1. The most general Hamiltonian (modulo addition by an unimportant constant) for a single qubit system is

$$H = \frac{\omega}{2}\hat{n}\cdot\vec{\sigma}.$$

Here $\hat{n}$ is a unit vector and $\vec{\sigma}$ is the vector of Pauli matrices as discussed in class. Use this form of the Hamiltonian and the Bloch parameterization

$$\rho = \frac{1}{2}\left(I + \vec{P}\cdot\vec{\sigma}\right)$$

to show that the polarization vector evolves in time as

$$\dot{\vec{P}} = \omega(\hat{n}\times\vec{P}).$$

2. Suppose we have a set of qubits which are subject to depolarization at the rate, Γ, according to the equation

$$\dot{\rho} = -i[H,\rho] - \Gamma\left(\rho - \frac{1}{2}I\right)$$

where $H = \frac{\omega}{2}\hat{n}\cdot\vec{\sigma}$. Show that the master equation describing the polarization vector evolution becomes

$$\dot{\vec{P}} = \omega(\hat{n}\times\vec{P}) - \Gamma\vec{P}.$$

If a qubit is prepared at time $t=0$ with polarization vector $\vec{P}(0) = (1,0,0)$, find the polarization vector as a function of time if the Hamiltonian is $H = \omega\sigma_z/2$.

3. An arbitrary rotation may be parameterized by three angles, α, β and γ, called Euler angles. Consider a sequence of Euler rotations represented by

$$\begin{aligned} U(\alpha,\beta,\gamma) &= \exp\left(-i\alpha\sigma_z/2\right)\exp\left(-i\beta\sigma_y/2\right)\exp\left(-i\gamma\sigma_z/2\right) \\ &= \begin{pmatrix} e^{-i(\alpha+\gamma)/2}\cos(\beta/2) & -e^{-i(\alpha-\gamma)/2}\sin(\beta/2) \\ e^{i(\alpha-\gamma)/2}\sin(\beta/2) & e^{i(\alpha+\gamma)/2}\cos(\beta/2) \end{pmatrix} \end{aligned}$$

Because the rotations form a group, we expect this sequence of operations to be equivalent to a *single* rotation about some axis, $\hat{n}$, by an angle θ. Find $\hat{n}$ and θ so that .

$$U(\alpha,\beta,\gamma) = \exp\left(i\theta\hat{n}\cdot\sigma\right)$$

Hint: Explicitly write down the matrices for the above equation. Because the matrices are equal, corresponding entries in each matrix must be equal as well!
Note: This problem could also be done with SO(3) matrices, but it is a miserable calculation. This is a nice example of how many properties of SO(3) are easier to analyze in terms of the corresponding SU(2) matrices.