Particle Systems

$\ddot{x} = 1/m \left(F_g + F_i + F_e + F_r \right)$

F_g - gravity

F_i - inter particle -- beware $O(n^2)$ work

F_e - external Force ⟵ eg: Damping, Attractors, Repulsars

F_r - "rocket" Forces

Attach properties to particle that also evolve:

Temp, age, etc...

w/ Lot of particles: - sprays of water
- explosions
- Flocks of birds ⟵ * special rules
- smoke

Spring & Mass Systems -- "Linear Springs"

Basicly a particle system w/ special forces

F_{ij} - Force between x_i & x_j

$$F_{ij} = \frac{x_i - x_j}{\|x_i - x_j\|} \left(\|x_i - x_j\| - l_{ij} \right) K_{ij}$$

Spring strength

Spring length

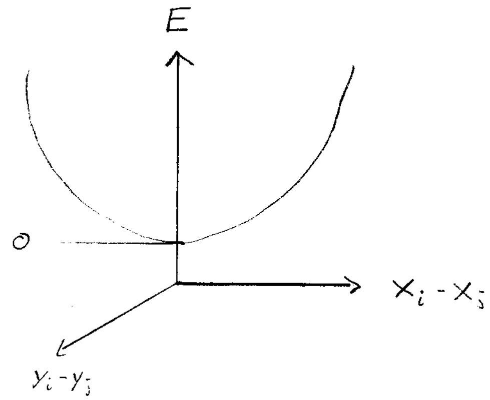
energy in spring

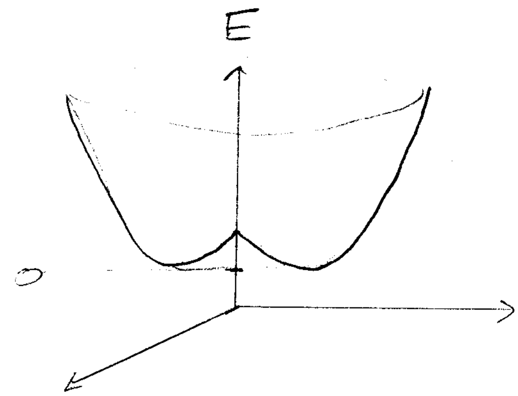
$E_{ij} = 1/2 \, 1/K \left(F_{ij} \cdot F_{ij} \right)$ ← think about Kd^2)

$d = \|x_i - x_j\| - l_{ij}$

IF $l_{ij} = 0$: $\quad F_{ij} = (x_i - x_j) K_{ij}$

Energy plots:





E_{ij} when $l_{ij} = 0$

linear Frc.
Quadratic Energy

$l_{ij} \neq 0$

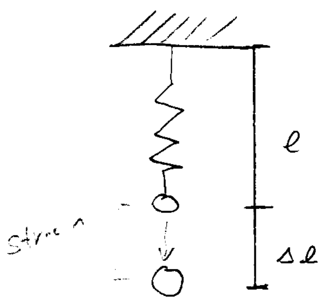
Non-linear Frc
Non-Quadratic energy

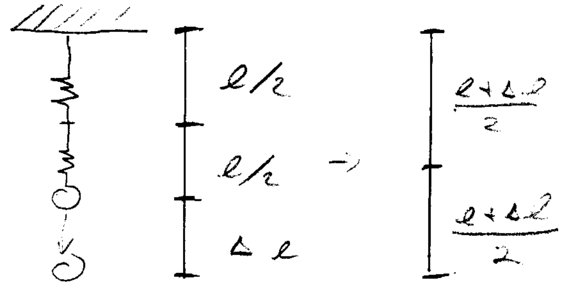
Damping Force:

$$F = -\dot{x} \quad \leftarrow \text{"mass proportional damping"}$$

$$F = \frac{x_i - x_j}{\|x_i - x_j\|^2} (x_i - x_j) \cdot (\dot{x}_i - \dot{x}_j) \quad \leftarrow \text{"stiffness prop. damping"}$$

