1 Readings

Benenti, Casati, and Strini:

No Cloning Ch.4.2

Teleportation Ch. 4.5

2 Bell inequalities

See lecture notes from H. Mabuchi, Caltech, Ph 195A (2001) on next page

Nonlocality and Bell Inequalities

(Based on the discussion in Chris Isham's book, *Lectures on Quantum Theory: Mathematical and Structural Foundations* (Imperial College Press, 1995).)

Say we have two experimenters, Alice and Bob, whose labs are located many kilometers apart. Their labs are basically identical, actually, each consisting of one particle 'detector' that has one meter, one switch, and a bell. The meter is for reading out the result of a measurement (which we assume to be either ± 1), while the switch is used to select which of two types of measurements the experimenter would like to make. On Alice's side we'll label the two possibilities A and A', and on Bob's side B and B'. The bell rings each time a particle hits the detector, letting the experimenter know when he or she can read out the result of his/her selected measurement.

So where do these particles come from? Midway between Alice's lab and Bob's there is a 'pair source.' This source always produces particles in pairs, sending one to Alice and the other to Bob. We assume that the particles have some internal degree of freedom, which is what Alice's and Bob's detectors are designed to measure. The pair source prepares the internal states of the particles in some unknown, possibly random fashion.

The 'experiment' consists of the following procedure. The source prepares and emits one pair of particles per unit of time, so Alice and Bob know that they may expect to receive particles at a regular rate. Once per unit time, they each (independently) select a random setting for their switch, wait for their bell to ring, and then read off and write down the measurement result.

Hence after ten rounds, *e.g.*, Alice's and Bob's lab books might look something like this:

Alice		**Bob**		
A	-1	B'	-1	
A	$+1$	B'	-1	
A'	$+1$	B	$+1$	
A	$+1$	B	$+1$	
A'	-1	B	-1	
A'	-1	B'	$+1$	1
A	$+1$	B	-1	
A'	-1	B'	$+1$	
A	$+1$	B'	$+1$	
A	-1	B	$+1$	

Although this experimental scenario seems extremely general, it turns out that we have already specified enough to derive some important predictions about the statistics of Alice's and Bob's measurement records!

Let's start by making some reasonable assumptions about the overall behavior of the experiment:

1. ***Local determinism*** – we might like to believe that the result of Alice's measurement (either A or A') is *locally* determined by the physical state of the particle she receives from the pair source. It should not depend on the state of Bob's particle, since in this scenario Bob could be really far away! And the result of Alice's measurement certainly should not depend on Bob's choice of measurement – that is, whether Alice's meter reads $+$ or -1 should not depend on whether Bob has his switch set to B or B'...
2. ***Objective reality*** – Even though Alice (and Bob) must choose to make one measurement or the other (A or A') on any given particle, each particle 'knows' what its value is for both measurements. That is, sufficient information to determine the outcome of either measurement is encoded in the internal state of each particle.

Under these assumptions, we can write down the following model for this experiment. In each round, the pair souce produces a pair of particles with the following information encoded in their internal states:

$$A_n = \pm 1, \quad A'_n = \pm 1, \quad B_n = \pm 1, \quad B'_n = \pm 1. \qquad 2$$

Here the four possible measurement labels are treated as random variables, with the subscript labelling the round. As a logical consequence of local determinism and objective realism, we can assume the existence of a *joint probability distribution* $P(A, A', B, B')$. Hence, it should be meaningful to consider correlation functions of all four random variables simultaneously, and these correlation functions should be measurable by Alice and Bob.

Consider the following function of the random variables,

$$g_n = A_nB_n + A'_nB_n + A_nB'_n - A'_nB'_n. \qquad 3$$

Were we to tabulate the 16 possible values of g_n, we would magically find that $g_n = \pm 2$. However, an easier way to see this is to note that the last term in the sum is equal to the product of the first three, since $A_n^2 = (A'_n)^2 = B_n^2 = (B'_n)^2 = +1$:

$$\begin{aligned} A'_nB'_n &= (A_nB_n)(A'_nB_n)(A_nB'_n) \\ &= A_n^2 B_n^2 A'_n B'_n. \end{aligned} \qquad 4$$

Then if $A'_nB'_n = +1$, the set $\{A_nB_n, A'_nB_n, A_nB'_n\}$ has either zero or two -1's, hence $g_n = A_nB_n + A'_nB_n + A_nB'_n - A'_nB'_n$ must be either $+2$ or -2. If on the other hand $A'_nB'_n = -1$, the set must have either zero or two $+1$'s, hence g_n must be either -2 or $+2$.

