1. Consider a superposition on n bit strings $\left|\psi\right\rangle = \frac{1}{\sqrt{M}}\sum_{x:\,u\cdot x=0}\left|x\right\rangle$ where u is a fixed non-zero n bit string and $u\cdot x = u_1x_1 + u_2x_2\ldots u_nx_n (mod\ 2)$. What is M? (hint: how many different n-bit strings x exist? If u is not all zeros, then for what fraction of the possible strings x do we have $u\cdot x = 0$? Hint to the hint: Remember that $u\cdot x$ can only take the values 0 or 1).

   Apply the n-bit Hadamard transform to $\left|\psi\right\rangle$ and let the result be $\left|\phi\right\rangle = \sum_x \alpha_x\left|x\right\rangle$ (the somewhat cryptic subscript on the summation sign means "sum over all strings x for which $u\cdot x = 0$"). What is $\alpha_{00\ldots0}$? What is $\left|\phi\right\rangle$? (hint: remember that for n-bit strings i and j, we have $H^{\otimes n}\left|i\right\rangle = 2^{-n/2}\sum_j(-1)^{i\cdot j}\left|j\right\rangle$)

2. Show that the trace of an operator is independent of the basis in which it is evaluated.

3. We have seen that non-commuting observables cannot be exactly specified simultaneously. In this question you will quantify this statement and find just how sharply they can be defined.

   a) Let A and B be two Hermitian operators, and write their commutator as $AB - BA = iC$. Prove that the operator $C \equiv [A,B]/i$ must be a Hermitian operator.

   b) The variance of A in state $\left|\psi\right\rangle$ is defined as

   $$(\Delta A)^2 = \left\langle\psi\right|(A - <A>)^2\left|\psi\right\rangle.$$

   Use the Schwarz inequality to prove that

   $$(\Delta A)^2(\Delta B)^2 \geq |\left\langle\psi\right|(A - <A>)(B - <B>)\left|\psi\right\rangle|^2.$$

   c) Rewrite the operator $F = (A - <A>)(B - <B>)$ in terms of the Hermitian operators $i(F - F^\dagger)$ and $F + F^\dagger$. Now go on and make use of the fact that Hermitian operators have real expectation values to prove that

   $$(\Delta A)^2(\Delta B)^2 \geq \frac{1}{4} <C>^2$$

   d) Evaluate this uncertainty relation for $A = x$ (the position operator) and $B = p$ (the momentum operator). Can you apply this to $A = E$ and $B = t$?

4. Consider a qubit subject to the Hamiltonian

   $$H = \frac{1}{2}\begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$$

   a) Calculate the states of definite energy. What is the energy of each of these states?

   b) Now suppose the state of the qubit at time 0 is $\left|\psi(0)\right\rangle = \left|0\right\rangle$. What is its state at time t?

5. You are given ρ, the density matrix describing a two qubit system.

   a) Suppose we perform a projective measurement in the computational basis of the second qubit. Let $P_0 = \left|0\right\rangle\left\langle 0\right|$ and $P_1 = \left|1\right\rangle\left\langle 1\right|$ be the projectors onto the $\left|0\right\rangle$ and $\left|1\right\rangle$ states of the second qubit, respectively,. Let ρ' be the density matrix which would be assigned to the system after this measurement by an observer who did not know the measurement result. Show that

   $$\rho' = P_0\rho P_0 + P_1\rho P_1.$$

b) Show that the reduced density matrix for the first qubit is not affected by the measurement, i.e., $Tr_2(\rho) = Tr_2(\rho')$.

c) How does this relate to the principle of deferred measurement that we discussed in class?