Scene Graphs - Simple version

Stack Contains "graphics state"
- Trans Form
- Materiel Props
- Active lights
- Other stuFF

Nodes Contain
- Geometry
- StuFF that may change graphics state

Depth-First Traversal
- Push stack when going deeper
- Pop when comming back up

Variation (ie OpenInventor)
- Explicit nodes to pop/push stack.

Euler Angles

$$\Theta = [\Theta_x, \Theta_y, \Theta_z]$$

$$R(\Theta) = R_{\hat{z}}(\Theta_z) \cdot R_{\hat{y}}(\Theta_y) \cdot R_{\hat{x}}(\Theta_x)$$

→ Allows tumbling

→ Gimbal Lock (inconvenient singularities)

→ Non-unique representation of orientations

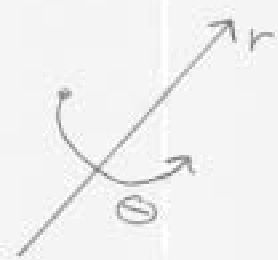
* Generally they should not be used

(U.I.s may be an exception

Angular Displacement -- AKA Exponential maps

Axis of rotation $\hat{r} = \frac{r}{\|r\|}$

Angle of rotation $\Theta = \|r\|$



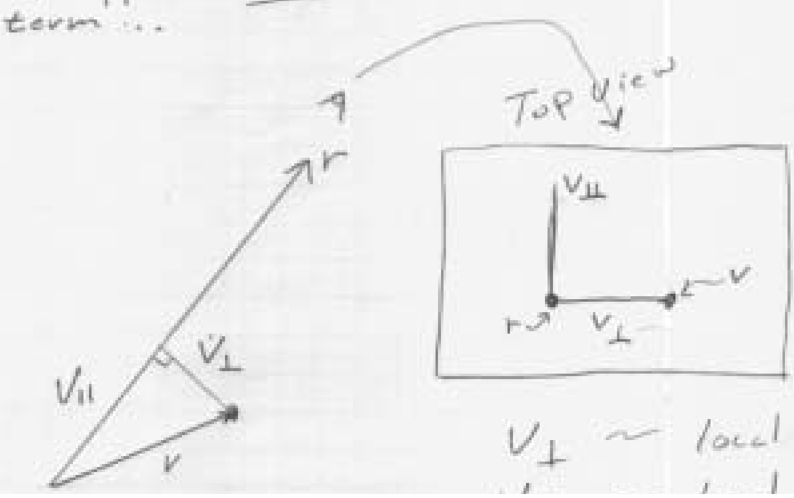
$$v' = V_{\parallel} + V_{\perp} \cos\Theta + V_{\perp\!\!\perp} \sin\Theta$$

$= \hat{r}(\hat{r} \cdot v) +$ ← $V_{\parallel}$

$-\hat{r} \times (\hat{r} \times v) \cos\Theta +$ ← $V_{\perp}$

$\hat{r} \times v \quad \sin\Theta$ ← $V_{\perp\!\!\perp}$

* Note v only appears once in each term ...



$V_{\perp}$ ~ local X Axis

Let's call the opperation $(r\times)$ "r cross-"
a matrix

$$(r\times) = \begin{bmatrix} 0 & -r_3 & r_2 \\ r_3 & 0 & -r_1 \\ -r_2 & r_1 & 0 \end{bmatrix}$$

random side note:
$(r\times) = R(\hat{r}, 90°)$

So $r \times v = (r\times)\cdot v$ ← easy to verify by hand.

$$v' = \underbrace{(\hat{r}\hat{r}^T)v}_{v_{\parallel}} - \underbrace{(\hat{r}\times)(\hat{r}\times)v\cos\theta}_{v_{\perp}} + \underbrace{(\hat{r}\times)v\sin\theta}_{v_{\perp\!\!\perp}}$$

$$= \Big(\hat{r}\hat{r}^T - \cos\theta(\hat{r}\times)(\hat{r}\times) + \sin\theta(\hat{r}\times)\Big)v$$

$$= R(\hat{r}, \theta)v$$

* Recall that rot mat.s are unique.

