Marginalia

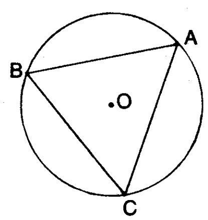
More on randomness

Mark Kac

A variation of the old saying "ars longa vita brevis" is applicable to randomness, and I could probably devote many more columns to exploring the various aspects of this fascinating subject. But life being short and the patience of my readers limited, this discussion will have to do for the foreseeable future.

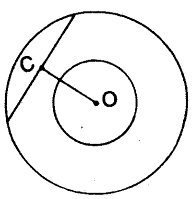
Let me begin by considering an innocent-sounding problem. A chord is chosen "at random" in a circle of radius R. What is the probability that it is shorter than the side of an equilateral triangle inscribed in the circle (i.e., $R\sqrt{3}$)?

Here is a solution. A chord is determined by its two endpoints, one of which, A, may as well be fixed. Taking A as one of the vertices of the inscribed equilateral triangle, we easily determine the other two, B and C, by this construction:



in which the arcs AB, BC, and CA are each subtended by an angle of 120°. For a chord with one of its endpoints at A to be shorter than ($R\sqrt{3}$), the second endpoint must be chosen on either the arc AB or the arc CA. It thus follows almost at once that the desired probability is 2/3. Neat and simple.

Unfortunately there is another solution which is equally neat and simple. We begin with the observation that a chord is completely determined by its center C. All one has to do is to connect C with the center of the circle, O, and draw a chord through C which is perpendicular to this line:



It is clear that for the chord to be shorter than ($R\sqrt{3}$) its center C must lie outside the circle with the same center O as the original one but with half the radius ($R/2$). Since the area of the annulus between the two circles is 3/4 of the area of the bigger circle, the desired probability is 3/4.

Which of the two answers is correct? Mathematically both are. The difference is due to different interpretations of the phrase "at random." In the first case it is assumed that all endpoints of the chord (one endpoint having been chosen and fixed) are uniformly distributed on the circumference of the circle; in the second case the assumption is that the centers of the chords are uniformly distributed throughout the interior of the circle.

All the example shows is that the phrase "at random" does not have an absolute meaning. And yet when Bertrand discussed this problem in a book based on his lectures at the Sorbonne (*1*) it caused a mini-crisis in probability theory and became known as the Bertrand paradox.

While from a mathematical point of view there is no paradox nor in fact any difficulty, a question may be raised as to whether there is an *empirical* way of deciding which of the many answers (there are infinitely many ways of defining "at random," each yielding an answer of its own) is the "right" one.

This is not a well-defined question until one knows something about the empirical device which is used to draw the chords "at random." An analysis of such a device will (we hope!) lead us to the proper interpretation of the phrase "at random" and allow us to calculate the desired probability, which can then be compared with an appropriate frequency derived from empirical data.

Now, however, one may become embroiled in questions such as: Should the device which draws chords "at random" be itself in some sense "random"? And if so, how can one tell whether it is indeed "random"? These questions have already been considered in an earlier column (*Am. Sci.* 71:405–06), and I am only repeating them in a different context. We are, in short, back to the initial question: What is random?

I wish I could follow Mark Twain's advice to tell the truth, thereby "gratifying some and astonishing the rest," but I cannot, because the question as posed has no answer. Rather than engage in a lengthy (and futile) discussion of the semantics of randomness I will consider an example which might help to clarify the subtle and hitherto unresolved (perhaps even unresolvable) issues.

The example is an experiment first suggested in 1914 by the great Polish physicist Marian Smoluchowski, who had worked out the theory behind it. Smoluchowski's suggestion was taken up by several investigators, but I shall refer specifically to Kappler's experiment of 1931 because his paper (*2*) includes a number of excellent reproductions of the results.

The experiment consisted of observing the motion of a tiny mirror suspended on a quartz fiber in a vessel containing air. By a simple and ingenious method it is possible to magnify the deflections of the mirror, converting the angular displacements into linear ones and photographing the results. A 30-second tracing obtained in this way is reproduced in Figure 1. One cannot fail to be impressed by such an *ad oculus* demonstration of the existence of molecules, for if it were not for molecules of air hitting the little mirror "at random" the tracing would have been purely sinusoidal.

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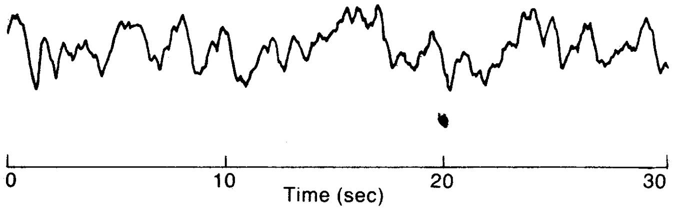


Figure 1. By recording the deflections of a mirror suspended from a quartz fiber in a vessel containing air, Kappler arrived at a pictorial representation of the "random" collisions of molecules of air with the mirror. However, a pattern indistinguishable from this "random" one can be produced by a purely deterministic procedure. The two patterns will yield the same results when analyzed statistically. (After Kappler 1931.)

