The Frequency Domain, without tears

Many slides borrowed from Steve Seitz

Somewhere in Cinque Terre, May 2005

CS194: Intro to Computer Vision and Comp. Photo
Angjoo Kanazawa & Alexei Efros, UC Berkeley, Fall 2022
Salvador Dali
“Gala Contemplating the Mediterranean Sea, which at 30 meters becomes the portrait of Abraham Lincoln”, 1976
What about near the edge?

- the filter window falls off the edge of the image
- need to extrapolate
- methods:
  - clip filter (black)
  - wrap around (circular)
  - copy edge
  - reflect across edge

Source: S. Marschner
Salvador Dali
“Gala Contemplating the Mediterranean Sea, which at 30 meters becomes the portrait of Abraham Lincoln”, 1976
Spatial Frequencies and Perception

Campbell-Robson contrast sensitivity curve
A nice set of basis

Teases away fast vs. slow changes in the image.

This change of basis has a special name…
Jean Baptiste Joseph Fourier (1768-1830)

had crazy idea (1807):

Any univariate function can be rewritten as a weighted sum of sines and cosines of different frequencies.

Don’t believe it?

• Neither did Lagrange, Laplace, Poisson and other big wigs
• Not translated into English until 1878!

But it’s (mostly) true!

• called Fourier Series

...the manner in which the author arrives at these equations is not exempt of difficulties and... his analysis to integrate them still leaves something to be desired on the score of generality and even rigour.
A sum of sines

Our building block:

\[ A \sin(\omega x + \phi) \]

Add enough of them to get any signal \( f(x) \) you want!

How many degrees of freedom?

What does each control?

Which one encodes the coarse vs. fine structure of the signal?
Fourier Transform

We want to understand the frequency $\omega$ of our signal. So, let's reparametrize the signal by $\omega$ instead of $x$:

$$f(x) \rightarrow \text{Fourier Transform} \rightarrow F(\omega)$$

For every $\omega$ from 0 to $\infty$, $F(\omega)$ holds the amplitude $A$ and phase $\phi$ of the corresponding sine $A \sin(\omega x + \phi)$

- How does $F$ hold both?

$$F(\omega) = R(\omega) + iI(\omega)$$

$$A = \pm \sqrt{R(\omega)^2 + I(\omega)^2}$$

$$\phi = \tan^{-1} \frac{I(\omega)}{R(\omega)}$$

We can always go back:

$$F(\omega) \rightarrow \text{Inverse Fourier Transform} \rightarrow f(x)$$
Time and Frequency

example: \( g(t) = \sin(2\pi f t) + (1/3)\sin(2\pi(3f) t) \)
example: \( g(t) = \sin(2\pi f t) + \frac{1}{3}\sin(2\pi (3f) t) \)
Frequency Spectra

g(t) = \sin(2pf t) + (1/3)\sin(2p(3f) t)

[Graphs and frequency spectrum diagram]
Frequency Spectra

Usually, frequency is more interesting than the phase
Frequency Spectra
Frequency Spectra
Frequency Spectra
Frequency Spectra
Frequency Spectra
Frequency Spectra

\[ A \sum_{k=1}^{\infty} \frac{1}{k} \sin(2\pi kt) \]
Frequency Spectra

(a) $f(x)$

(b) $|F(u)|$

(c) $f(x)$ and $|F(u)|$
FT: Just a change of basis

\[ \mathbf{M} \ast f(x) = F(\omega) \]
IFT: Just a change of basis

\[ M^{-1} \ast F(\omega) = f(x) \]
Finally: Scary Math

Fourier Transform: \( F(\omega) = \int_{-\infty}^{+\infty} f(x)e^{-i\omega x} \, dx \)

Inverse Fourier Transform: \( f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(\omega)e^{i\omega x} \, d\omega \)
Finally: Scary Math

Fourier Transform: \[ F(\omega) = \int_{-\infty}^{+\infty} f(x)e^{-i\omega x} \, dx \]

Inverse Fourier Transform: \[ f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(\omega)e^{i\omega x} \, d\omega \]

...not really scary: \[ e^{i\omega x} = \cos(\omega x) + i \sin(\omega x) \]

is hiding our old friend: \[ \sin(\omega x + \phi) \]

phase can be encoded by sin/cos pair

\[ P \cos(x) + Q \sin(x) = A \sin(x + \phi) \]

\[ A = \pm \sqrt{P^2 + Q^2} \quad \phi = \tan^{-1}\left(\frac{P}{Q}\right) \]

So it’s just our signal \( f(x) \) times sine at frequency \( \omega \)
Extension to 2D

Image as a sum of basis images
Extension to 2D

in Matlab, check out: imagesc(log(abs(fftshift(fft2(im)))));
Fourier analysis in images

Intensity Image

Fourier Image

http://sharp.bu.edu/~slehar/fourier/fourier.html#filtering
Signals can be composed

http://sharp.bu.edu/~slehar/fourier/fourier.html#filtering
More: http://www.cs.unm.edu/~brayer/vision/fourier.html
Man-made Scene

Amplitude Spectrum

what does phase look like, you ask? (less visually informative)
The importance of Phase
Can change spectrum, then reconstruct

Local change in one domain, courses global change in the other
Low and High Pass filtering
The Convolution Theorem

The greatest thing since sliced (banana) bread!

• The Fourier transform of the convolution of two functions is the product of their Fourier transforms

\[ F[g \ast h] = F[g]F[h] \]

• The inverse Fourier transform of the product of two Fourier transforms is the convolution of the two inverse Fourier transforms

\[ F^{-1}[gh] = F^{-1}[g] \ast F^{-1}[h] \]

• Convolution in spatial domain is equivalent to multiplication in frequency domain!
2D convolution theorem example

\[ f(x, y) \]

\[ h(x, y) \]

\[ g(x, y) \]

\[ /F(s_x, s_y) / \]

\[ /H(s_x, s_y) / \]

\[ /G(s_x, s_y) / \]
Filtering

Why does the Gaussian give a nice smooth image, but the square filter give edgy artifacts?
Fourier Transform pairs

Spatial domain

\[ f(x) \]

- box(x)

- gauss(x; \sigma)

- sinc(s)

Frequency domain

\[ F(s) = \int_{-\infty}^{\infty} f(x)e^{-i2\pi sx} \, dx \]

- sinc(s)

- gauss(s; 1/\sigma)

- box(x)
Box Filter
Low-pass, Band-pass, High-pass filters

low-pass:

High-pass / band-pass:
Edges in images
Low Pass vs. High Pass filtering

Image

- 

Smoothed

Details
Filtering – Sharpening

Image + $\alpha$

"Sharpened" $\alpha=1$

=
Filtering – Sharpening

Image + α = “Sharpened” α=0

Details
Filtering – Sharpening

Image + $\alpha$ = “Sharpened” $\alpha=2$
Filtering – Sharpening

Image

Details

\[ \text{"Sharpened" } \alpha = 0 \]
Filtering – Extreme Sharpening

Image + α Details

“Sharpened” α=10
Unsharp mask filter

\[ f + \alpha(f - f \ast g) = (1 + \alpha)f - \alpha f \ast g = f \ast ((1 + \alpha)e - \alpha g) \]

- Image
- Blurred image
- Unit impulse (identity)
- Unit impulse
- Gaussian
- Laplacian of Gaussian
5 min recap

Fourier Transform in 5 minutes: The Case of the Splotched Van Gogh, Part 3

https://www.youtube.com/watch?v=JciZYrh36LY