In any case, it follows that

$$\left|\frac{1}{N}\sum_{n=1}^{N} g_n\right| = \frac{1}{N}\left|\sum_{n=1}^{N} A_n B_n + \sum_{n=1}^{N} A'_n B_n + \sum_{n=1}^{N} A_n B'_n - \sum_{n=1}^{N} A'_n B'_n\right| \leq 2. \qquad 5$$

This is one form (due to Clauser, Horne, Shimony, and Holt) of Bell's famous inequality.

It should be noted that at this point, all we have relied on in our derivation is basic probability theory! Hence the Bell Inequality is a ***model-independent*** prediction about measurement statistics in a world that is locally deterministic and allows objective realism.

Hence experimental violations of the Inequality actually tell us something about Nature, not just quantum theory!

As it turns out, one can actually go to the lab and perform experiments of precisely the type described above, and find that this inequality is strongly violated! For example, see

- G. Weihs *et al*, "Violation of Bell's Inequality under Strict Einstein Locality Conditions," Phys. Rev. Lett. **81**, 5039-5043 (1998);
- W. Tittel *et al*, "Violation of Bell Inequalities by Photons More Than 10 km Apart," Phys. Rev. Lett. **81**, 3563-3566 (1998);
- A. Aspect, "Bell's inequality test: more ideal than ever," Nature **398**, 189-190 (1999).

In experiments of this type, the key is to construct a source that produces pairs of photons an *entangled* state such as

$$|\Psi_{ab}\rangle = \frac{1}{\sqrt{2}}(|0_a 1_b\rangle - |1_a 0_b\rangle). \qquad 6$$

In each round of the experiment, Alice's two measurements correspond to the observables $\mathbf{A} = \boldsymbol{\sigma}_z^a$ and $\mathbf{A}' = \cos\phi\,\boldsymbol{\sigma}_z^a + \sin\phi\,\boldsymbol{\sigma}_x^a$, where

$$\boldsymbol{\sigma}_z^a = |0_a\rangle\langle 0_a| - |1_a\rangle\langle 1_a|,$$
$$\boldsymbol{\sigma}_x^a = |0_a\rangle\langle 1_a| + |1_a\rangle\langle 0_a|. \qquad 7$$

On Bob's side we choose $\mathbf{B} = \boldsymbol{\sigma}_z^b$ and $\mathbf{B}' = \cos\phi\,\boldsymbol{\sigma}_z^b - \sin\phi\,\boldsymbol{\sigma}_x^b$. The eigenvalues of **A** and **B** are clearly ± 1, and it turns out that those of $\mathbf{A}'$ and $\mathbf{B}'$ are also ± 1. For example, the eigenstates of $\cos\phi\,\boldsymbol{\sigma}_z + \sin\phi\,\boldsymbol{\sigma}_x$ are simply

$$|\tilde{0}\rangle = \cos\frac{\phi}{2}|0\rangle + \sin\frac{\phi}{2}|1\rangle,$$
$$|\tilde{1}\rangle = \sin\frac{\phi}{2}|0\rangle - \cos\frac{\phi}{2}|1\rangle. \qquad 8$$

Hence $\mathbf{A}'$ corresponds to projectors on a basis that is rotated from that of **A** by an angle $\phi/2$ (and similarly a rotation of $-\phi/2$ for $\mathbf{B}, \mathbf{B}'$).

Now we can compute the necessary correlation functions using the standard quantum probability rules:

$$
\begin{aligned}
\frac{1}{N}\sum_{n=1}^{N} A_n B_n &= \langle \mathbf{A} \otimes \mathbf{B} \rangle \\
&= \left\langle \mathbf{P}_0^a \mathbf{P}_0^b \right\rangle + \left\langle \mathbf{P}_1^a \mathbf{P}_1^b \right\rangle - \left\langle \mathbf{P}_0^a \mathbf{P}_1^b \right\rangle - \left\langle \mathbf{P}_1^a \mathbf{P}_0^b \right\rangle \\
&= -1.
\end{aligned} \qquad 9
$$