It is also difficult to escape the feeling that in looking at Kappler's tracing one is in the presence of chance incarnate and that the tracing could have been produced only by a random mechanism. Indeed, Smoluchowski's theory, so completely confirmed by Kappler's experiment, was based on the assumption that the mirror is hit "at random" by molecules of air. But the phrase "at random" is only a mathematical camouflage for the rules of calculating the statistical properties of the motion of the mirror. These rules must, of course, be consistent with the mechanism of the collisions between the molecules and the mirror, and therefore the arbitrariness of "at random" in Bertrand's example is absent in this case. But there are still no grounds for concluding that the molecular bombardment is in some operationally definable way "random." It couldn't be, because the molecules, the mirror, the quartz fiber, and the walls of the container form a dynamical system which, as long as we stay within the realm of classical physics, is subject to purely deterministic laws. There just isn't room for chance or randomness or any such concept, and it is still a largely unresolved question why statistical methods in the dynamics of many-body systems are so successful.

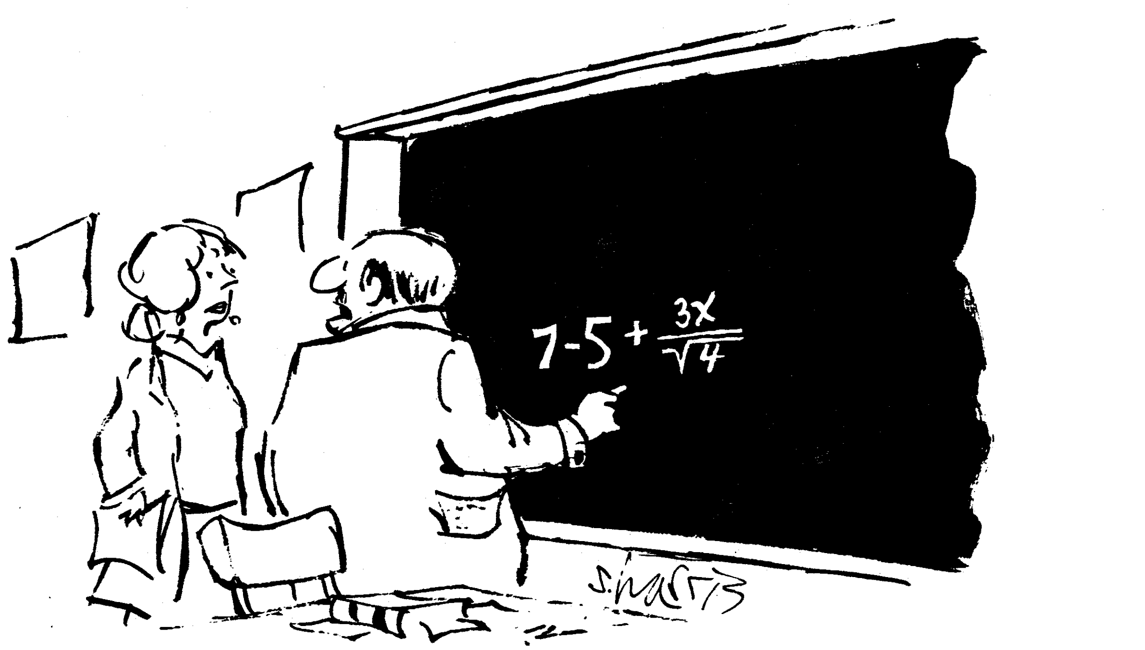
Worse yet, it is possible to make a perfect "counterfeit" of Kappler's tracings if one knows Smoluchowski's theory and the pertinent physical parameters (the moment of inertia of the mirror, the temperature, and so on) of the experiment. The theory predicts that the displacement of the mirror is what is called a stationary Gaussian process. Such a process is determined by its so-called covariance function and this function is also given by the theory in terms of the pertinent physical parameters. Now, if I have this information I can choose a sequence of numbers $\lambda_1, \lambda_2, \lambda_3, \ldots$, and a "scale" α such that where n and t are sufficiently large the graph of the function

$$\chi_n(t) = \alpha \frac{\cos\lambda_1 t + \cos\lambda_2 t + \ldots + \cos\lambda_r t}{\sqrt{n}}$$

will be indistinguishable from a typical Kappler tracing of the same duration. Not only will this graph *look* like the "random" tracing in Figure 1, but when it is subjected to the same statistical analysis it will yield the same results (*3*). The moral should be abundantly clear: given a Kappler tracing and a graph of a properly prepared $\chi_n(t)$ there will be no way to tell which one was produced by molecular shocks and which one was contrived with malice aforethought by the author of this column. So what is random?

References

1. J. Bertrand. 1907. *Calcul des Probabilités.* 2nd ed. New York: Chelsea Publishing Company. The example in question is found on pp. 4–5.
2. E. Kappler. 1931. *Annalen der Physik* 11:242.
3. For a precise formulation see my article: 1981. In the search for the meaning of independence. In *The Making of Statisticians*, ed. J. Gani, pp. 68–69. Springer-Verlag.



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