A problem

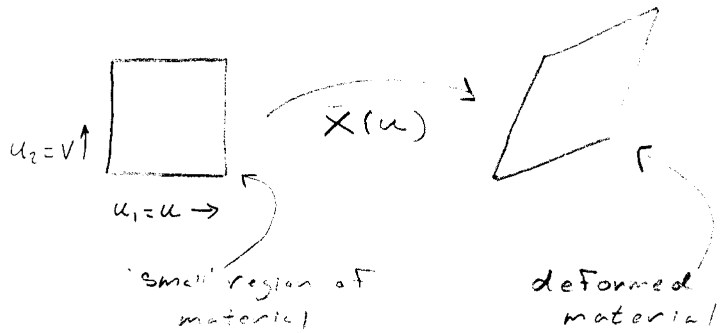




$$F = -k \Delta l \quad \xleftrightarrow{?} \quad F = -k \frac{\Delta l}{2}$$

* Strain $\sim \Delta l / l_0$
* Still 1D

Non-1D object



$X(u)$ defines deformation

* Assume $u \perp v$ in undeformed param.

Greens Strain:

$$\varepsilon_{ij} = \left(\underbrace{\frac{\partial X_k}{\partial u_i} \frac{\partial X_k}{\partial u_j}} - \delta_{ij} \right) \frac{1}{2}$$

↳ AKA 1st metric tensor of 1st Fund. Form

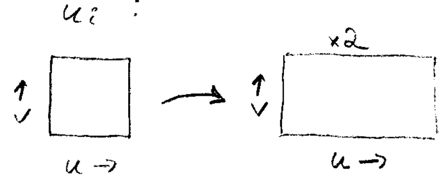
Consider if $X(u) = u$ ie not deformed

then $\frac{\partial X}{\partial u_i}$ are unit length & orthogonal

So ⟶ $\frac{\partial X_k}{\partial u_i} \frac{\partial X_k}{\partial u_j} = \delta_{ij}$

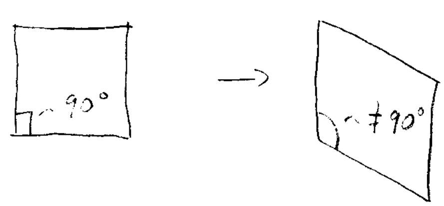
⟶ $\varepsilon_{ij} = \frac{1}{2}(\delta_{ij} - \delta_{ij}) = 0$

Scale by λ in a direction aligned with u_i:



$$\varepsilon_{ij} = \overset{\text{F.F.}}{\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}} - \overset{\delta}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

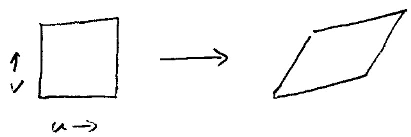
Sheer changes the angles:



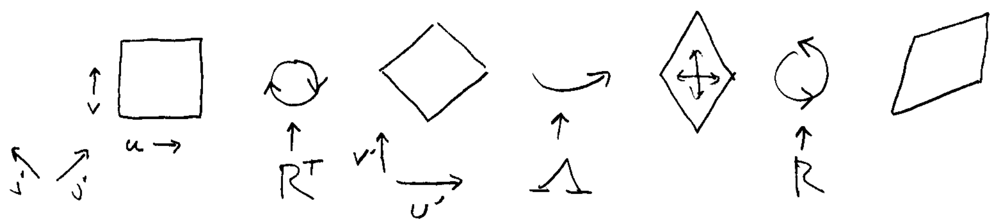
Since ε is real & symetric

$$\varepsilon = R \Lambda R^T \quad \leftarrow \text{eigen decomp}$$

But this is just a transform to diff. coordinate system, so -



This can be seen as the following



Another way to see what ε is

$$\varepsilon_{ij}\, du_i\, du_j = \left((\partial_i X_k)(\partial_j X_k) - \delta_{ij}\right) du_i\, du_j$$

$$= (du_i\, \partial_i X_k)(du_j\, \partial_j X_k) - du_k\, du_k$$

$$= dX_k\, dX_k - du_k\, du_k$$

$$= |dX_k|^2 - |du|^2$$

Diff of square dist.

