1 Spin Resonance

How do we control qubit states in the lab? If $|psi(t)\rangle = \alpha(t)|0\rangle + \beta(t)|1\rangle$, how do we deterministically change α and β?

We know that the Hamiltonian evolves things in time, so if we turn on a field then the Hamiltonian will evolve the state via $e^{-i\hat{H}t/\hbar}$.

For a static magnetic field this allows us to rotate qubit state from one point on the Bloch sphere to another via rotations:

$$\hat{R}_i(\Delta\theta) = e^{-i\hat{S}_i\Delta\theta/\hbar}, \Delta\theta = \frac{eB_o}{m}\Delta t, \vec{B} = B_o\hat{x}_i$$

Question: How can we maintain energy level splitting between $|0\rangle$ and $|1\rangle$ and *control* the rate at which a qubit rotates between states? (i.e. change it at a rate different from $\omega_o = \frac{eB_o}{m}$.)

Answer: Spin Resonance gives us a new level of control (most clearly seen in NMR).

How it works: Turn on a big DC field B_o and a little AC field $\vec{B}$ sin(ω_o t) that is tuned to the resonance $\omega_o = \frac{eB_o}{m}$:

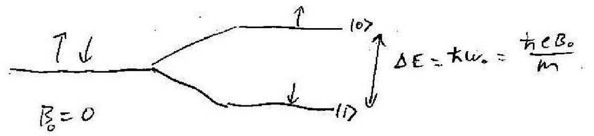

Figure 1:

The small AC field induces controlled mixing between $|0\rangle$ and $|1\rangle$... "SPIN FLIPS".

We must solve the Schrodinger equation to understand what is going on:

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = \hat{H}|\psi(t)\rangle$$

It is convenient to use column vector notation:

$$|psi(t)\rangle = \alpha(t)|0\rangle + \beta(t)|1\rangle = \begin{pmatrix} \alpha(t) \\ \beta(t) \end{pmatrix}$$

What's the Hamiltonian? $\hat{H} = -\vec{\mu}\cdot\vec{B} = \frac{e}{m}\vec{S}\cdot\vec{B}$

We now let the magnetic field be composed of the large bias field and a small oscillating transverse field:

$$\vec{B} = B_o\hat{z} + B_1 cos\omega_o t\hat{x}$$

With this we obtain the Hamiltonian:

$$\hat{H} = \frac{e}{m}B_o\hat{S}_z + \frac{e}{m}B_1 cos\omega_o t\hat{S}_x$$

Now use 2×2 matrix formulation, where the Pauli matrices ($\hat{S}_z = \frac{\hbar}{2}\sigma_z$, etc.) are of course eminently useful:

$$\hat{H} = \frac{e}{m}B_o \cdot \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{e}{m}B_1 cos\omega_o t \cdot \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

The two terms sum to give the following 2×2 Hamiltonian matrix (expressed in the $\hat{S}_z$ basis):

$$\hat{H} = \frac{e\hbar}{2m}\begin{pmatrix} B_o & B_1 cos\omega_o t \\ B_1 cos\omega_o t & -B_o \end{pmatrix}$$

Now we can plug this Hamiltonian into the Schr. equation and solve for $|psi\rangle$.

A bit of intuition on QM: If you construct a Hamiltonian matrix out of some basis, then the matrix element H_{ij} tells us how much application of the Hamiltonian tends to send a particle from state $|j\rangle$ to state $|i\rangle$. (The units are of course energy $\Rightarrow$ rate of transitions $\propto$ frequency $\propto \frac{E}{\hbar} \propto \frac{H_{ij}}{\hbar}$.)

So, if we only had $\vec{B} = B_o\hat{z}$ and $\vec{B}_1 = 0$, then what would the rate of spin flip transitions be?

$$rate_{i\leftarrow j} \propto \langle i|\hat{H}|j\rangle = |1\rangle\hat{H}|0\rangle = H_{21} = 0!$$

So, we can conclude that we NEED to have a field perpendicular to the large bias field $\vec{B} = B_o\hat{z}$ to induce "spin flips" or to mix up $|0\rangle$ and $|1\rangle$ states in $|\psi\rangle$. This is perhaps more obvious in case of spin, but not as obvious for other systems. It is important to develop our quantum mechanical intuition which can easily get lost in the math!

Now let's solve the Schr. equation for Spin Resonance.

$$\hat{H}|\psi(t)\rangle = i\hbar\frac{\partial}{\partial t}\begin{pmatrix} \alpha(t) \\ \beta(t) \end{pmatrix} = \frac{e\hbar}{2m}\begin{pmatrix} B_o & B_1 cos\omega_o t \\ B_1 cos\omega_o t & -B_o \end{pmatrix}\begin{pmatrix} \alpha(t) \\ \beta(t) \end{pmatrix}$$

We get two coupled differential equations. First, we define $\omega_o = \frac{eB_o}{m}$ and $\omega_1 = \frac{eB_1}{2m}$, where the latter quantity is defined with a seemingly annoying factor of 1/2. It'll make sense later, though.

$$i\frac{\partial\alpha(t)}{\partial t} = \frac{\omega_o}{2}\alpha(t) + \omega_1 cos(\omega_o t)\beta(t)$$

$$i\frac{\partial\beta(t)}{\partial t} = \omega_1 cos(\omega_o t)\alpha(t) - \frac{\omega_o}{2}\beta(t)$$