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what is $\dfrac{\partial v'}{\partial\theta}$?

$$\frac{\partial v}{\partial\theta} = \Big(0 + \sin\theta(\hat{r}\times)(\hat{r}\times) + \cos\theta(\hat{r}\times)\Big)v$$

@ $\theta = 0$ $\quad \dfrac{\partial v'}{\partial\theta} = \hat{r}\times v$

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what is $\hat{r}\times v'$?

$$\hat{r}\times v' = \hat{r}\times v_{\parallel} - \hat{r}\times v_{\perp}\cos\theta + \hat{r}\times v_{\perp\!\!\perp}\sin\theta$$

$$= \Big(0 - (\hat{r}\times)(\hat{r}\times)(\hat{r}\times)\cos\theta + (\hat{r}\times)(\hat{r}\times)\sin\theta\Big)v$$

$$= \Big((\hat{r}\times)\cos\theta + (\hat{r}\times)^2\sin\theta\Big)v$$

$$= \frac{\partial v'}{\partial\theta}$$

Note $(\hat{r}\times)^3 = -(\hat{r}\times)$

$$\boxed{\star \Rightarrow \frac{\partial v'}{\partial\theta} = (\hat{r}\times)v'}$$

Cont.

$$\frac{\partial v'}{\partial \theta} = (\hat{r}\times)v'$$

$\hookrightarrow$ looks like $\frac{\partial X}{\partial t} = KX \Rightarrow X = Ce^{Kt}$

So

$$* \rightarrow \boxed{\Rightarrow v' = e^{(\hat{r}\times)\theta} v}$$

Since $v' @ \theta = 0$ is v

$e^{(\hat{r}\times)0} = e^0 = I$

But what does e^A
where A is a matrix mean?

Recall

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{i=0}^{\infty} \frac{x^i}{i!}$$

For example - Euler ~1???

$$e^{i\theta} = 1 + \frac{i\theta}{1!} + \frac{-\theta^2}{2!} + \frac{-i\theta^3}{3!} + \frac{\theta^4}{4!} + \cdots$$

$$= \left(1 + \frac{-\theta^2}{2!} + \frac{\theta^4}{4!} + \cdots\right) + i\left(\theta + \frac{-\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots\right)$$

$$= \cos\theta + i \sin\theta$$

Try same w/ $e^{(\hat{r}\times)\theta}$:

$$e^{(\hat{r}\times)\theta} = I + \frac{(\hat{r}\times)\theta}{1!} + \frac{(\hat{r}\times)^2\theta^2}{2!} + \frac{(\hat{r}\times)^3\theta^3}{3!} + \frac{(\hat{r}\times)^4\theta^4}{4!}$$

but $(\hat{r}\times)^3 = -(\hat{r}\times)$

$$= I + \frac{(\hat{r}\times)\theta}{1!} + \frac{(\hat{r}\times)^2\theta^2}{2!} - \frac{(\hat{r}\times)\theta^3}{3!} - \frac{(\hat{r}\times)^2\theta^4}{4!} + \cdots$$

$$= I + \left(\frac{\theta}{1} - \frac{\theta^3}{3!} + \ldots\right)(\hat{r}\times) + \left(\frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \ldots\right)(\hat{r}\times)^2$$

$$= I + (\hat{r}\times)\sin\theta - (\hat{r}\times)^2(\cos\theta - 1)$$

$$e^{(\hat{r}\times)\theta}v = v' = \left(I + (\hat{r}\times)\sin\theta - (\hat{r}\times)^2(\cos\theta - 1)\right)v$$

$$= \left(I + (\hat{r}\times)^2 + (\hat{r}\times)\sin\theta - (\hat{r}\times)^2\cos\theta\right)v$$

$$= \left(\hat{r}\hat{r}^T + (\hat{r}\times)\sin\theta - (\hat{r}\times^2)\cos\theta\right)v$$

$$= R(\hat{r},\theta)v$$

$$R = R(\hat{r},\theta) = e^{(\hat{r}\times)\theta} = e^{(r\times)}$$

No "hat"

$$\text{so } (r\times) = \ln R = \frac{\theta}{2\sin\theta}\left(R - R^T\right)$$

$$\theta = \cos^{-1}\left(\underbrace{\frac{Tr(R) - 1}{2}}\right) \leftarrow \underline{R3}$$

guess where we get this from...

Hint in R2 the -1 goes away.

Trivia About Exp Map

- Allows tumbling
- No gimbal lock
- Orientations are r w/ $\|r\| \le \pi$ (Ball of Radius π)
- "Nearly" unique representation of orientations
- Singularities $2K\pi$ & $(2K+1)\pi$ $\quad K \in \mathbb{Z}^+$
- Nice for interpolation

Quaternions - Hamilton 1843

Vector in R3

$$q = (z_1, z_2, z_3, s) = (z, s)$$

$$= z_1 i + z_2 j + z_3 k + s$$

$$i^2 = j^2 = k^2 = -1$$

$$ij = k \qquad ji = -k$$

$$jk = i \quad \text{but} \quad kj = -i$$

$$ki = j \qquad ik = -j$$

Both Quats.

