C/CS/Phys C191 Measurements, Multiple Qubits 9/3/09
Fall 2009 Lecture 3

1 Readings - as in previous lecture

Benenti et al.: Chs. 2, 3.1 - 3.4

Kaye et al.: Chs. 2, 3.1 - 3.4, 4.1-4.2

2 Hilbert Spaces

Consider a discrete quantum system that has k distinguishable states (e.g. a system that can be in one of k distinct energy states. The state of such a system is a unit vector in a k dimensional complex vector space $\mathscr{C}^k$, which we refer to as Hilbert space. The k distinguishable states form an orthogonal basis for the vector space - say denoted by $\{\left|1\right\rangle, \ldots, \left|k\right\rangle$. Here we are using the standard inner-product over $\mathscr{C}^k$ to define orthogonality. Recall that the inner-product of two vectors $\left|\phi\right\rangle = \sum_i \alpha_i \left|i\right\rangle$ and $\left|\psi\right\rangle = \sum_i \beta_i \left|i\right\rangle$ is $\sum_i \bar{\alpha_i}\beta_i$.

Dirac's Braket Notation

We have already introduced the ket notation for vectors.

If $|v\rangle = \sum_i \alpha_i |i\rangle$ and $|w\rangle = \sum_i \beta_i |i\rangle$, then we have already observed that

$$(\vec{v}, \vec{w}) = \begin{pmatrix} \overline{\alpha_1} & \overline{\alpha_2} & \cdots & \overline{\alpha_d} \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \cdots \\ \beta_d \end{pmatrix}.$$

We denote the row vector $(\overline{\alpha_1} \cdots \overline{\alpha_d})$ by $\langle v|$ and the inner product $(\vec{v}, \vec{w})$ by $\langle v|w\rangle$.

$\langle v|$ is a *bra*, and $|w\rangle$ is a *ket*, so $\langle v|w\rangle$ is a *braket*.

What is a bra?

The bra of $|v\rangle$ is NOT a vector in some complex vector space. The bra is defined by the inner product $\langle a||b\rangle$ which maps the vector $\langle v|$ onto the set of complex numbers.

$$L_a(|b\rangle) = \langle a||b\rangle \in \mathscr{C}, \tag{1}$$

i.e.

$$\langle a| \equiv L_a \tag{2}$$

is a map from V to $\mathscr{C}$. Properties: (1) linearity: if $|c\rangle = |a\rangle + |b\rangle$, then $\langle c| = \langle a| + \langle b|$. (2) multiplication by a constant $c|c\rangle \mapsto \langle a|c^*$.

Usually we normalize kets. For

$$|v\rangle = \frac{v_x}{\sqrt{v_x^2 + v_y^2}}|0\rangle + \frac{v_y}{\sqrt{v_x^2 + v_y^2}}|1\rangle, \tag{3}$$

we work with vector

$$|\tilde{v}\rangle = v_x|0\rangle + v_y|1\rangle, \tag{4}$$

that has property $\langle\tilde{v}||\tilde{v}\rangle = 1$.

To demonstrate the utility of the bra-ket notation, let $|v\rangle$ be a vector of norm 1. Define $P = |v\rangle\langle v|$. Then for any $|w\rangle$ we have $P|w\rangle = |v\rangle\langle v|w\rangle$, so P is the projection operator onto $|v\rangle$. Note that $P^2 = |v\rangle\langle v|v\rangle\langle v| = P$ since $|v\rangle$ has norm 1. If the notion of a projection in a complex valued vector space is hard for you to visualize, consider the simpler situation with real vectors in 2D. We can choose basis vectors **i** and **j**

$$|\mathbf{i}\rangle = \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) \text{ and } |\mathbf{j}\rangle = \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right). \tag{5}$$

The scalar projection of a ket onto a basis vector is given by the inner (dot) product:

$$\langle i||v\rangle = \left(\begin{smallmatrix}1 & 0\end{smallmatrix}\right)\left(\begin{smallmatrix}a\\b\end{smallmatrix}\right) = a. \tag{6}$$

We can also write

$$|v\rangle = \left(\begin{smallmatrix}a\\b\end{smallmatrix}\right) = \langle i||v\rangle|i\rangle + \langle j||v\rangle|j\rangle = |i\rangle\langle i||v\rangle + |j\rangle\langle j||v\rangle = I|v\rangle. \tag{7}$$

Mathematical properties of Hilbert space

A Hilbert space is a complex vector space endowed with an inner-product and which is complete under the induced norm. The vector space axioms give us notions of span and linear independence of a set of vectors. However, to endow the vector space with geometry — the notion of angle between two vectors and the norm or length of a vector, we must define an inner-product — whose properties are listed below. The third property — completeness — is trivially satisfied for a finite dimensional system, so we will not bother to define it here.