Strain is geometric, like distance
Stress is like Force ...

$$\sigma_{ij} = C_{ijkl}\,\varepsilon_{kl}$$ ← General linear relation
lots of redundancy
+ isotropy → only two
free parameters

$$\sigma_{ij} = \lambda\,\delta_{ij}\,\varepsilon_{kk} + 2\mu\,\varepsilon_{ij}$$

λ, μ -- Lamé constants

→ Rigidity
→ Incompressibility.

Analogies

$$\Delta d \to \varepsilon$$
$$f \to \sigma$$
$$f = -kd \to \sigma_{ij} = C_{ijkl}\,\varepsilon_{kl}$$

$$\eta = \tfrac{1}{2}\,\sigma_{ij}\,\varepsilon_{ij}$$ ← Elastic Enery Density

$$\sigma_{ij}\,\hat{n}_i = T_j$$ ← Traction T on surface normal to $\hat{n}$

Let Ω be some region where we have material:

e.g. a bowl.

$X(u)$ is defined on $u \in \Omega$

$$E = \int_\Omega \eta \, du$$

↑ total elastic energy

η is a function of $X(u)$'s derivatives

So if $x(u) = x(u, c) = \sum c_i b(u)$

then $\eta = \eta(u) = \eta(u, c)$

* so $E = E(c)$

Generalized Force:

$X(u)$ changes with c_i's ← c controls shape

F_{c_i} ← generalized Force on c_i

$$F_{c_i} = \frac{\partial E}{\partial c_i} = \int_\Omega \frac{\partial \eta}{\partial c_i} \, du$$

* What basis to use? -- This is an important question it makes most of the diff between

FEM
F.D.
SPH
etc...

* How to evaluate the integrals?

Damping Forces

Already saw simple damping on poin mass

& internal damping for a spring

-- Want same generalization
For solid bit of material

Strain Rate

$$\frac{d\,\varepsilon_{ij}}{dt} = \dot{\varepsilon}_{ij}$$

$$= \frac{1}{2}\left(\frac{\partial X_k}{\partial u_i} \frac{\partial \dot{X}_k}{\partial u_j} + \frac{\partial X_k}{\partial u_j} \frac{\partial \dot{X}_k}{\partial u_i} \right)$$

Isotropic

$$\sigma_{ij} = D_{ijkl}\,\dot{\varepsilon}_{kl} = \phi\,\dot{\varepsilon}_{kk} + \psi\,2\,\dot{\varepsilon}_{ij}$$

Stress due to strain rate ...
Not same that we saw before

$K = \frac{1}{2}\,\dot{\varepsilon}_{ij}\,\sigma_{ij}$ ← Kinetic Energy Density (Internal)

$E = \int_{\Omega} K \, du$ ← Total internal kinetic energy

$F_{c_i} = \dfrac{\partial E}{\partial \dot{c}_i}$ ← Damping Force on C_i

See 99 paper on Fracture by O'Brien & Hodgins for more info

Standard Form

$$K(d) + C(d, \dot{d}) + M(d, \dot{d}, \ddot{d}) = F_{ext}$$

Linearize (assume no internal Force @ $d / \dot{d} = 0$)

$$Kd + C\dot{d} + M\ddot{d} = F_{ext}$$

For non-linear system K, C & possibly M change as d changes, but we can analyze the linearized system:

eg assume no damping ($C=0$) & no external Forces ($F_{ext}=0$)

$$Kd + M\ddot{d} = 0$$

$$-(M^{-1}K)d = \ddot{d}$$

Assume $W \Delta W^{-1} = -M^{-1}K$ (eigen decomp)

Modal decomp

$$W \Delta W^{-1} d = \ddot{d}$$

$$\Delta W^{-1} d = W^{-1}\ddot{d}$$

$z = W^{-1} d$

$$\Delta z = \ddot{z}$$

Diagonal!!

