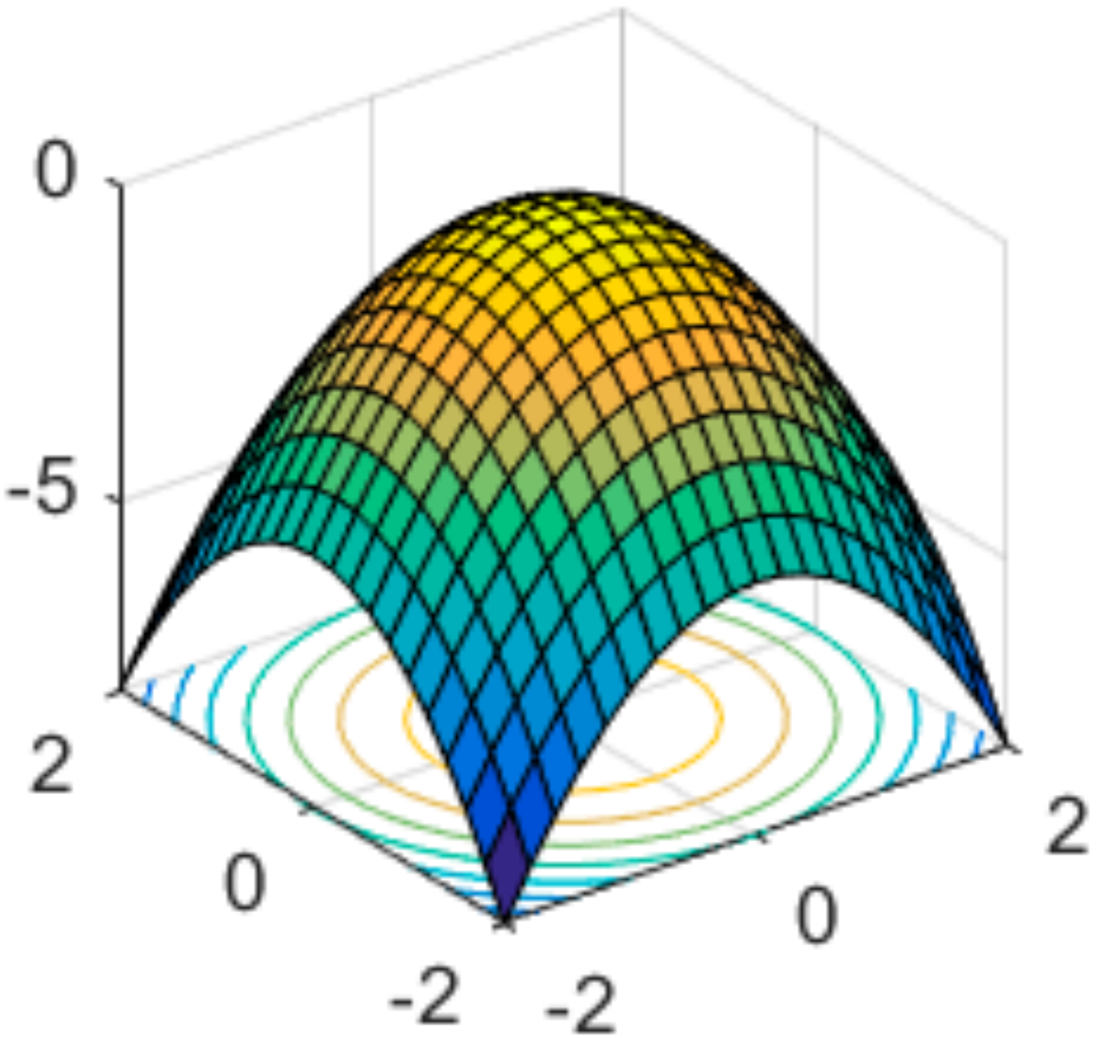


- Could evaluate $g(w_0 + h)$ and $g(w_0 - h)$
  - Then step in best direction

- Or, evaluate derivative: $\frac{\partial g(w_0)}{\partial w} = \lim_{h \to 0} \frac{g(w_0 + h) - g(w_0 - h)}{2h}$

