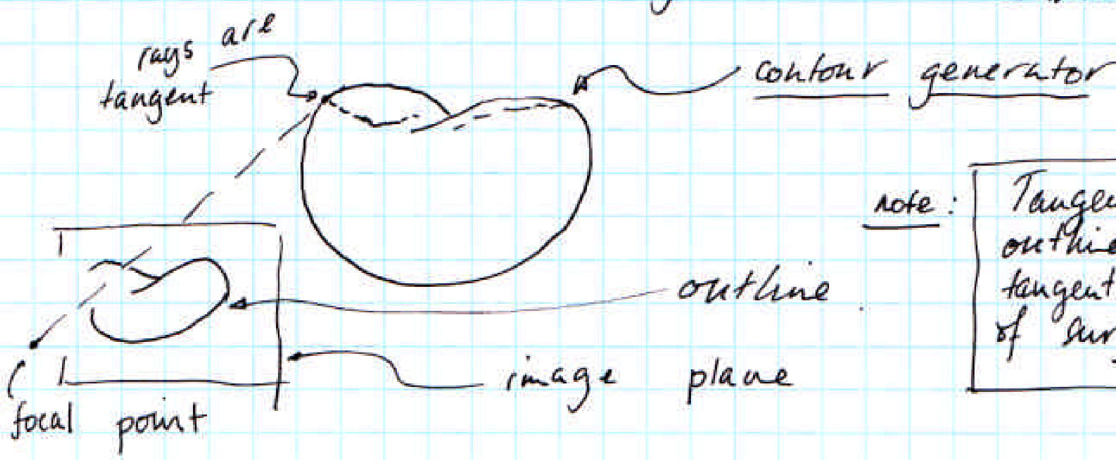
⑦

Contours, outlines, etc.

The points where a surface turns away from the eye are interesting

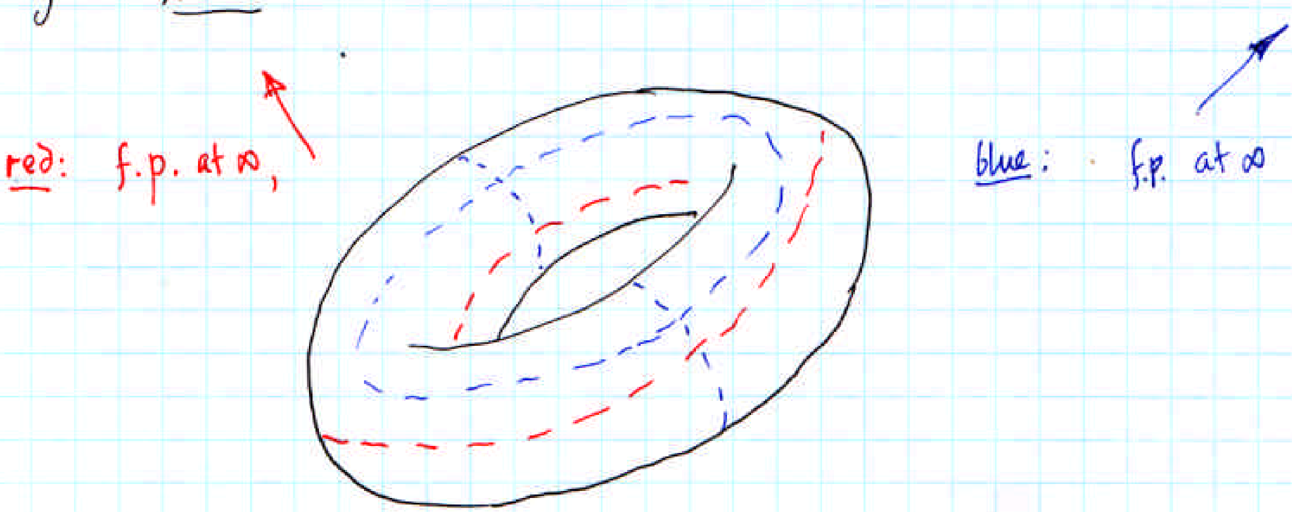
- hard to ray-trace
- sudden change in what is visible