- An *inner product* on a (complex) vector space V is a map $(\cdot,\cdot) : V \times V \to \mathscr{C}$ satisfying for each $\vec{u},\vec{v},\vec{w} \in V$ and $\alpha,\beta \in \mathscr{C}$:

  (i) $(\vec{v},\vec{v}) \geq 0$, and $(\vec{v},\vec{v}) = 0$ if and only if $\vec{v} = \vec{0}$;

  (ii) $(\alpha\vec{u} + \beta\vec{v},\vec{w}) = \alpha(\vec{u},\vec{w}) + \beta(\vec{v},\vec{w})$;

  (iii) $(\vec{v},\vec{w}) = \overline{(\vec{w},\vec{v})}$.

  An *inner product space* is a vector space together with an inner product.

- Vectors $\vec{v},\vec{w} \in V$ are *orthogonal* if $(\vec{v},\vec{w}) = 0$.

- A *basis* for V is a set $\{\vec{v}_1,\cdots,\vec{v}_d\}$ such that each $\vec{v} \in V$ can be written uniquely in the form $\vec{v} = \alpha_1\vec{v}_1 + \cdots + \alpha_n\vec{v}_n$. The basis is said to be *orthonormal* if $(\vec{v}_i,\vec{v}_j) = \delta_{ij}$ for each i,j. (Here $\delta_{ij} = 1$ if $i = j$ and 0 if $i \neq j$.)

Note that we can associate to each inner product space a canonical norm, defined by $\|\vec{v}\| = \sqrt{(\vec{v},\vec{v})}$. A *Hilbert space* is an inner product space which is complete with respect to its norm. If V is finite-dimensional (i.e. it has a finite basis), then completeness is automatically satisfied. Furthermore, there is only one Hilbert space of each dimension (up to isomorphism.)

3 Measurement as a projection

Recall that the state of a single qubit can be written as a superposition over the possibilities 0 and 1: $\left|\psi\right\rangle = \alpha\left|0\right\rangle + \beta\left|1\right\rangle$. Measuring in the standard basis, then, there is probability $|\alpha|^2$ that we get 0 and the new state is $\left|\psi'\right\rangle = \left|0\right\rangle$, and probability $|\beta|^2$ that we get 1 and $\left|\psi'\right\rangle = \left|1\right\rangle$. Note that **a single measurement** of the observable A on a state $\left|\Psi\right\rangle$ in the basis (representation) of eigenstates of $\hat{A}$ will yield the value a_i, with probability $P_\Psi(a_i) = |\left\langle a_i\middle|\Psi\right\rangle|^2$. To determine this probability, the measurement must be repeated many times! A single measurement alone is not sufficient.

This procedure defines the measurement operator

$$\hat{M}_i = \left|a_i\right\rangle\left\langle a_i\right|$$

that acts on the state $\left|\Psi\right\rangle$. The normalized state after measurement is then easily seen to be equal to

$$\frac{\hat{M}_i\left|\Psi\right\rangle}{\sqrt{\left\langle\Psi\middle|M_i^\dagger M_i\middle|\Psi\right\rangle}}.$$

For a measurement in the $\left|a_i\right\rangle$ basis this is given by

$$\left|i\right\rangle\frac{\left\langle i\middle|\Psi\right\rangle}{\sqrt{\left\langle\Psi\middle|M_i^\dagger M_i\middle|\Psi\right\rangle}},$$

where we have abbreviated

$$\left|a_i\right\rangle \equiv \left|i\right\rangle.$$

For example, suppose we have the linear superposition

$$\left|\Psi\right\rangle = \alpha_1\left|1\right\rangle + \alpha_2\left|2\right\rangle + \alpha_3\left|3\right\rangle + ... + \alpha_k\left|k\right\rangle.$$

Making a single measurement of the observable A on $\left|\Psi\right\rangle$ will result in the outcome a_i with probability

$$P_\Psi(a_i) = |\alpha_i|^2$$

and the resulting state after the measurement is equal to

$$\left|i\right\rangle\left(\frac{\alpha_i}{|\alpha_i|}\right).$$

The measurement of the observable has "collapsed" the state $\left|\Psi\right\rangle$ to a single eigenstate $\left|i\right\rangle \equiv \left|a_i\right\rangle$ of $\hat{A}$ (recall these constitute an orthonormal basis).

