1. Recall the "reflection about the mean operator" in Grover's algorithm, which is given by $G = H_{2^n} A H_{2^n}$, where A is a $2^n \times 2^n$ diagonal matrix with -1's on the diagonal, except the top left entry which is 1. i.e. $A = -I + 2|0^n\rangle\langle 0^n|$.

   Show that if G is applied to the state $|\psi\rangle = \sum_x \alpha_x |x\rangle$ then the result is $|\phi\rangle = \sum_x \beta_x |x\rangle$, where $\beta_x = 2\mu - \alpha_x$. Here $\mu = \frac{\sum_x \alpha_x}{2^n}$ is the arithmetic mean of all the amplitudes.

   Justify in what sense β_x represents the reflection about the mean?

2. Given an efficient quantum circuit for computing A. You may assume that you are given a quantum circuit U_f for computing a suitably chosen function f on n bits.

3. Prove the following relationships involving the Pauli matrices,

   (a) $\sigma_x \sigma_z \sigma_x = -\sigma_z$

   (b) $[\sigma_i, \sigma_j] = 2i\varepsilon_{ijk}\sigma_j$

   (c) $\exp(i\theta\hat{n}\cdot\vec{\sigma}) = \cos(\theta)I + i\sin(\theta)\hat{n}\cdot\vec{\sigma}$

   Here ε_{ijk} is the Levi-Civita symbol, $\hat{n}$ is a unit vector, $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ is the vector of Pauli matrices, and I is the identity matrix.

4. We saw in the lecture that spins will precess in the presence of a magnetic field. Using the Zeeman Hamiltonian,

   $$H = \frac{1}{2}\gamma B \sigma_z,$$

   describe the time evolution of the expectation values, $\langle\sigma_x(t)\rangle, \langle\sigma_x(t)\rangle, \langle\sigma_x(t)\rangle$ in terms of their values at time t=0.

   Hint: It may help to use the relations you proved in problem 3.

5. A mathematically nice, but unphysical, way to detect entanglement is to use the state inversion operator, $\mathscr{T}$, which, for all states, $\hat{n}$, acts as

   $$\mathscr{T}|\hat{n}\rangle = |-\hat{n}\rangle.$$

   Why can't this operator be realized physically?