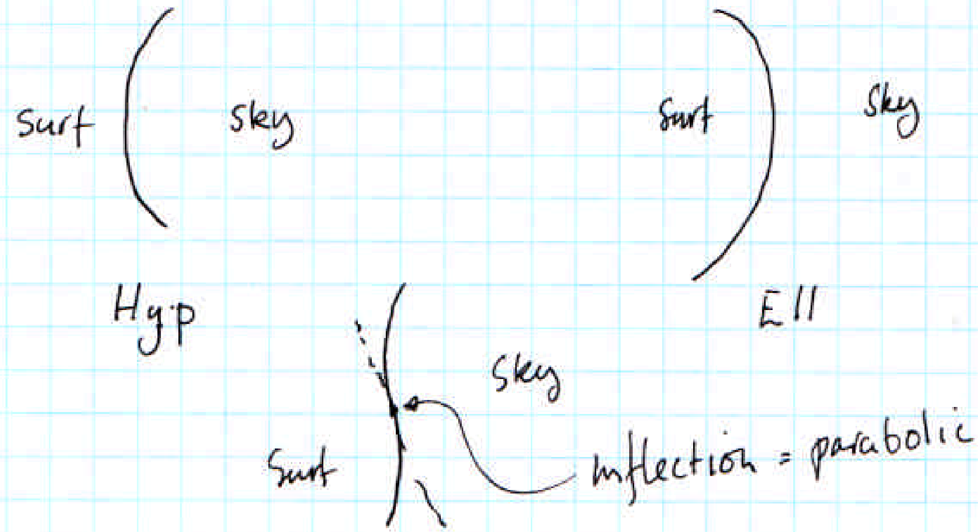
note: Tangent to outline is tangent plane of surface

for geometric purposes, we think of the surface as translucent; c.g. moves w/ focal point

e.g. torus:



(8) Curvature and outline:



→ (8a)

Some eqns + calcs

Ignore this bit right now

• form: local Taylor series – look up calc.s in book

• plane curves:

Ignore this bit right now

① a circle: $(r\sin\theta,\ r - r\cos\theta)$ at $\theta = 0$

$\approx \left(r\theta + O(\theta^3),\ \frac{r\theta^2}{2} + O(\theta^4\right.$

Ignore this bit right now

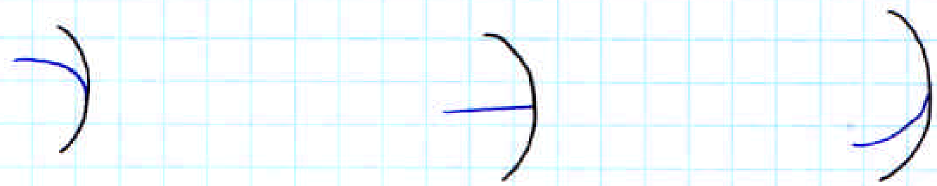
$$\approx \left(t\ ,\ \left(\frac{1}{r}\right)\frac{t^2}{2}\right) + \text{small term}$$

near $t = 0$

8a

· Tangency at outlines

- draw a curve on the surface — what does it look like at the outline?



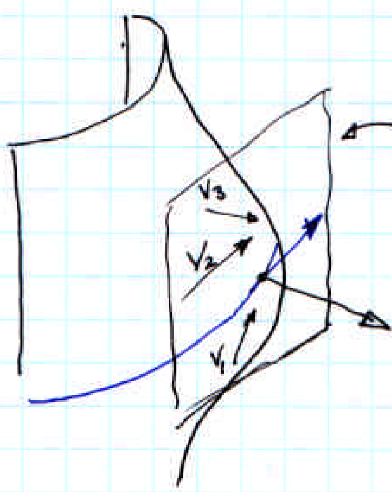
① ② ③

Never:

→ Not tangent, not normal.

Why:

· curve lies on surface ⇒ its tangent is tangent to surf.



tangent plane.

· view dir is in tangent plane.

· in V_1, V_3 curve tangent proj to line
— same line as T.P.

· in V_2, to pt. on curve

86

Duality

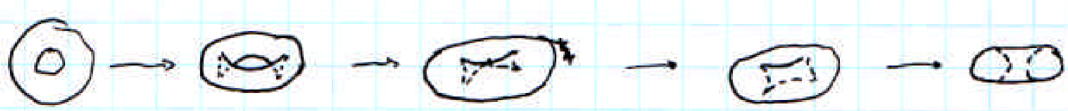
Point $(x, y, z, 1)$ (H.C.'s)

plane $ax+by+cz+d=0$

$\therefore$ (a, b, c, d) H.C.'s for plane

Usual	Dual
plane $(ax+by+cz+d=0)$	point (a,b,c,d)
point $(x_0, y_0, z_0, 1)$	plane; all points (a, b, c, d) such that $ax_0+by_0+cz_0+d$.
surface	dual surface: each point is a plane tangent to the original surface
outline:	plane section of dual:

implies the dual is a nasty surface!