We can formalize this by noticing that a measurement can be written as a projector. A projector $P_i = \left|i\right\rangle\left\langle i\right|$ takes a ket $\left|\psi\right\rangle$ and replaces it by its component $\left|i\right\rangle$, with amplitude $\left\langle i\middle|\psi\right\rangle$. The spectral resolution of the identity defines a set of projectors. For a general expansion $\left|\psi\right\rangle = \sum_j c_j\left|j\right\rangle$ and an orthonormal basis $\{\left|i\right\rangle\}$, we have the corresponding resolution of the identity:

$$I = \sum_i \left|i\right\rangle\left\langle i\right| = \sum_i P_i$$

E.g., $I = |0\rangle\langle 0| + |1\rangle\langle 1|$ for a two state basis.

Hence

$$P_i|\psi\rangle = |i\rangle\langle i|\,\psi\rangle = \sum_j c_j\langle i|\,j\rangle|j\rangle = \sum_j \delta_{i,j}\,c_j|i\rangle = c_i|i\rangle$$

Note: operators may generally be written in the form $O = \{\langle a|\,b\rangle\}_{\{a,b\}}$.

4 Measurement in general basis

More generally, we can measure the qubit in any orthonormal basis simply by projecting $|\psi\rangle$ onto the two basis vectors. See Figure 1.

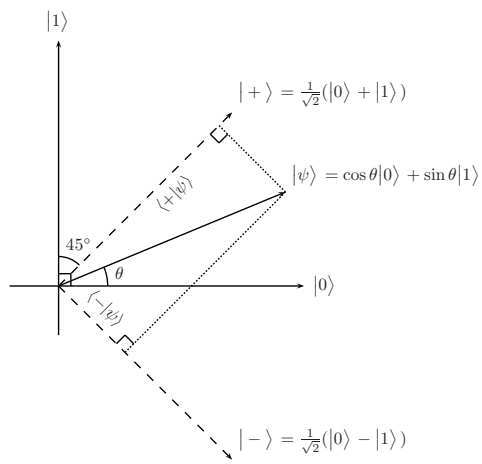


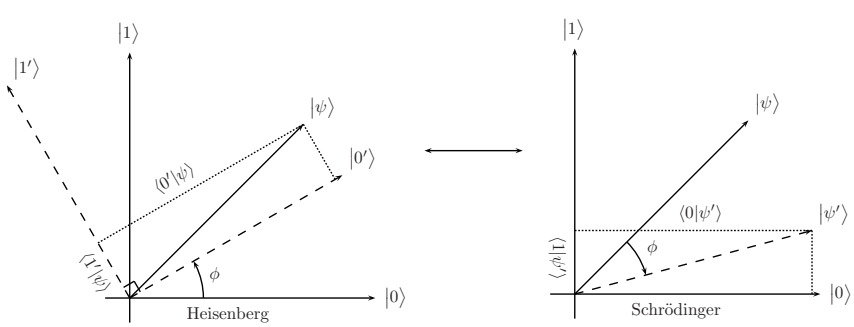
Figure 1:

The new state of the system $|\psi'\rangle$ is the outcome of the measurement. An important alternate basis is the Hadamard basis:

$$|+\rangle = \tfrac{1}{\sqrt{2}}|0\rangle + \tfrac{1}{\sqrt{2}}|1\rangle \tag{8}$$

$$|-\rangle = \tfrac{1}{\sqrt{2}}|0\rangle - \tfrac{1}{\sqrt{2}}|1\rangle \quad . \tag{9}$$

Alternatively, instead of measuring the system in a rotated basis, we can rotate the system (in the opposite direction) and measure it in the original, standard basis.



