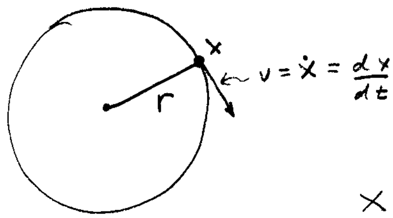
# Intro to Simulation

Simple System:




speed in Rad/s

$$x = [\cos \omega t, -\sin \omega t]\, r$$

$$v = \dot{x} = [-\sin \omega t, -\cos \omega t]\, \omega r \quad \text{assume } r \text{ is const}$$

$$a = \ddot{x} = [-\cos \omega t, \sin \omega t]\, \omega^2 r = -x\omega^2$$

Think about First order system
<u>ie</u> assume we know $\dot{x}(t)$

$$\Rightarrow \quad \dot{x} = \dot{x}(t)$$

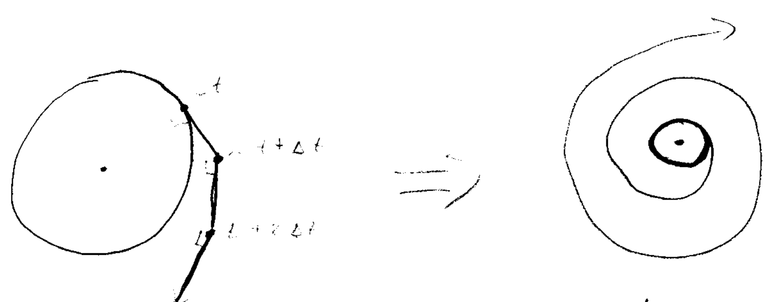
$$x_t = x_0 + \int_0^t \dot{x}(\tau)\, d\tau$$

$$\boxed{\underbrace{x_{t+\Delta t}}_{\text{what we want to know}} = \underbrace{x_t}_{\text{known}} + \underbrace{\int_t^{t+\Delta t} \dot{x}(\tau)\, d\tau}_{\text{we can estimate this}}}$$

## Euler's Integration (Forward Euler)

$$x_{t+\Delta t} = x_t + \underbrace{\Delta t\, \dot{x}_t}_{\text{Assume } \dot{x} \text{ is const over } \Delta t}$$

what happens?




spiral outwards
"Unstable" - diverges

② How Fast?

<u>Backwards Euler</u> – An implicit method

$$X_{t+\Delta t} = X_t + \Delta t \, \underline{\dot{X}_{t+\Delta t}}$$

Compare to Fwd. Euler

For this <u>toy</u> example we know $\dot{X}(t)$ ...

Spirel inwards

Normally $\dot{X}$ is some function of $X$:

$$\dot{X} = K(X)$$

So how do we know $\dot{X}_{t+\Delta t}$ if we don't know $X_{t+\Delta t}$

<u>Semi-Implicit:</u>

$$\dot{X}_{t+\Delta t} \approx \frac{\partial \dot{X}}{\partial X_t} \cdot \Delta X + \dot{X}_t \qquad \Delta X = X_{t+\Delta t} - X_t$$

Jacobian of $\dot{X}$ wrt $X$ @ $X_t$

So: $X_{t+\Delta t} = X_t + \Delta t \, \dot{X}_{t+\Delta t}$

$$X_{t+\Delta t} \approx X_t + \Delta t \left( \frac{\partial \dot{X}}{\partial X_t} \cdot \Delta X + \dot{X}_t \right)$$

$$\cancel{X_t} + \Delta X \approx \cancel{X_t} + \Delta t \left( J \cdot \Delta X + \dot{X}_t \right)$$