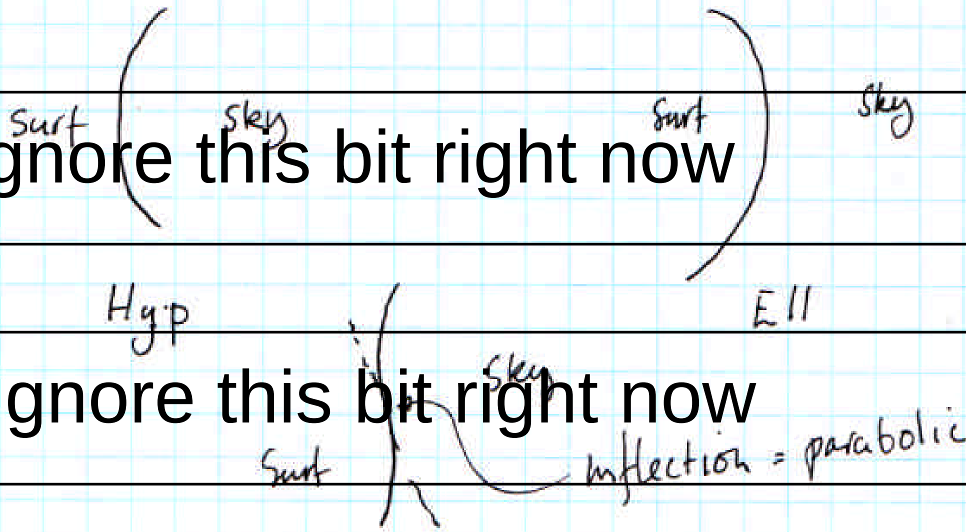
e.g.

(8) Curvature and outline:

Ignore this bit right now

Ignore this bit right now

Ignore this bit right now



→ (8a)

Some eqns + calcs

- why: Gaussian curvature 'bout everything.
- form: local Taylor series – look up calc.s in book

- plane curves:

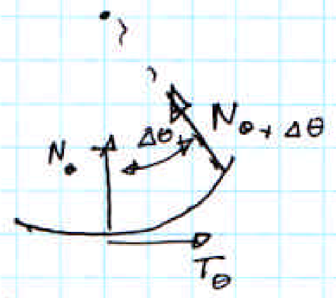
① a circle: $(r\sin\theta,\ r - r\cos\theta)$ at $\theta = 0$

$$\approx \left(r\theta + O(\theta^3),\ \frac{r\theta^2}{2} + O(\theta^4\right.$$

$$\approx \left(t\ ,\ \left(\frac{1}{r}\right)\frac{t^2}{2}\right) \quad + \text{ small term}$$

near $t = 0$

. a circle's normal:



distance along curve $\simeq r\Delta\theta = \Delta s$

$$N_{\theta+\Delta\theta} \simeq N_\theta + \Delta\theta \, T_\theta$$

$$\therefore \lim_{\Delta s \to 0} \left(\frac{\Delta N}{\Delta s} \cdot T \right) = \frac{dN}{ds} \cdot T = \frac{1}{r}$$

$\frac{1}{r}$ is called the curvature of the ~~curve~~ circle.

Now if we can write a curve as

$$\left(t, \ \frac{1}{2} \kappa t^2 + O(t^3) \right) \quad \text{near } t=0$$

then the best-fitting circle has curvature κ

and we say the curve has curvature κ

. obviously, a second derivative

. Do this by rotating, translating coordinate system and reparametrising.

(10) by analogy, etc. we have for any curve

$$\boxed{\frac{dN}{ds} \cdot T = k,}$$

N a unit normal, s is arclength, T tangent

- Space curves: we can choose a plane through the tangent, in which the curve is "most like" a plane curve

$$\left(t \ , \ \frac{k}{2} t^2 + \cdots \ , \ \frac{\tau}{3!} t^3 + \cdots \right) \quad \text{near } t=0$$

τ is called $\boxed{\text{TORSION}}$

11 Surfaces, and the second fundamental form

- choose c-sys so that N is z-axis and origin is pt.
- can write

$$\left(u,\ v,\ {}^{0+0}-\frac{1}{2}\left[au^2 + 2buv + cv^2\right] + O(^3)\right)$$

write $au^2 + 2buv + cv^2$ as $\begin{pmatrix} u \\ v \end{pmatrix}^T M \begin{pmatrix} u \\ v \end{pmatrix}$

$$M = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

called: <u>second fundamental form.</u>

<u>Notice:</u>

<u>Tangents:</u>

$T_u \quad (1, 0, -(au + bv))$

$T_v \quad (0, 1, -(bu + cv))$.

