1. Consider the normalized states

$$\begin{pmatrix} \cos\theta_1 \\ \sin\theta_1 \end{pmatrix}$$

and

$$\begin{pmatrix} \cos\theta_2 \\ \sin\theta_2 \end{pmatrix}.$$

Find the condition on θ_1 and θ_2 such that the superposition

$$\begin{pmatrix} \cos\theta_1 \\ \sin\theta_1 \end{pmatrix} + \begin{pmatrix} \cos\theta_2 \\ \sin\theta_2 \end{pmatrix}$$

is normalized to unity ("properly normalized").
(*Hint*: Remember that $\cos(\theta_1+\theta_2) = \cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2$)

2. The kets $|h\rangle$ and $|v\rangle$ are states of horizontal and vertical photon polarization, respectively. Consider the states

$$|\psi_1\rangle = -\frac{1}{2}\left(|h\rangle + \sqrt{3}|v\rangle\right), \qquad |\psi_2\rangle = -\frac{1}{2}\left(|h\rangle - \sqrt{3}|v\rangle\right), \qquad |\psi_3\rangle = |h\rangle,$$

What are the relative orientations of the plane of polarization for these three states?

3. a) Find the eigenvectors, eigenvalues, and diagonal representations of the Pauli matrices I, X, Y, and Z, where

$$I \equiv \sigma_0 \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad X \equiv \sigma_x \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad Y \equiv \sigma_y \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad Z \equiv \sigma_z \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Show that $X^2 = Y^2 = Z^2 = I$ (so that calculating X^n, Y^n or Z^n becomes really simple...)

b) Find the points on the Bloch sphere that correspond to the normalized eigenvectors of the Pauli matrices.

c) Find the action of the Z operator on a general qubit state $|\Psi\rangle = \alpha|0\rangle + \beta|1\rangle$ and describe this action on the Bloch sphere (i.e. how does the vector representing $|\Psi\rangle$ get rotated on the Bloch sphere?).

d) Form the matrix representation of the exponentiated operator $e^{-i\gamma Z/2}$ and show how this exponentiated operator acts on the Bloch sphere vector for $|\Psi\rangle$.

e) Similarly form the matrix representations of the exponentiated operators $e^{-i\gamma X/2}$ and $e^{-i\gamma Y/2}$. Show explicitly how these act on the vector at the North pole of the Bloch sphere, i.e. on the qubit state $|\Psi\rangle = |0\rangle$, specifying the nature of the resulting rotation.

4. You are given one of two quantum states of a single qubit: either $|\phi\rangle = |0\rangle$ or $|\psi\rangle = \cos\theta|0\rangle + \sin\theta|1\rangle$. You want to make a single measurement that best distinguishes between these two states, i.e., you want to find the best basis for making a measurement to distinguish the two states. So let's measure the qubit in basis $|v\rangle = \alpha|0\rangle + \beta|1\rangle, |v^{\perp}\rangle$, where α, β are to be determined for optimal success. For outcome $|v\rangle$ we guess that the qubit was in state $|0\rangle$; for outcome $|v^{\perp}\rangle$ we guess that the

qubit was in state $|\psi\rangle$. Determine the optimal measurement basis given this procedure. You can take α and β to be real numbers, in which case the normalization $|\alpha|^2 + |\beta|^2 = 1$ implies that you can write α and β as e.g. $\alpha = \sin\gamma$ and $\beta = \cos\gamma$.

Hint: you will need to first construct the probability of a correct measurement in this situation. You should convince yourselves that this is given by

$$\Pr[\text{qubit was}|0\rangle]\,\Pr[\text{measure}|v\rangle|\text{qubit was}|0\rangle] + \Pr[\text{qubit was}|\psi\rangle]\,\Pr[\text{measure}|v^{\perp}\rangle|\text{qubit was}|\psi\rangle].$$

where, e.g.,

$$\Pr[\text{measure}|v\rangle|\text{qubit was}|0\rangle] \quad = |\langle v|\,0\rangle|^2.$$

If the state you are presented is either $|\phi\rangle$ or $|\psi\rangle$ with 50% probability each, what is the probability that your measurement correctly identifies the state?