Columns of W are vibrational modes of system:

A discplacement in that shape produces an acceleration in that shape as well...

See 2002 paper by O'Brien et al on sound for more info.

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* Linearizing Green's strain is same as using Cauchy's

$$\varepsilon_{ij} = \tfrac{1}{2}\left(\frac{\partial x_k}{\partial u_i}\frac{\partial x_k}{\partial u_j} - \delta_{ij}\right)$$

Finite

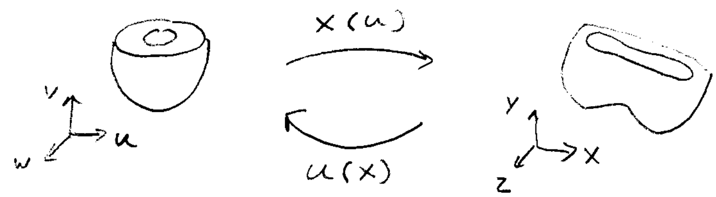
$$\varepsilon_{ij} = \tfrac{1}{2}\left(\frac{\partial x_i}{\partial u_j} + \frac{\partial x_j}{\partial u_i}\right) - \delta_{ij}$$

Infinitesimal

Fluids

Eulerian -vs- Lagrangian

Do you model $x(u)$? - Lagrange
or
Do you model $u(x)$? - Euler



u -- material coodinates

x -- world coordinates

Solids care about original positions,
Fluids don't ...

IE: For a Fluid $\mu = 0$
(& λ goes into a pressure term)

* $\longrightarrow$ Only need to track $\dot{x}$ (or $\dot{u}$)

Consider a region of space $\rightarrow \square$ e

How do the properties of material in e change w/ time?

$V(x)$ - Velocity of stuff @ x (eg stuff in e)

$$\frac{dV}{dt} = \frac{\partial V}{\partial t} + \underbrace{(V \cdot \nabla) V}_{\text{advection}}$$

ie

$$\frac{dV_x}{dt} = \frac{\partial V_x}{\partial t} + V \cdot \nabla V_x$$

("V · grad of V_x")

So what is $\frac{\partial V}{\partial t}$?

→ Pressure term $-\nabla P$

-- Stuff goes high to low pressure

→ Viscosity -- Just like internal friction

$$\dot{\varepsilon}_{ij} = \tfrac{1}{2}\left(\frac{\partial x_k}{\partial u_i}\frac{\partial \dot{x}_k}{\partial u_j} + \frac{\partial x_k}{\partial u_j}\frac{\partial \dot{x}_k}{\partial u_i} \right)$$

$$= \tfrac{1}{2}\left(\partial_i \dot{x}_j + \partial_j \dot{x}_i \right)$$ ← if $x = u$ why?? !?

$$\sigma_{ij} = \phi\, \delta_{ij}\, \dot{\varepsilon}_{kk} + 2\psi\, \dot{\varepsilon}_{ij}$$

$$= \phi\, \delta_{ij}\, \tfrac{1}{2}\left(\partial_k \dot{x}_k + \partial_k \dot{x}_k\right) + \psi\left(\partial_i \dot{x}_j + \partial_j \dot{x}_i\right)$$

$$= \phi\, \delta_{ij}\, \partial_k \dot{x}_k + \psi\left(\partial_i \dot{x}_j + \partial_j \dot{x}_i\right)$$

$$F_i = \frac{\partial \sigma_{ij}}{\partial x_j} = \phi\, \delta_{ij}\, \partial_k \partial_j \dot{x}_k + \psi\left(\partial_i \partial_j \dot{x}_j + \partial_j \partial_j \dot{x}_i\right)$$

$$= \phi\, \partial_k \partial_i \dot{x}_k + \psi\left(\partial_i \partial_j \dot{x}_j + \partial_j \partial_j \dot{x}_i\right)$$