<u>Normal:</u>

$$N = \frac{((au+bv), (bu+cv), 1)}{\|\qquad\|}$$

$$\left.\begin{array}{ll} \frac{dN}{du} \cdot T_u = a & \\ \frac{dN}{dv} \cdot T_u = b & \\ \frac{dN}{du} \cdot T_v = b & \frac{dN}{dv} \cdot T_v = c \end{array}\right] \text{at } u=0, v=0$$

(12) Now consider a <u>directional</u> derivative of Normal

$\nabla_X N$. $\qquad X = m T_u + n T_v$

$$= m \frac{dN}{du} + n \frac{dN}{dv}.$$

$\nabla_X N \cdot Y \qquad$ where $\quad Y = p T_u + q T_v$.

$$(mp)\left(\frac{dN}{du} \cdot T_u\right) + np\left(\frac{dN}{dv} \cdot T_v\right) + mq\left(\frac{dN}{du} \cdot T_v\right) + nq\left(\frac{dN}{dv}\right.$$

$$= \boxed{(m \;\; n)\begin{pmatrix} a & b \\ b & c \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix}}$$

write $\boxed{\nabla_X N \cdot Y = \mathrm{II}(X, Y)}$

(13) Now, look at II in special c-sys

- rotate to diag II

$$\left(x, y, -\frac{1}{2}\left(k_1 x^2 + k_2 y^2\right) + O(3)\right)$$

Notice:

~~• on elliptic surfaces~~,

Normal is

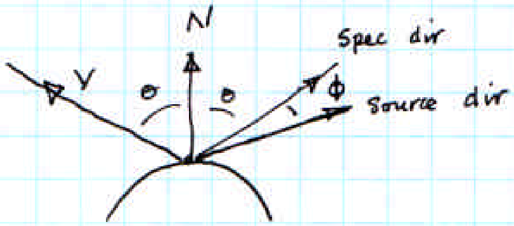
$$\frac{(k_1 x, k_2 y, 1)}{\sqrt{1 + k_1^2 x^2 + k_2^2 y^2}} + O(\ )$$

from this we could get

$$\boxed{K = k_1 k_2}$$ — Gaussian curvature

$$\boxed{H = k_1 + k_2}$$ — mean curvature

(13) What do specularities look like?



Phong's model Spec. Int $\propto (\cos\Phi)^n$

Now we want pts where $(\cos\Phi)^n$ = const.

- fix source dir.
- ~~implies~~ at s, normal N_s, source dir = Spec dir.
- want p such that N_p means $(\cos\varphi)^n$ = const.
- $\Rightarrow$ p such that $\boxed{N_p \cdot N_s = \text{const}}$
- set up csys as above, such that N_s = z-axis, s is origin.

$$\therefore \quad N_p \cdot N_s = \frac{1}{\sqrt{1 + K_1^2 u^2 + K_2^2 v^2}}$$

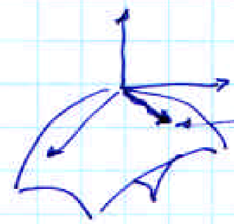
$$\therefore \quad K_1^2 u^2 + K_2^2 v^2 = \text{const}.$$

(15)

Rodriguez? formula

- consider a direction $d = \cos\theta, \sin\theta$ on tangent plane
- slice surface to get curve — what is its curvature?

in plane:



vec $= \alpha(\cos\theta, \sin\theta,$

$\alpha\cos\theta,\ \alpha\sin\theta,\ -\frac{\alpha^2}{2}(k_1\cos^2\theta + k_2$

or, along curve

$\alpha,\quad -\frac{\alpha^2}{2}(k_1\cos^2\theta + k_2\sin^2\theta)$

$\therefore$ k_1 is max / min, k_2 is other.

$\therefore$ Priviliged orthonormal direction field on surface — where form is diagonalised — gives

- lines of curvature.