5 Sequential Measurements

This linear superposition $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ is part of the private world of the qubit. For us to know the state of the qubit, whether this is realized by an electron, a photon, or an electron spin, we must make a measurement. Measuring $|\psi\rangle$ in the standard basis $\{|0\rangle, |1\rangle\}$ yields $|0\rangle$ with probability $|\alpha|^2$, and $|1\rangle$ with probability $|\beta|^2$.

One important aspect of the measurement process is that it alters the state of the quantum system: the effect of the measurement is that the new state is exactly the outcome of the measurement. I.e., if the outcome of the measurement is $|0\rangle$, then following the measurement, the qubit is in state $|0\rangle$. This implies that you cannot collect any additional information about the amplitudes α_j by repeating the measurement on the resuling state. You need to make many identical measurements on a set (ensemble) of equivalent states.

Repeated measurements on a state may however be useful for other reasons. We shall examine this with analysis of the measurement process for photon polarization. The polarization of a photon can be measured by using a polaroid or a calcite crystal. These materials act as filters that select only one component of the electric field vector. See the section on polarization in ScienceTrek at http://www.colorado.edu/physics/2000. A polaroid sheet (suitably oriented) transmits x-polarized photons $|x\rangle$ and absorbs y-polarized photons $|y\rangle$. Thus a photon that is in a superposition $|\phi\rangle = \alpha|x\rangle + \beta|y\rangle$ is transmitted with probability $|\alpha|^2$ if the polaroid sheet is oriented to transmit x and with probability $|\beta|^2$ is the sheet is oriented to transmit y. In the former case the final state is $|x\rangle$, in the latter case it is $|y\rangle$.

Consider passing a photon in state $|\psi\rangle$ through 2 polaroid filters, first an x filter, then a y filter. After the first filter we have $|x\rangle$ with prob. $|\alpha|^2$. After the second filter we have nothing, with prob. 1. Where has the photon gone? During passage through the first filter it was absorbed by the first filter with prob. $|\beta|^2$. If it got through this first filter, it was absorbed by the second filter with prob. 1. Note that the experiment may also be interpreted as the results of identical experiments on many identical photons in state $|\psi\rangle$.

Now consider what happens if we interpose a third polaroid sheet at a 45 degree angle between the first two. Now a photon that is transmitted by the first sheet makes it through the next two with probability $1/4$. Why is this? The polarization of light after the first filter is $|x\rangle$. The second filter is oriented at 45 degrees, i.e., it will pass photons with polarization orientation $\vec{v} = \frac{1}{\sqrt{2}}(\vec{x}+\vec{y})$. So let's express $|x\rangle$ in the basis $|v\rangle$ and its orthogonal complement $|v^\perp\rangle$. This is also known as the $|+\rangle, |-\rangle$ basis.

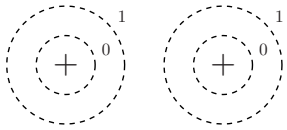
$$|x\rangle = \frac{1}{\sqrt{2}}\Big(|v\rangle + |v^\perp\rangle\Big)$$

Now the light passing through the first filter is in state $|x\rangle$ with probability $|\alpha|^2$. The probability this light passes the second filter is equal to the probability that a $|0\rangle$ qubit ends up in $|+\rangle$ when measured in the $|+\rangle, |-\rangle$ basis. Reading off from the above equation, we see that this probability is 1/2. Those photons that do pass successfully through the second filter now have a resulting polarization $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$. The probability of this state now passing the third filter oriented in y is then 1/2. What is the overall probability of having a photon pass successfully through all 3 sheets? Answer: $|\alpha|^2/4$, obtained by multiplying the three probabilities for successive passage. The final photon state is $|y\rangle$.

Note that one effect of these measurements is to effectively rotate the plane of polarization of the photon - measurements can thus provide a way to make operations on qubits, although these are not unitary operations.