$$\Delta X - \Delta t J \cdot \Delta X \approx \Delta t \, \dot{X}_t$$

$$(I - \Delta t J) \Delta X \approx \Delta t \, \dot{X}_t$$

$$\Delta X \approx (I - \Delta t J)^{-1} \Delta t \, \dot{X}_t$$

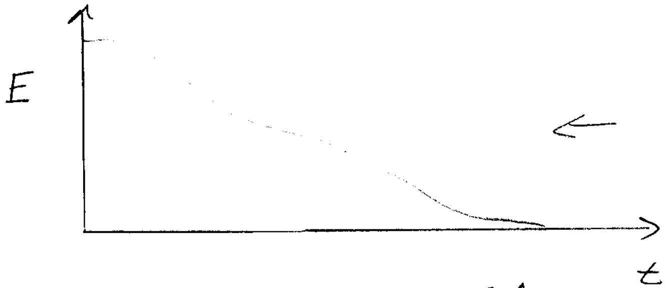
ΔX from Forward Euler

$$\Delta X_B \approx (I - \Delta t J)^{-1} \Delta X_F$$

as $\Delta t \to 0$
$\Delta X_B \to \Delta X_F$

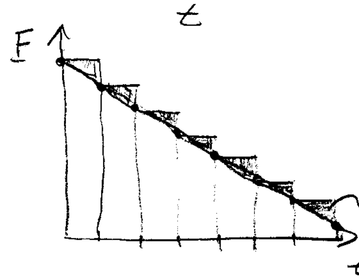
## Another Perspective

* Not quite true but close enough to make the point



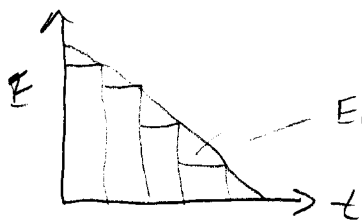

Energy decreases monotonically w/ time

Explicit
ie Fwd. Euler



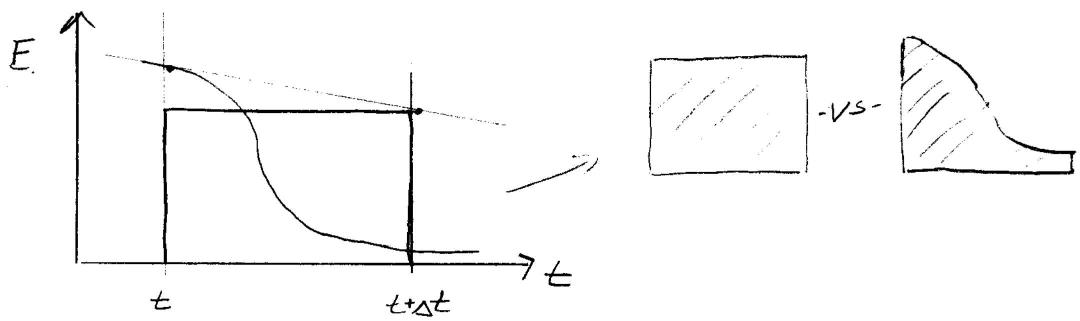

Errors <u>add</u> energy

(Semi-) Implicit
ie Bkwd. Euler




Errors <u>lose</u> energy

* Semi-Implicit may still go unstable:




** Forward Euler is about the worst thing method you could use

See text on numerical integration of ODEs

(or N.R. in C. is you want the "Cliff Notes" version)

* For physical systems what we really have a second order ODE.

  i.e. we know $\ddot{x}(x)$

  ↳ eg a spring

$$F = m\ddot{x}$$
$$F = -K(x - x_0)$$
$$\rightarrow \ddot{x} = -\frac{K(x - x_0)}{m}$$

Can turn into First order with

$$q = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}, \quad \dot{q} = \begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix}$$

but

$$x_{t+\Delta t} = x_t + \Delta t\, \dot{x}_t + \tfrac{1}{2} \Delta t^2 \ddot{x}_t = x_t + \tfrac{1}{2}\Delta t(\dot{x}_t + \dot{x}_{t+\Delta t})$$
$$\dot{x}_{t+\Delta t} = \dot{x}_t + \Delta t\, \ddot{x}_t$$

↑ a modified Fwd. Euler

is better than

$$q_{t+\Delta t} = q_t + \Delta t\, \dot{q}_t \Rightarrow \begin{aligned} x_{t+\Delta t} &= x_t + \Delta t\, \dot{x}_t \\ \dot{x}_{t+\Delta t} &= \dot{x}_t + \Delta t\, \ddot{x}_t \end{aligned}$$

* Think about why we can detect jerk in physical motions