(except $k_1 = k_2$, umbilic)

(15) Qualitative info about II:

Elliptic: $k_1, k_2 > 0$ or $k_1, k_2 < 0$ (not really two cases)

Hyp: ~~k~~ different signs

Parab: one is zero

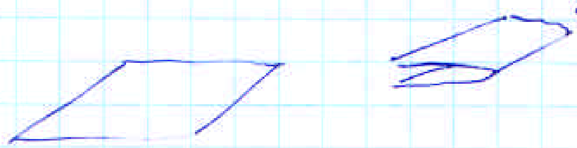
~~Hyp~~: Ell: $(\nabla_X N \cdot X)$ is always +ve ~~...~~

$\therefore$

Par: $(\nabla_X N \cdot X)$ +ve or zero

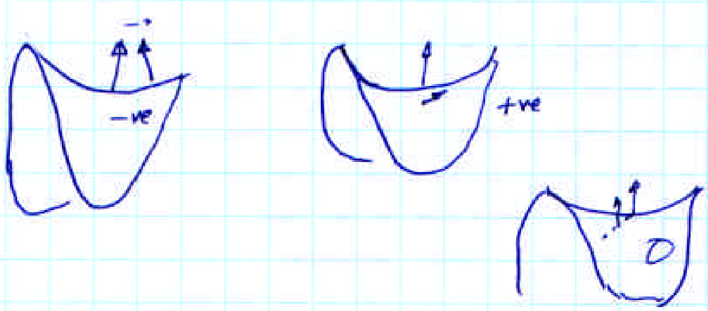
$\frac{(0,\ k_2 y,\ 1)}{\sqrt{\quad}}$ $\therefore$ move along x, no chan[ge] y, chang[e]

hence, fold.

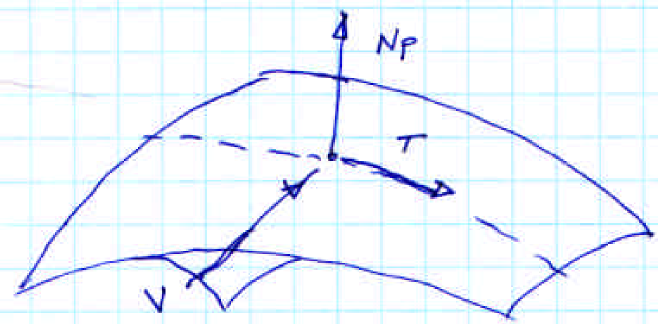


Hyp: $(\nabla_X N \cdot X)$ can be +ve, -ve or 0

assymptotic direction



(16) Direction of C. Gen on surface



Now $\qquad N \cdot V = 0$

$$\therefore \quad \nabla_T N \cdot V + \underbrace{N \cdot \nabla_T V}_{} = 0$$

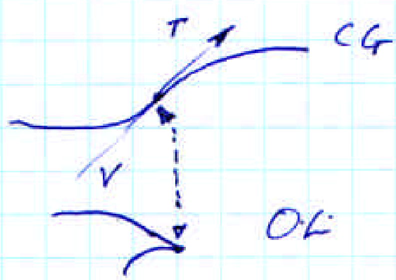
orth: obviously zero, because V is const

persp: also zero (ex:!)

$\therefore \quad II(T, V) = 0$

what happens if V ~~T~~ is assymptotic?
$(T = V)$

and outline cusps.



(17) N.P.R.

3 issues:

- how do we hatch? ON curve field
  - umbilics – ignore
  - hatch evenly, then remove
- how many hatch curves do we show?
  - Depends on shading e.g.
    - 0 · bright → usual / contrast
    - 1 · MID ~~dark~~
    - 2 · shadow
    - · undercuts (dense cross hatching)
- how do we compute?
  - silhouette $(N \cdot V = 0)$
  - lines of curvature.

Afterword

I have avoided some issues in these notes and lectures. To do the geometry I described rigorously, one has to be able to describe how to move curves and surfaces into the configuration I used. With plane curves, this is easy --- one constructs a coordinate system with origin at the point of interest, one axis the normal and the other axis the tangent. I now construct a parametric form in this coordinate system, where when the parameter is zero, the curve passes through the origin; I take a Taylor series, and do whatever rearranging is necessary to get the form

$$\left(t, \kappa\left(\frac{t^2}{2}\right)\right) + \text{higher order terms}$$

We actually did this for the circle; while it is a sweat, it is in principle straightforward. In practice, it is quite easy to develop equations for the curvature in terms of derivatives of a parametrisation. If you use my characterization of curvature, that

$$\frac{dN}{ds} \cdot T = \kappa$$

then you will find that, for a parametric curve

$$(x(t), y(t))$$

the curvature is given by

$$\frac{\left(x''y' - y''x'\right)}{\left(\left(x'\right)^2 + \left(y'\right)^2\right)^{(3/2)}}$$

where the dashes denote derivatives with respect to the parameter, we don't care about a possible missing minus sign, and the ugly typesetting is entirely Microsoft's fault.