6 Two qubits

Now let us examine the case of two qubits. Consider the two electrons in two hydrogen atoms:



Since each electron can be in either of the ground or excited state, classically the two electrons are in one of four states – 00, 01, 10, or 11 – and represent 2 bits of classical information. Quantum mechanically, they are in a superposition of those four states:

$$\left|\psi\right\rangle = \alpha_{00}\left|00\right\rangle + \alpha_{01}\left|01\right\rangle + \alpha_{10}\left|10\right\rangle + \alpha_{11}\left|11\right\rangle \quad ,$$

where $\sum_{ij}|\alpha_{ij}|^2 = 1$. Again, this is just Dirac notation for the unit vector in $\mathscr{C}^4$:

$$\begin{pmatrix} \alpha_{00} \\ \alpha_{01} \\ \alpha_{10} \\ \alpha_{11} \end{pmatrix}$$

where $\alpha_{ij} \in \mathscr{C}$, $\sum|\alpha_{ij}|^2 = 1$.

Measurement of 2 qubits

If the two electrons (qubits) are in state $\left|\psi\right\rangle$ and we measure them, then the probability that the first qubit is in state i, and the second qubit is in state j is $P(i,j) = |\alpha_{ij}|^2$. Following the measurement, the state of the two qubits is $\left|\psi'\right\rangle = \left|ij\right\rangle$. What happens if we measure just the first qubit? What is the probability that the first qubit is 0? In that case, the outcome is the same as if we had measured both qubits: $\Pr\{\text{1st bit } = 0\} = |\alpha_{00}|^2 + |\alpha_{01}|^2$. The new state of the two qubit system now consists of those terms in the superposition that are consistent with the outcome of the measurement – but normalized to be a unit vector:

$$\left|\phi\right\rangle = \frac{\alpha_{00}\left|00\right\rangle + \alpha_{01}\left|01\right\rangle}{\sqrt{|\alpha_{00}|^2 + |\alpha_{01}|^2}}$$

.

A more formal way of describing this partial measurement is that the state vector is projected onto the subspace spanned by $\left|00\right\rangle$ and $\left|01\right\rangle$ with probability equal to the square of the norm of the projection, or onto the orthogonal subspace spanned by $\left|10\right\rangle$ and $\left|11\right\rangle$ with the remaining probability. In each case, the new state is given by the (normalized) projection onto the respective subspace.

Tensor products (informal):

Suppose the first qubit is in the state $\left|\phi_1\right\rangle = \alpha_1\left|0\right\rangle + \beta_1\left|1\right\rangle$ and the second qubit is in the state $\left|\phi_2\right\rangle = \alpha_2\left|0\right\rangle + \beta_2\left|1\right\rangle$. How do we describe the joint state of the two qubits?

$$\begin{aligned} \left|\phi\right\rangle &= \left|\phi_1\right\rangle \otimes \left|\phi_2\right\rangle \\ &= \alpha_1\alpha_2\left|00\right\rangle + \alpha_1\beta_2\left|01\right\rangle + \beta_1\alpha_2\left|10\right\rangle + \beta_1\beta_2\left|11\right\rangle \quad . \end{aligned}$$

We have simply multiplied together the amplitudes of $|0\rangle_1$ and $|0\rangle_2$ to determine the amplitude of $|00\rangle_{12}$, and so on. The two qubits are not entangled with each other and measurements of the two qubits will be distrbuted independently.

Given a general state of two qubits can we say what the state of each of the individual qubits is? The answer is usually no. For a random state of two qubits is entangled — it cannot be decomposed into state of each of two qubits. In the next lecture we will study the Bell states, which are maximally entangled states of two qubits.

The significance of tensor products

Classically, if we put together a subsystem that stores k bits of information with one that stores l bits of information, the total capacity of the composite system is $k+l$ bits.

From this viewpoint, the situation with quantum systems is extremely paradoxical. We need k complex numbers to describe the state of a k-level quantum system. Now consider a system that consists of a k-level subsystem and an l-level subsystem. To describe the composite system we need kl complex numbers. One might wonder where nature finds the extra storage space when we put these two subsystems together.

An extreme case of this phenomenon occurs when we consider an n qubit quantum system. The Hilbert space associated with this system is the n-fold tensor product of $\mathscr{C}^2 \equiv \mathscr{C}^{2^n}$. Thus nature must "remember" of 2^n complex numbers to keep track of the state of an n qubit system. For modest values of n of a few hundred, 2^n is larger than estimates on the number of elementary particles in the Universe.

This is the fundamental property of quantum systems that is used in quantum information processing.

Finally, note that when we actually a measure an n-qubit quantum state, we see only an n-bit string - so we can recover from the system only n, rather than 2^n, bits of information.