$$q \cdot p = (\underbrace{z_p s_q + z_q s_p + z_p \times z_q}_{z}, \underbrace{s_p s_q - z_p \cdot z_q}_{s})$$

$$q^* = (-z, s)$$

$$\|q\|^2 = s^2 + \|z\|^2 = q \cdot q^*$$

Vector $v \in R^3$ as a quat:

$$v = (v, 0)$$

Orientation as a quat:

$$r = (\hat{r} \sin\frac{\theta}{2}, \cos\frac{\theta}{2})$$

$v' = R(r)v = r \cdot v \cdot r^* \leftarrow$ expand & compare to $v' = e^{(r\times)}v$

$r = r_1 \cdot r_2 \leftarrow$ Compose rotations

Trivia

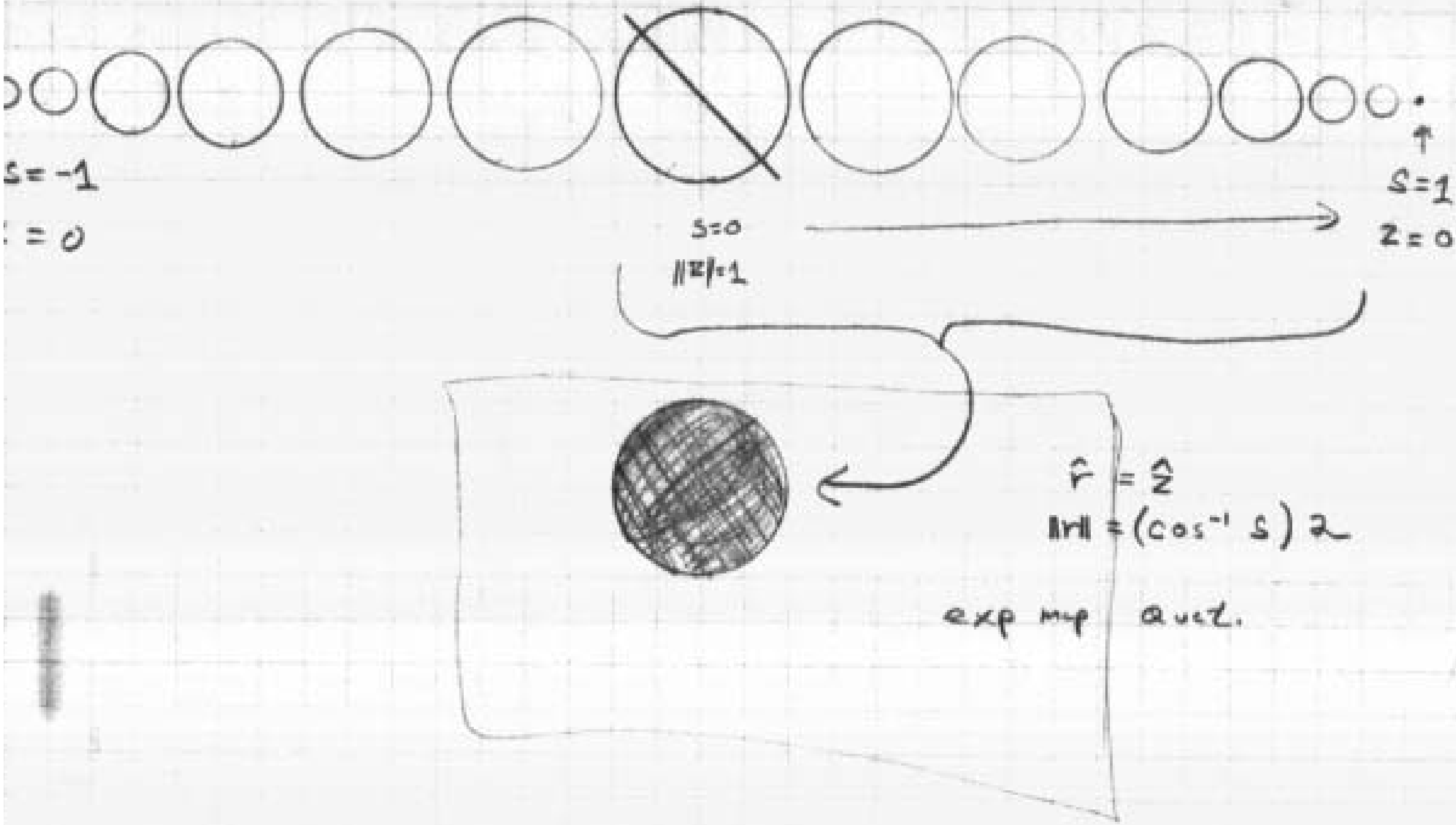
No Tumbling

No gimbal lock -- No singularities at all

Orientations are "surface" of 3-manifold sphere in R4

Double rep of Orientations

Nice for interpolating



s = -1
= 0
s=0
||z||=1
s=1
z=0
r̂ = ẑ
||r|| = (cos⁻¹ s) 2
exp map quat.