To solve we make a substitution. This may seem weird, but it involves the recognition that the system has a natural rotating frame in which the system should be viewed.

$$a(t) = \alpha(t)e^{i\omega t/2}$$

$$b(t) = \alpha(t)e^{-i\omega t/2}$$

Now we're going to use a dubious approximation, but it involves a recognition that ω_o is much larger than ω_1 and these fast rotations average to zero on the timescales $1/\omega_1$ (which are the relevant experimental timescales). Anyway, here's the dubious approximation:

$$cos(\omega_o t)e^{i\omega_o t} \approx \frac{1}{2}$$

Using these definitions and dubious approximations and we obtain the following differential equation for a(t) (and correspondingly b(t)):

$$\frac{\partial^2 a(t)}{\partial t^2} + \frac{\omega_1^2}{4}a(t) = 0$$

This is a familiar second order differential equation. Our initial conditions have yet to be specified, but let's say $\alpha(0) = \beta(0) = 0$. This gives the following solution:

$$\begin{pmatrix} \alpha(t) \\ \beta(t) \end{pmatrix} = \begin{pmatrix} e^{-i\frac{\omega_o}{2}t} cos\frac{\omega_1}{2}t \\ -e^{+i\frac{\omega_o}{2}t} sin\frac{\omega_1}{2}t \end{pmatrix}$$

What does this mean geometrically? Let's go to the Bloch sphere! Our generalized Bloch vector looks like:

$$|\psi\rangle = cos\frac{\theta}{2}|0\rangle + e^{i\phi} sin\frac{\theta}{2}|1\rangle$$

Our time-dependent state which is a solution to the Schr. equation looks like:

$$|\psi(t)\rangle = cos\frac{\omega_1 t}{2}|0\rangle + e^{i(\omega_o+\pi)} sin\frac{\omega_1 t}{2}|1\rangle$$

Geometrically we can say that $\phi = \omega_o t + \pi$, so we conclude that the qubit is spinning around $\hat{z}$ at a rate ω_o.

What about θ? $\theta = \omega_1 t$, so we're crawling up the sphere at a rate $\omega_1 = \frac{eB_1}{m}$ at the same time we're spinning rapidly about $\hat{z}$ at the fast ω_o, the *Larmor frequency*. We can control ω_1 precisely by changing the amplitude of B_1.

Even though ω_0 is very large, ω_1 can be very small. If we're really good, we can flip spins by applying a "π-pulse": $\omega_1 \Delta t = \pi$.

Note: As spins flip out of ground state they suck energy out of the "RF field" ($B_1 cos\omega_o$). This is easily detected and forms the basis of NMR.

Now let's talk about a little bit of quantum weirdness. What happens if we take the spin wavefunction of a particle, break it into two pieces and let it interfere with itself? How do you do this? Use a classic 2-slit experiment. You can get strange interference effects.

Imagine the following strange device:

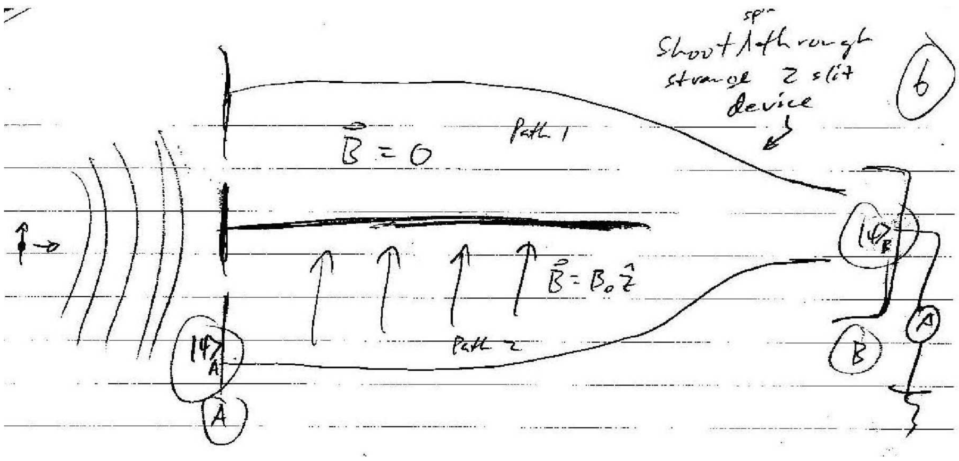

Figure 2:

The particle starts in spin up $|\psi\rangle_A = |0\rangle$. So what we will do is shoot spins through the device and measure the number of spins that get through to B:

$$|\psi\rangle_B = e^{-i\hat{H}t/\hbar} = |\psi\rangle_A = |\psi\rangle_{path1} + |\psi\rangle_{path2}$$

This is the classic description of interference where we superpose to quantum states and see if they constructively or destructively interfere. But what are the quantum states for the two paths?

$$|\psi\rangle_{path1} = |0\rangle$$

and

$$|\psi\rangle_{path2} = e^{-i\frac{\hat{S}_z}{\hbar}\Delta\phi}|0\rangle$$

where $\Delta\phi = \frac{eB_o}{m}\Delta t$ and Δt is the transit time.

Now let's suppose that B_o and Δt are tuned so that $\Delta\phi = 2\pi$. What happens?

This is the subject of a homework problem.