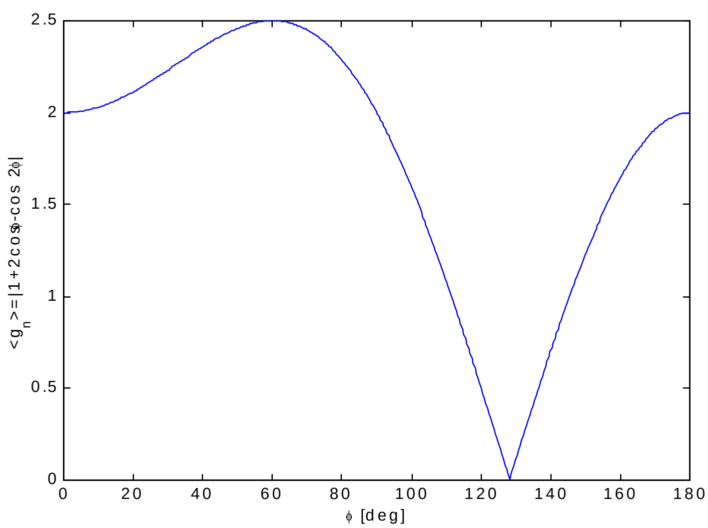
Similarly,

$$
\begin{aligned}
\frac{1}{N}\sum_{n=1}^{N} A_n B'_n &= \langle \mathbf{P}_0^a \cos\phi\, \boldsymbol{\sigma}_z^b \rangle - \langle \mathbf{P}_0^a \sin\phi\, \boldsymbol{\sigma}_x^b \rangle - \langle \mathbf{P}_1^a \cos\phi\, \boldsymbol{\sigma}_z^b \rangle + \langle \mathbf{P}_1^a \sin\phi\, \boldsymbol{\sigma}_x^b \rangle \\
&= -\frac{1}{2}\cos\phi - \frac{1}{2}\cos\phi = -\cos\phi. \\
\frac{1}{N}\sum_{n=1}^{N} A'_n B_n &= \left\langle \mathbf{P}_0^b \cos\phi\, \boldsymbol{\sigma}_z^a \right\rangle + \left\langle \mathbf{P}_0^b \sin\phi\, \boldsymbol{\sigma}_x^a \right\rangle - \left\langle \mathbf{P}_1^b \cos\phi\, \boldsymbol{\sigma}_z^a \right\rangle - \left\langle \mathbf{P}_1^b \sin\phi\, \boldsymbol{\sigma}_x^a \right\rangle \\
&= -\cos\phi. \\
\frac{1}{N}\sum_{n=1}^{N} A'_n B'_n &= \langle \cos^2\phi\, \boldsymbol{\sigma}_z^a \boldsymbol{\sigma}_z^b \rangle + \langle \cos\phi \sin\phi\, \boldsymbol{\sigma}_z^a \boldsymbol{\sigma}_x^b \rangle - \langle \cos\phi \sin\phi\, \boldsymbol{\sigma}_x^a \boldsymbol{\sigma}_z^b \rangle \\
&\quad - \left\langle \sin^2\phi\, \boldsymbol{\sigma}_x^a \boldsymbol{\sigma}_x^b \right\rangle \\
&= \frac{\cos^2\phi}{2}(-1-1) - \frac{\sin^2\phi}{2}(-1-1) = \sin^2\phi - \cos^2\phi \\
&= -\cos 2\phi.
\end{aligned} \qquad 10
$$

Finally, we can construct the overall quantity

$$
\begin{aligned}
\frac{1}{N}\left|\sum_{n=1}^{N} g_n\right| &= |-1 - 2\cos\phi + \cos 2\phi| \\
&= |1 + 2\cos\phi - \cos 2\phi|.
\end{aligned} \qquad 11
$$

Plotting this, we find that the Bell Inequality is violated ($\langle g_n \rangle > 2$) for $0 < \phi < 90^\circ$:



So what's going on here? From the graph we see that our Bell Inequality can be violated when the two possible measurements that Alice and Bob can perform correspond to projections on nonorthogonal bases. Hence what is being exploited here is the extra-strong "quantum correlation" between two particles that have been prepared in an entangled state such as

$$|\Psi_{ab}\rangle = \frac{1}{\sqrt{2}}(|0_a 1_b\rangle - |1_a 0_b\rangle). \qquad 12$$

—

3 No Cloning Theorem

A quantum operation which copied states would be very useful. For example, we considered the following problem in Homework 1: Given an unknown quantum state, either $|\phi\rangle$ or $|\psi\rangle$, use a measurement to guess which one. If $|\phi\rangle$ and $|\psi\rangle$ are not orthogonal, then no measurement perfectly distinguishes them, and we always have some constant probability of error. However, if we could make many copies of the unknown state, then we could repeat the optimal measurement many times, and make the probability of error arbitrarily small. The no cloning theorem says that this isn't physically possible. Only sets of mutually orthogonal states can be copied by a single unitary operator.