Stokes Fluid → $\sigma_{kk} = 0$ → $\phi = -\frac{2}{3}\psi$

$$= (\phi + \psi)\, \partial_i \partial_j \dot{x}_j + \psi\, \partial_j \partial_j \dot{x}_i$$

$$= \frac{\psi}{3}\, \partial_i \partial_j \dot{x}_j + \psi\, \partial_j \partial_j \dot{x}_i$$

$$= \frac{\psi}{3}\, \nabla(\nabla \cdot \dot{x}) + \psi\, (\nabla \cdot \nabla)\, \dot{x}$$

ψ is coeff of viscosity

Note -- viscosity is mostly negligible

Mass conservation :

$$\frac{d\rho}{dt} = -\nabla \cdot (\rho \dot{x})$$

$$0 = -\nabla \cdot (\rho \dot{x})$$ ← if incompressible

Rigid Body Dynamics

Translational Parst are like a point mass

- Rotations

E is kinetic energy from rotational part of motion

$$E = \int_\Omega \tfrac{1}{2} \rho \, V \cdot V \, du$$

V is vel of point in local coord sys

$$= \int_\Omega \tfrac{1}{2} \rho \, V_i V_i \, du$$

ω is angular velocity in local coords

$$= \int_\Omega \rho \tfrac{1}{2} \varepsilon_{ijk} \omega_j x_k \varepsilon_{iab} \omega_a x_b \, du$$

x is pos in local coords

$$H = \frac{\partial E}{\partial \omega}$$

H momentum (angular) (work/it)

$$H_p = \frac{\partial E}{\partial \omega_p}$$

$$= \int_\Omega \rho \tfrac{1}{2} \left(\varepsilon_{ijk} \delta_{jp} x_k \varepsilon_{iab} \omega_a x_b + \varepsilon_{ijk} \omega_j x_k \varepsilon_{iab} \delta_{pa} x_b \right) du$$

$$= \int_\Omega \rho \tfrac{1}{2} \left(\varepsilon_{ipk} x_k \varepsilon_{iab} \omega_a x_b + \varepsilon_{ijk} \omega_j x_k \varepsilon_{ipb} x_b \right) du$$

$$= \int_\Omega \rho \, \varepsilon_{ipk} x_k \underline{\varepsilon_{iab} \omega_a x_b} \, du$$

$$= \int_\Omega \rho \quad X \times (\omega \times X) \quad du$$

$$\boxed{= \int_\Omega \rho \; X \times V \; du}$$

$$= \omega_a \int_\Omega \rho \, \varepsilon_{ipk} x_k \varepsilon_{iab} x_b \, du$$

$$= \omega_a \int_\Omega \rho \left(\delta_{pa} \delta_{kb} - \delta_{pb} \delta_{ka} \right) x_k x_b \, du$$

$$= \omega_a \int_\Omega \rho \left(\delta_{pa} x_k x_k - x_a x_p \right) du,$$

Inertia Tensor

$$* \quad \boxed{H_p = I_{ap} \omega_a}$$

$$H^w = I^w \omega^w$$

w - world
m - material (aka local))

$$= R \underbrace{I^m}_{\text{const}} R^T \omega^w$$

$* \; I^w = R I^m R^T$

0 - why? $\omega \times R \times \omega = 0$

$$\dot{H}^w = \dot{R} I^m R^T \omega^w + R I^m \dot{R}^T \omega^w + R I^m R^T \dot{\omega}^w$$

$$\dot{R} = [\omega^w \times R]$$

$$= \omega^w \times R I^m R^T \omega^w + R I^m R^T \dot{\omega}^w$$

aka $\alpha = \dot{\omega}$

$$\dot{H}^w = 0 = \omega^w \times H^w + I^w \alpha^w$$

No ext ~~Frc~~ Torque

$$\alpha^w = (I^w)^{-1} (-\omega^w \times H^w)$$

or

$$\alpha^m = (I^m)^{-1} ((I^m \omega^m) \times \omega^m)$$