C/CS/Phys C191 **2 Qubit gates and EPR** 9/11/07
Fall 2007 Lecture 5

# 1 Two-qubit gate: CNOT

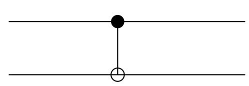
The controlled-not (CNOT) gate exors the first qubit into the second qubit ($|a,b\rangle \to |a, a\oplus b\rangle = |a, a+b \bmod 2\rangle$). Thus it permutes the four basis states as follows:

$$00 \to 00 \qquad 01 \to 01$$
$$10 \to 11 \qquad 11 \to 10 \ .$$

As a unitary $4\times 4$ matrix, the CNOT gate is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

In a quantum circuit diagram, the CNOT gate has the following representation. The upper wire is called the control bit, and the lower wire the target bit.



It turns out that this is the only two qubit gate we need to think about . . .

# 2 Bell states (EPR pairs)

There are four Bell states:

$$\begin{aligned} |\Phi^\pm\rangle &= \tfrac{1}{\sqrt{2}}\left(|00\rangle \pm |11\rangle\right) \\ |\Psi^\pm\rangle &= \tfrac{1}{\sqrt{2}}\left(|01\rangle \pm |10\rangle\right) \ . \end{aligned}$$

These are maximally entangled states on two qubits. They cannot be product states because there are no cross terms.

Consider one of the Bell states (also known as a EPR pair):

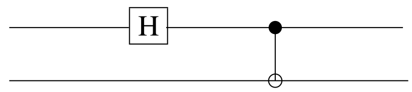
$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

Measuring the first qubit of $|\Psi^-\rangle$ in the standard basis yields a 0 with probability 1/2, and 1 with probability 1/2. Likewise, measuring the second qubit of $|\Psi^-\rangle$ yields the same outcomes with the same probabilities. Thus measuring one, and only one, qubit of this state yields a perfectly random outcome.

However, determining either qubit *exactly* determines the other. For example, if qubit 1 is measured and gives a 0, this projects the Bell state onto the state $\left|01\right\rangle$ and the second qubit is then definitely a 1.

Furthermore, measurement of $\left|\Psi^-\right\rangle$ in any basis will yield opposite outcomes for the two qubits. To see this, check that $\left|\Psi^-\right\rangle = \frac{1}{\sqrt{2}}\left(\left|vv^\perp\right\rangle - \left|v^\perp v\right\rangle\right)$, for any $\left|v\right\rangle = \alpha\left|0\right\rangle + \beta\left|1\right\rangle$, $\left|v^\perp\right\rangle = \bar{\alpha}\left|1\right\rangle - \bar{\beta}\left|0\right\rangle$.

We can generate the Bell states with a Hadamard gate and a CNOT gate. Consider the following diagram:




The first qubit is passed through a Hadamard gate and then both qubits are entangled by a CNOT gate.

If the input to the system is $|0\rangle \otimes |0\rangle$, then the Hadamard gate changes the state to

$$\tfrac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |0\rangle = \tfrac{1}{\sqrt{2}}|00\rangle + \tfrac{1}{\sqrt{2}}|10\rangle \ ,$$

and after the CNOT gate the state becomes $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, the Bell state $|\Phi^+\rangle$. In fact, one can verify that the four possible inputs produce the four Bell states:

$$|00\rangle \mapsto \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = |\Phi^+\rangle; \qquad |01\rangle \mapsto \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) = |\Psi^+\rangle;$$

$$|10\rangle \mapsto \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) = |\Phi^-\rangle; \qquad |11\rangle \mapsto \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) = |\Psi^-\rangle.$$

# 3 EPR Paradox

In 1935, Einstein, Podolsky and Rosen (EPR) wrote a paper "Can quantum mechanics be complete?" [Phys. Rev. 47, 777, Available online via PROLA: `http://prola.aps.org/abstract/PR/v47/i10/p777_1`]

For example, consider coin-flipping. We can model coin-flipping as a random process giving heads 50% of the time, and tails 50% of the time. This model is perfectly predictive, but incomplete. With a more accurate experimental setup it would in principle be possible to follow the dynamics of the coin and to determine precisely the range of initial parameters for which the coin ends up heads, and the range for which it ends up tails.

We saw above that for a Bell state, when you measure first qubit, the second qubit is completely determined. However, if two qubits are far apart, then the second qubit must have had a determined state in some time interval *before* measurement, since the speed of light is finite. Moreover this holds in any basis. This appears analogous to the coin flipping example, i.e., there might be a more complete description which allows the qubit states to be predicted. EPR therefore suggested that there is a more complete theory where"God does not throw dice".

EPR made two assumptions:

i) reality principle - the values of physical quantities have physical reality independent of whether a measurement of them is made or not.

ii) locality principle - the result of a measurement on one system cannot instantaneously influence the result of a measurement on the second system

These two assumptions give rise to a contradiction, nicely illustrated by the analysis of the two qubits in a Bell state, due to Bohm. See Benenti et al., Sec. 2.5. A source emits the Bell state

$$\left|\Psi^-\right\rangle = \frac{1}{\sqrt{2}}(\left|01\right\rangle - \left|10\right\rangle)$$

and sends one qubit to Alice, and one to Bob. If Alice measures her qubit in the standard basis and e.g., gets a 1, then Bob will get a 0 upon measuring in the standard basis. On the other hand, if Alice measures her qubit in the Hadamard basis $\{\left|+\right\rangle, \left|-\right\rangle\}$, and gets a +, then Bob will get a - in the Hadamard basis. However, the states $\left|0\right\rangle$ and $\left|-\right\rangle$ are not the same, they differ by a Hadamard rotation. Which state Bob ends up with depends on the measurement made by Alice. This contradicts the assumption of locality. EPR concluded that such situations imply that quantum mechanics is not a complete theory of the physical world.

What would a more complete theory look like? Here is the most extravagant framework... When the entangled state is created, the two particles each make up a (very long!) list of all possible experiments that they might be subjected to, and decide how they will behave under each such experiment. When the two particles separate and can no longer communicate, they consult their respective lists to coordinate their actions. To describe such behavior one would have to invoke the existence of 'local hidden variables' that are not evident in the quantum description.

It was not until 1964, almost three decades later, that a verifiable and quantitative measure of the local realism assumption was provided. This was given by Bell, who constructed correlation functions of the measurements of Alice and Bob that satisfy a strict inequality under the assumptions of local realism. However, the quantum analog of the correlation functions can violate the inequality for certain choices of measurement basis. The Bell inequality was subsequently tested experimentally in 1981 by Aspect and co-workers, using Bell (EPR) pairs constructed from photon polarization states. Aspect et al. found that indeed nature does not obey the Bell inequalities and so violates local realism. This is consistent with the predictions of quantum mechanics for EPR pairs summarized above and supports the view that nothing *can* be known about the quantum state until a measurement is made. It also tells us that the quantum correlations in an EPR pair are 'stronger' than classical correlations. A detailed analysis of the Bell inequality for $\left|\Psi^-\right\rangle$ can be found at http://minty.caltech.edu/Ph195/downloads.htm (lecture 10/24, pp. 11-15).

For further reading on the EPR paradox, Bell's inequality, and the experimental verification of violation of this by quantum systems, see

1. Styer, Ch. 6

2. D. Mermin, Physics Today vol. 38(4), April 1985, pp. 38-47.