There are two ways to prove the no cloning theorem. The first follows from the norm preserving property of the inner product, the second from the linearity of quantum mechanics.

No Cloning Assume we have a unitary operator U_{cl} and two quantum states $|\phi\rangle$ and $|\psi\rangle$ which U_{cl} copies, i.e.,

$$\begin{aligned} |\phi\rangle \otimes |0\rangle &\xrightarrow{U_{cl}} |\phi\rangle \otimes |\phi\rangle \\ |\psi\rangle \otimes |0\rangle &\xrightarrow{U_{cl}} |\psi\rangle \otimes |\psi\rangle \ . \end{aligned}$$

Then $\langle\phi|\psi\rangle$ is 0 or 1.

Proof 1: $\langle\phi|\psi\rangle = (\langle\phi| \otimes \langle 0|)(|\psi\rangle \otimes |0\rangle) = (\langle\phi| \otimes \langle\phi|)(|\psi\rangle \otimes |\psi\rangle) = \langle\phi|\psi\rangle^2$. In the second equality we used the fact that U, being unitary, preserves inner products. □

Proof 2: Suppose there exists a unitary operator U_{cl} that can indeed clone an unknown quantum state $|\phi\rangle = \alpha|0\rangle + \beta|1\rangle$. Then

$$\begin{aligned} |\phi\rangle|0\rangle \xrightarrow{Ucl} |\phi\rangle|\phi\rangle &= (\alpha|0\rangle + \beta|1\rangle)(\alpha|0\rangle + \beta|1\rangle) \\ &= \alpha^2|00\rangle + \beta\alpha|10\rangle + \alpha\beta|01\rangle + \beta^2|11\rangle \end{aligned}$$

But now if we use U_{cl} to clone the expansion of $|\phi\rangle$, we arrive at a different state:

$$(\alpha|0\rangle + \beta|1\rangle)|0\rangle \xrightarrow{U_{cl}} \alpha|00\rangle + \beta|11\rangle .$$

Here there are no cross terms. Thus we have a contradiction and therefore there cannot exist such a unitary operator U_{cl}. □

Note that it is however possible to clone a known state such as $|0\rangle$ and $|1\rangle$.

4 Teleportation

Contrary to its sci-fi counterpart, quantum teleportation is rather mundane. Quantum teleportation is a means to replace the *state* of one qubit with that of another. It gets its out-of-this-world name from the fact that the state is "transmitted" by setting up an entangled state-space of three qubits and then removing two qubits from the entanglement (via measurement). Since the information of the source qubit is preserved by these measurements that "information" (i.e. state) ends up in the final third, destination qubit. This occurs, however, without the source (first) and destination (third) qubit ever directly interacting. The interaction occurs via entanglement. Figure 1 (see below) shows the set up for quantum teleportation, and Figure 2 (see below) presents a quantum circuit implementing teleportation of a one-qubit state.

Suppose $|\psi\rangle = a|0\rangle + b|1\rangle$ and given an EPR pair $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, the state of the entire system is:

$$\frac{1}{\sqrt{2}}\left[a|0\rangle\left(|00\rangle + |11\rangle\right) + b|1\rangle\left(|00\rangle + |11\rangle\right)\right] = \frac{1}{\sqrt{2}}\begin{bmatrix} a \\ 0 \\ 0 \\ a \\ b \\ 0 \\ 0 \\ b \end{bmatrix}$$

Perform the *CNOT* operation and you obtain

$$\frac{1}{\sqrt{2}}\left[a|0\rangle\left(|00\rangle + |11\rangle\right) + b|1\rangle\left(|10\rangle + |01\rangle\right)\right] = \frac{1}{\sqrt{2}}\begin{bmatrix} a \\ 0 \\ 0 \\ a \\ 0 \\ b \\ b \\ 0 \end{bmatrix}$$

Next we apply the H gate. However, as an aside, lets examine what happens when we apply the H gate to $|0\rangle$ and to $|1\rangle$. Recall that:

$$H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$H|0\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$H|1\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Thus, applying H to our system we have:

$$|\varphi\rangle = \frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{2}}a\left(|0\rangle + |1\rangle\right)\left(|00\rangle + |11\rangle\right) + \frac{1}{\sqrt{2}}b\left(|0\rangle - |1\rangle\right)\left(|10\rangle + |01\rangle\right)\right] = \frac{1}{2}\begin{bmatrix} a \\ b \\ b \\ a \\ a \\ -b \\ -b \\ a \end{bmatrix}$$

