1. Show that the trace of an operator is independent of the basis in which it is evaluated.

2. When is $e^{(\hat{A}+\hat{B})} = e^{\hat{A}}e^{\hat{B}}$? In the case that they are not equal, estimate the difference to first order in the commutator $[A,B]$. Show all reasoning explicitly.

3. We have seen that non-commuting observables cannot be exactly specified simultaneously. In this question you will quantify this statement and find just how sharply they can be defined.

   a) Let A and B be two Hermitian operators, and write their commutator as $AB - BA = iC$. Prove that the operator $C \equiv [A,B]/i$ must be a Hermitian operator.

   b) The variance of A in state $|\psi\rangle$ is defined as

   $$(\Delta A)^2 = \langle\psi|(A- <A>)^2|\psi\rangle.$$

   Use the Schwarz inequality to prove that

   $$(\Delta A)^2(\Delta B)^2 \geq |\langle\psi|(A- <A>)(B- <B>)|\psi\rangle|^2.$$

   c) Rewrite the operator $F = (A- <A>)(B- <B>)$ in terms of the Hermitian operators $i(F - F^\dagger)$ and $F + F^\dagger$. Now go on and make use of the fact that Hermitian operators have real expectation values to prove that

   $$(\Delta A)^2(\Delta B)^2 \geq \frac{1}{4} <C>^2$$

   d) Evaluate this uncertainty relation for $A = x$ (the position operator) and $B = p$ (the momentum operator). Can you apply this to $A = E$ and $B = t$?

4. Consider the total spin of a system having 2 electrons: $\hat{S}_T = \hat{S}_1 + \hat{S}_2$. Show that the Bell state $|\psi_-\rangle = |\uparrow\rangle_1|\downarrow\rangle_2 - |\downarrow\rangle_1|\uparrow\rangle_2$ has a net spin of zero. In other words, show that it is an eigenstate of $\hat{S}_T^2$ with eigenvalue = 0. $|\uparrow\rangle$ here means spin up, and $|\downarrow\rangle$ means spin down (you can take this to be along the z-axis, but as you showed in problem set 2, the Bell state here is rotationally invariant, so it doesn't actually matter which axis you use).

   Hint: Use $\hat{S}_T^2 = \left(\hat{S}_1 + \hat{S}_2\right)^2 = \hat{S}_1^2 + \hat{S}_2^2 + 2\hat{S}_1 \cdot \hat{S}_2$, *and recall that* $\hat{S}_1 \cdot \hat{S}_2 = \hat{S}_{1x}\hat{S}_{2x} + \hat{S}_{1y}\hat{S}_{2y} + \hat{S}_{1z}\hat{S}_{2z}$.