Quantum Error Correction Codes-From Qubit to Qudit

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Outline

- Introduction to quantum error correction codes (QECC)
- Qudits and Qudit Gates
- Generalizing QECC to Qudit computing

Need for QEC in Quantum Computation

- Sources of Error
  - Environment noise
    - Cannot have complete isolation from environment → entanglement with environment → random changes in environment cause undesirable changes in quantum system
  - Control Error
    - e.g. timing error for X gate in spin resonance
- Cannot have reliable quantum computer without QEC

Error Models

- Bit flip |0> → |1>, |1> → |0> Pauli X
- Phase flip |0> → |0>, |1> → -|1> Pauli Z
- Bit and phase flip Y = iXZ
- General unitary error operator

  I, X, Y, Z form a basis for single qubit unitary operator. Correctable if I, X, Y, Z are.

QECC

- Achieved by adding redundancy.
  - Transmit or store n qubits for every k qubits.
- 3 qubit bit flip code

  Simple repetition code |0> → |000>, |1> →|111> that can correct up to 1 bit flip error.
- Phase flip code
  - Phase flip in |0>, |1> basis is bit flip in |+>, |-> basis. a|0> + b|1> → a|0>-b|1> ←→ (a+b)|+> + (a-b)|-> (a-b)|+> + (a+b) |->
  - 3 qubit bit flip code can be used to correct 1 phase flip error after changing basis by H gate.

QECC

- Shor code: combine bit flip and phase flip codes to correct arbitrary error on a single qubit

  |0> → (|000>+|111>) (|000>+|111>) (|000>+|111>)/2sqrt(2)

  |1> → (|000>-|111>) (|000>-|111>) (|000>-|111>)/2sqrt(2)

Stabilizer Codes

- Group theoretical framework for QEC analysis
- Pauli Group
  - I, X, Y, Z form a basis for operator on single qubit
  - G_1= {aE | a is 1, -1, i, -i and E is I, X, Y, Z} is a group
  - G_n is n-fold tensor of G_1
- S: an Abelian (commutative) subgroup of Pauli Group G_n
- Stabilized: g|φ> = |φ> (i.e. eigenvalue = 1)
- Codespace: stabilized by S
  - g|φ> = |φ> for all g in S.
  - Decode by measuring generators of S.
  - Correct errors in G_n that anti-commute with at least one g in S.

Stabilizer Codes – Examples

- The 3 qubit bit flip code: S $\{Z_1Z_2, Z_2Z_3\}$

  |000> and |111> stabilized by S.
- The 5 qubit code [5, 1, 3]
  - S: XZZXI, IXZZX, XIXZZ, ZXIXZ

Qudits

- A qudit is a generalization of the qubit to a d-dimensional Hilbert space.
- The qutrit is a three-state quantum system.
  - The computation basis is then a set of three (orthogonal) kets

    {|0>, |1>, |2>}
  - An arbitrary qutrit is a linear combination of these three states

    |ψ>=α|0>+β|1>+γ|2>
  - Examples: Three energy levels of a particular atom. A spin-1 massive boson.
- To represent an integer k in a qutrit system, one writes k as a sum of powers of 3:

$$k = \sum_j p_j 3^j$$

- The trinary representation is then $p_n p_{n-1} \ldots p_1 p_0$
- So, for example, the number 65 can be written

$$65 = 2 \cdot 3^3 + 1 \cdot 3^2 + 0 \cdot 3^1 + 2 \cdot 3^0$$

  so the trinary representation is 2102. This will be encoded into a register of qutrits.
- This can be easily generalized to a Hilbert space of dimension *d*.

Why Qudits?

- Classically, a d-nary system allows for more efficient way to store data.
- For example, the number 157 only requires three digits but requires eight bits (10011101).
- In quantum computing, the increase is even more dramatic.
- Unfortunately, it is clearly much more difficult to construct a computer that uses qudits rather than qubits.

Qudit Gates

- The Pauli operators for a d-dimensional Hilbert space are defined by their action on the computational basis:
  - X: |j> →|j+1 (mod d)>
  - Z: |j> →ω^j |j> where ω= exp(2πi/d)
- The elements of the Pauli group, *P*, are given by

$$E_{r,s} = X^r Z^s$$

  where r,s = 0,1,…,d-1 (note that are d^2 of these).
- As is the case for d=2, these operators form a basis for U(d).
- The matrix representations of X and Z for the qutrit are:

$$X = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad Z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{2\pi i/3} & 0 \\ 0 & 0 & e^{4\pi i/3} \end{pmatrix}$$

Qudit Stabilizers

- As with d=2, the stabilizer *S* of a code is an Abelian subgroup of *P*.
- If *d* is prime, constructing codes is a straightforward generalize from qubits.
- The 3 qudit bit flip code:

$$S = \{Z_1(Z_2)^{-1}, Z_2(Z_3)^{-1}\}$$

  |000>, |111>, … |d-1, d-1, d-1> stabilized by S.
- The 5 qudit code [[5, 1, 3]]
  - S: XZZXI, IXZZX, XIXZZ, ZXIXZ, same as qubit.
- If the stabilizer on *n* qudits has *n* – *k* generators, then *S* will have d^{n-k} elements and the coding space has *k* qudits. This is not true for composite *d*.

Summary

- Abelian subgroups of the Pauli group can be used to correct errors arising on quantum computing.
- Qudits are the higher-dimensional analogue of qubits.
- The generalization of stabilizer groups to qudits from qubits is easy when d is prime.

References

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