  - Tells which direction to step into

2-D Optimization



Source: offconvex.org

Gradient Ascent

- Perform update in uphill direction for each coordinate
- The steeper the slope (i.e. the higher the derivative) the bigger the step for that coordinate
- E.g., consider: $g(w_1, w_2)$

  - Updates:

  $$w_1 \leftarrow w_1 + \alpha * \frac{\partial g}{\partial w_1}(w_1, w_2)$$

  $$w_2 \leftarrow w_2 + \alpha * \frac{\partial g}{\partial w_2}(w_1, w_2)$$

  - Updates in vector notation:

  $$w \leftarrow w + \alpha * \nabla_w g(w)$$

  with: $\nabla_w g(w) = \begin{bmatrix} \frac{\partial g}{\partial w_1}(w) \\ \frac{\partial g}{\partial w_2}(w) \end{bmatrix}$ **= gradient**

Gradient Ascent

- Idea:
  - Start somewhere
  - Repeat: Take a step in the gradient direction

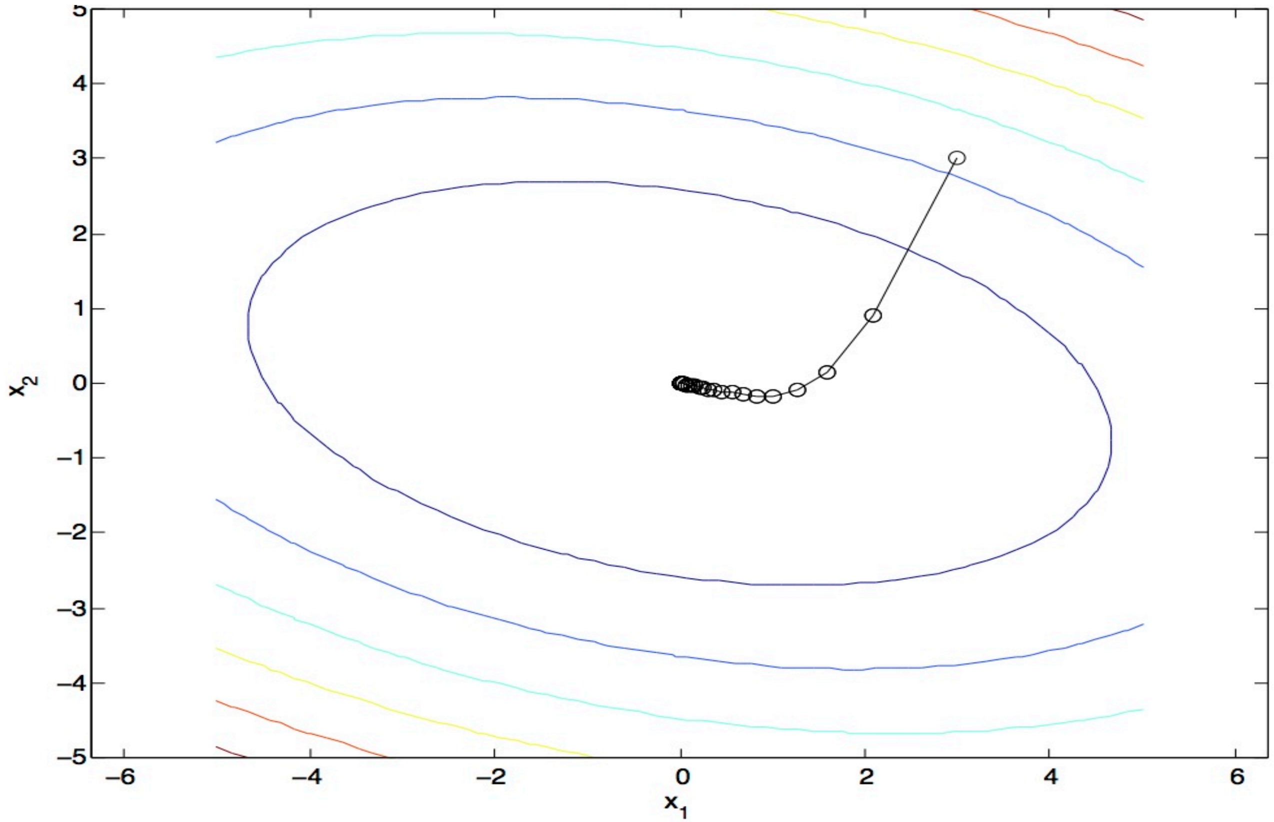
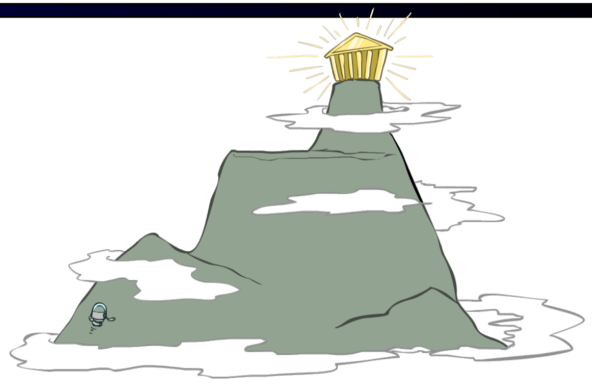


