1. Let \( f : \{0,1\}^n \rightarrow \{0,1\} \). Let \( U_f \) be the unitary transformation (quantum circuit) that on input \(|x\rangle |b\rangle\) (where \( x \in \{0,1\}^n \) and \( b \in \{0,1\} \)), outputs \(|x\rangle |b \oplus f(x)\rangle\). Show that if the answer bit \( b \) is initialized to \(|-\rangle\) then on input \( \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \), the output is \( \frac{(-1)^{f(x)}}{\sqrt{2}} |x\rangle |-\rangle \). i.e. the answer bit remains unchanged, and effectively the transformation that is achieved is: \(|x\rangle \rightarrow (-1)^{f(x)} |x\rangle\).

2. Let \( f : \{0,1\}^n \rightarrow \{0,1\}^n \) be a bijection on the \( n \)-bit strings. We showed in class that if there is an efficient classical circuit (say of size \( m \)) for computing \( f(x) \) on input \( x \), then there is an efficient reversible circuit (of size \( O(m) \)) that outputs \( x \cdot f(x) \cdot 0^k \) on input \( x \cdot 0^{n+k} \), where \( k = O(m) \).

Now suppose that you are also given an efficient classical circuit for computing \( f^{-1} \) (of size \( m' \)). Show that there is an efficient reversible circuit that outputs \( f(x) \cdot 0^l \) on input \( x \cdot 0^l \). What is the size of your circuit and how big is \( l \)?

3. Consider the state \( |\psi\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle) \). What is the state that results from applying the 2-qubit Hadamard transform to \(|\psi\rangle\)?
Repeat for the state \( |\phi\rangle = \frac{1}{\sqrt{2}} (|01\rangle + |10\rangle) \).

4. Consider a superposition on \( n \) bit strings \( |\psi\rangle = \frac{1}{\sqrt{M}} \sum_{x=0}^{2^n-1} |x\rangle \) where \( u \) is a fixed non-zero \( n \) bit string and \( u \cdot x = u_1 x_1 + u_2 x_2 \ldots u_n x_n (mod \ 2) \). What is \( M \)?
Apply the \( n \)-qubit Hadamard transform to \(|\psi\rangle\) and let the result be \( |\phi\rangle = \sum_x \alpha_x |x\rangle \). What is \( \alpha_{00\ldots 0} \)?
What is \( |\phi\rangle \)?

5. What is the quantum fourier transform modulo \( M \) of the uniform superposition \( \frac{1}{\sqrt{M}} \sum_{j=0}^{M-1} |j\rangle \)?
What is the QFT modulo \( M \) of \(|j\rangle\)?

6. Let \( |\beta\rangle = \frac{1}{2^{n/2}} \sum_{x \in \{0,1\}^n} \beta_x |x\rangle \) be the Hadamard transform of the superposition \( |\alpha\rangle = \frac{1}{2^{n/2}} \sum_{x \in \{0,1\}^n} \alpha_x |x\rangle \)
Consider the "shift" by \( u \) of the superposition \( |\alpha\rangle: |\alpha'\rangle = \frac{1}{2^{n/2}} \sum_{x \in \{0,1\}^n} \alpha_x |x + u\rangle \) (where \( x + u \) means bitwise sum mod 2) and its Hadamard transform \( |\beta'\rangle = \frac{1}{2^{n/2}} \sum_{x \in \{0,1\}^n} \beta'_x |x\rangle \). Describe \(|\beta'\rangle\) as a function of \(|\beta\rangle\) and \( u \).