Now two things have happened in this choice of coordinate system: firstly, we got the curve to pass through the origin with its tangent along the x-axis; and --- what is more important --- we arranged the parameter to move along the curve at unit speed. In effect, this means that the coefficient of t in the first term was one --- if it were 2, then the coefficient of t^2 wouldn't be half the curvature, it would be twice the curvature. In the equation, this is dealt with by differentiating the normal with respect to arclength.

Now things are more interesting for surfaces. I moved the surface to a coordinate system where it looked like

$$\left(s, t, \left(\frac{1}{2}\right)\left(as^2 + 2bst + ct^2\right)\right) + \text{higher order terms}$$

This can always be done, but involves more interesting problems. In particular, I may have to rearrange the parametrisation of the surface so that the parameter curves meet at right angles at the point that interests me. Of course, this can always be done, but it is much more of a nuisance to do.

This is where the first fundamental form comes in. I didn't use it, because I didn't need it. In general, however, it is usually easier to correct your calculation for the fact that the two parameter curves (a) travel across the surface at different speeds and (b) are seldom orthogonal than it is to rearrange the parametrization. The first fundamental form is a record of the speed of the parameter curves and their angle to one another. The actual process of correction is given in textbooks, below, as is a series of expressions for Gaussian curvature, mean curvature, etc. The one thing that is worth memorizing is the definition of Gaussian curvature as the limit of a ratio of areas; I will show the correction for this case. If you get this definition, then it is actually quite easy to recall a formula for Gaussian curvature.

In particular, recall that when the parametrisation was orthonormal (i.e. unit speed parameter curves which are orthogonal) and I have rotated the surface so that the second fundamental form is diagonal, the Gaussian curvature is the product of the diagonal values. Now this is just a product of eigenvalues, so that when the parametrization is orthonormal, the Gaussian curvature is given by the determinant of the matrix

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

from above; write this determinant as det(II). Now if I move ds along the s parameter curve and dt along the t parameter curve, and the parameter curves are orthonormal, I create a little rectangle of area (ds dt) on the surface and of area (det(II)ds dt) on the Gauss map, so in this case (det(II)) would be the Gaussian curvature. Of course, if the parameter curves are not orthonormal, I need to correct the measurement on the surface. Now assume that the surface is x(s, t) --- x is a three vector.
Take the matrix I shall denote by I --- the first fundamental form --- whose entries are

$$\begin{pmatrix} x_s \cdot x_s & x_t \cdot x_s \\ x_t \cdot x_s & x_t \cdot x_t \end{pmatrix}$$

Now if I take a step ds along the s parameter curve and dt along the t parameter curve, the area swept out on the surface is (det(I) ds dt). This means that, for the case of a general parametrisation, the Gaussian curvature is

$$\left(\frac{\det(II)}{\det(I)}\right)$$

which simplifies to what we had before in the orthonormal case because there we have that I is the identity.

There are numerous other formulae for the adventurous. I’ve never bothered to memorize them, and just look them up. My own experience is that one needs a clear understanding of what these objects mean much more than one needs their equations; of all these, the Gauss map is the most important, which is why I made such a fuss about it. While it should be obvious that Gaussian curvature is really significant, I can’t recall any application in vision or graphics where mean curvature was an important issue.

Good textbooks:

Elementary Differential Geometry
by Barrett O'Neill
I’ve always found this easy to read and informative; apparently the exercises are full of typos, and some proofs are incomplete, though I’ve never noticed.

Lectures on Classical Differential Geometry
by Dirk Jan Struik
Helpful, but quite hard to read because of a complicated and now old-fashioned notation. Can get it very cheap.

Differential Geometry of Curves and Surfaces
by Manfredo P. Do Carmo,
Many people learned geometry from this; I found it a bit dull.

Schaum's Outline of Differential Geometry (Schaum's)
by Lipschutz,
The cover seems to have changed, but I think this is the Schaum’s book that I used to use to look up formulae, etc, and whenever I got confused. Outline books are often very nice indeed; this one concentrates on curves and surfaces, which annoys mathematicians but is good for us.