We can rewrite this expression as:

$$\begin{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \\ \begin{bmatrix} b \\ a \end{bmatrix} \\ \begin{bmatrix} a \\ -b \end{bmatrix} \\ \begin{bmatrix} -b \\ a \end{bmatrix} \end{bmatrix} = \tfrac{1}{2}\left[\left|00\right\rangle \left(a\left|0\right\rangle + b\left|1\right\rangle\right) + \left|01\right\rangle \left(a\left|1\right\rangle + b\left|0\right\rangle\right) + \left|10\right\rangle \left(a\left|0\right\rangle - b\left|1\right\rangle\right) + \left|11\right\rangle \left(a\left|1\right\rangle - b\left|0\right\rangle\right)\right],$$

which we can shorten to:

$$\tfrac{1}{2}\left[\left|00\right\rangle \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \left|\psi\right\rangle + \left|01\right\rangle \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \left|\psi\right\rangle + \left|10\right\rangle \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \left|\psi\right\rangle + \left|11\right\rangle i \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \left|\psi\right\rangle\right].$$

We recognise that the third qubit is now in a state given by the action of one of the well-known Pauli operators I, X, Y, Z on the unknown initial state $\left|\psi\right\rangle$ of qubit 1. The state of qubit 3 can also be written as:

$$\tfrac{1}{2}\left[\left|00\right\rangle I\left|\psi\right\rangle + \left|01\right\rangle X\left|\psi\right\rangle + \left|10\right\rangle Z\left|\psi\right\rangle + \left|11\right\rangle iY\left|\psi\right\rangle\right]$$

and alternatively as:

$$\left|\varphi\right\rangle = \tfrac{1}{2}\left[\left|00\right\rangle I\left|\psi\right\rangle + \left|01\right\rangle X\left|\psi\right\rangle + \left|10\right\rangle Z\left|\psi\right\rangle + \left|11\right\rangle XZ\left|\psi\right\rangle\right].$$

Notice that the two-qubit state of qubits 1 and 2 is different in each term. This result implies that we can measure the first and second qubit and obtain two classical bits which will tell us what transform was applied to the third qubit. Thus we can subsequently "fixup" the third qubit once we know the classical outcome of the measurement of the first two qubits. This fixup is fairly straightforward, either applying nothing, X, Z or both X and Z. (Recall that $X^2 = Y^2 = Z^2 = I$.)

Lets work through an example. Suppose the result of measuring qubits 1 and 2 is 10. Then from the above, qubit 3 must be in the state $Z\left|\psi\right\rangle$. The matrix representing the measurement operator is

$$M_{10} = \begin{bmatrix} 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \end{bmatrix}$$

$P(10) = \left\langle\varphi\right| M_{10}^{\dagger} M_{10} \left|\varphi\right\rangle = \left\langle\varphi\right| M_{10} \left|\varphi\right\rangle$, since here $M_{10}^{\dagger} M_{10}$. Thus:

$$M_{10}\left|\varphi\right\rangle = \tfrac{1}{2}\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ a \\ -b \\ 0 \\ 0 \end{bmatrix}$$

$$\text{Hence: } \langle \varphi | M_{10} | \varphi \rangle = \frac{1}{2}[a,b,b,a,a,-b,-b,a]\,\frac{1}{2}\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ a \\ -b \\ 0 \\ 0 \end{bmatrix} = \frac{1}{4}[a \cdot a^* + b \cdot b^*]$$

Recall that by definition of a qubit we know that $a \cdot a^* + b \cdot b^* = 1$, hence the probability of measuring 01 is $1/4$. The same is true for the other outcomes.

What have we done? We have inserted an unknown single qubit quantum state into a system of 3 qubits where the other two qubits shared some entanglement. We carried out some unitary operations on qubits 1 and 2, and then measured out these two qubits. The result is that the unknown quantum state has been migrated through entanglement to qubit 3, where it is can be recovered by making a single qubit unitary operation dependent on the two measured values from qubits 1 and 2.

Quantum teleportation has been termed "disembodied transfer of quantum information from one place to another" (S. Braunstein). It does not violate relativity: the source sends only classical information (the result of the measurements of qubits 1 and 2) and this must be done by conventional means, e.g., optical fiber. The source sends no information about the quantum state. Neither does it violate the no-cloning theorem since the quantum state is destroyed at the source and created at the destination. ie.,

$$|\psi\rangle |0\rangle \longrightarrow |x\rangle |\psi\rangle .$$

Here $|x\rangle$ is the state of qubit 1 after measurement.