Figure source: Mathworks

What is the Steepest Direction?



$$\max_{\Delta:\Delta_1^2+\Delta_2^2\leq\varepsilon} g(w+\Delta)$$

- First-Order Taylor Expansion: $$g(w+\Delta) \approx g(w) + \frac{\partial g}{\partial w_1}\Delta_1 + \frac{\partial g}{\partial w_2}\Delta_2$$
- Steepest Ascent Direction: $$\max_{\Delta:\Delta_1^2+\Delta_2^2\leq\varepsilon} g(w) + \frac{\partial g}{\partial w_1}\Delta_1 + \frac{\partial g}{\partial w_2}\Delta_2$$
- Recall: $$\max_{\Delta:\|\Delta\|\leq\varepsilon} \Delta^\top a \rightarrow \Delta = \varepsilon\frac{a}{\|a\|}$$
- Hence, solution: $$\Delta = \varepsilon\frac{\nabla g}{\|\nabla g\|}$$

Gradient direction = steepest direction!

$$\nabla g = \begin{bmatrix} \frac{\partial g}{\partial w_1} \\ \frac{\partial g}{\partial w_2} \end{bmatrix}$$

Gradient in n dimensions

$$\nabla g = \begin{bmatrix} \frac{\partial g}{\partial w_1} \\ \frac{\partial g}{\partial w_2} \\ \cdots \\ \frac{\partial g}{\partial w_n} \end{bmatrix}$$

Optimization Procedure: Gradient Ascent

- init w
- for iter = 1, 2, …

$$w \leftarrow w + \alpha * \nabla g(w)$$

- α: learning rate --- hyperparameter that needs to be chosen carefully
- How? Try multiple choices
  - Crude rule of thumb: update changes w about 0.1 – 1 %

Batch Gradient Ascent on the Log Likelihood Objective

$$\max_w \quad ll(w) = \max_w \quad \underbrace{\sum_i \log P(y^{(i)}|x^{(i)};w)}_{g(w)}$$

- init w
- for iter = 1, 2, …

$$w \leftarrow w + \alpha * \sum_i \nabla \log P(y^{(i)}|x^{(i)};w)$$

Stochastic Gradient Ascent on the Log Likelihood Objective

$$\max_w \quad ll(w) = \max_w \quad \sum_i \log P(y^{(i)}|x^{(i)}; w)$$

Observation: once gradient on one training example has been computed, might as well incorporate before computing next one

- init w
- for iter = 1, 2, …
  - pick random j

$$w \leftarrow w + \alpha * \nabla \log P(y^{(j)}|x^{(j)}; w)$$

Mini-Batch Gradient Ascent on the Log Likelihood Objective

$$\max_w \quad ll(w) = \max_w \quad \sum_i \log P(y^{(i)}|x^{(i)}; w)$$

Observation: gradient over small set of training examples (=mini-batch) can be computed in parallel, might as well do that instead of a single one

- init w
- for iter = 1, 2, …
  - pick random subset of training examples J

$$w \leftarrow w + \alpha * \sum_{j \in J} \nabla \log P(y^{(j)}|x^{(j)}; w)$$

Learning Rate Finder

- Calculate a good learning rate by trying learning rates over a range of possible values
- Plot the training loss at each of these epochs
- Pick a learning rate where the loss is declining the most before it hits the minimum: 5x10^-5 - 3x10^-4