Teleportation illustrates an equivalence between quantum bits (qubits), entanglement bits (e-bits), and classical bits (c-bits):

1 qubit ≡ 1 e-bit + 2 c-bits

Note the difference between making a FAX copy and creating a copy by quantum teleportation. With a FAX, i) the original is preserved, and ii) only a partial copy is obtained. With quantum teleportation, i) the original state is destroyed (but not the qubit), and ii) an exact copy of the quantum state results.

Accessible sources on quantum teleportation:

IBM web page: $\text{http}: //\text{wwww.research.ibm.com}/\text{quantum}_\text{i}\text{nfo}/\text{teleportation}$

C. Caves, Science 282 (23 October) 1998, p. 637

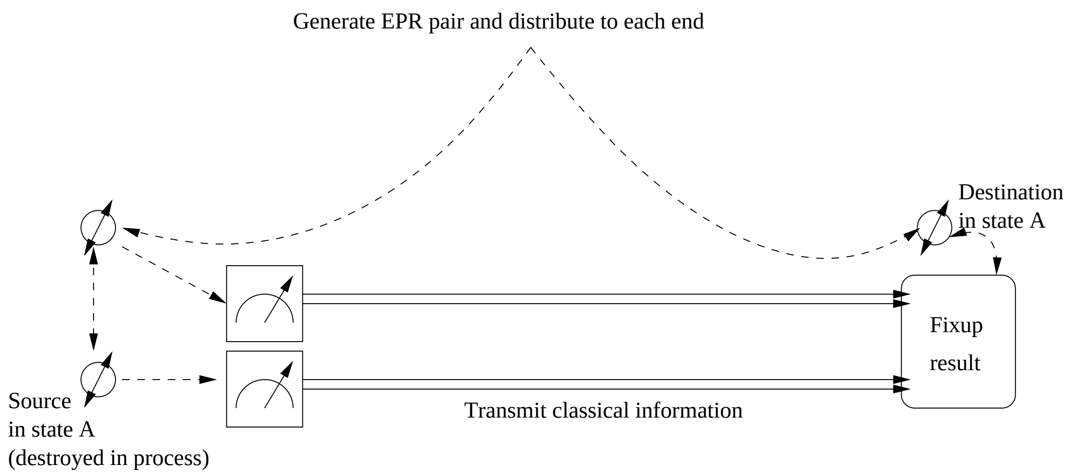


Figure 1: Teleportation requires pre-transmitting an EPR pair to the source and destination. The qubit containing the state to be "teleported" then interacts with one half of this EPR pair, creating a joint state space. Unitaries are performed in this joint state space and then these 2 qubits are measured. The resulting classical information of the measurement outcome is transmitted to the destination. This classical information is used to "fixup" the destination qubit with single qubit unitaries.

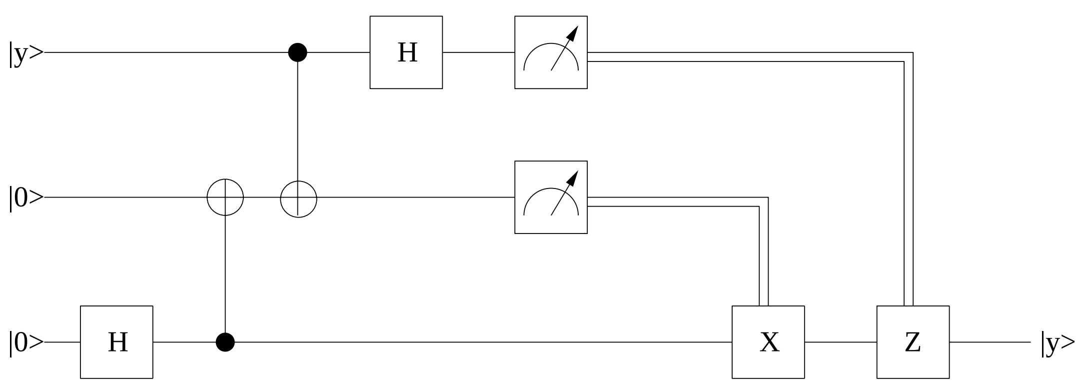


Figure 2: Quantum circuit implementing teleportation. The first two operations on qubits 2 and 3 at the bottom right form the EPR pair. Note that in this diagram single lines represent quantum data while double lines represent classical information.