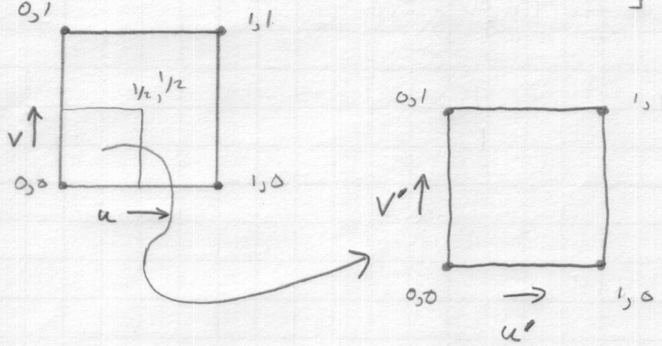
Catmull-Clark Subdivision (B-Spline Sub. Div.)
Ed Jim

$$x(u,v) = [1, u, u^2, u^3]\ \beta_B\ P\ \beta_B^T\ [1, v, v^2, v^3]^T$$

Tensor-product B-spline surface

β_B is B-spline basis matrix

$$\beta_B = 1/6 \begin{bmatrix} 1 & 4 & 1 & 0 \\ -3 & 0 & 3 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix}$$



$v = 1/2\ v'$

$u = 1/2\ u'$

$$x'(u', v') = x(u, v)$$
$$= x(1/2 u', 1/2 v')$$
$$= [1, 1/2 u', 1/4 u'^2, 1/8 u'^3]\ \beta_B\ P\ \beta_B^T\ [1, 1/2 v', 1/4 v'^2, 1/8 v'^3]^T$$
$$= U' S_1\ \beta_B\ P\ \beta_B^T\ S_1^T\ V'^T$$

$$\circledast\ S_1 = \begin{bmatrix} 1 & & & 0 \\ & 1/2 & & \\ & & 1/4 & \\ 0 & & & 1/8 \end{bmatrix}$$

Let $S_1 \beta_B P \beta_B^T S_1^T = \beta_B P_{11} \beta_B^T$

$$\underbrace{\beta_B^{-1} S_1 \beta_B}\ P\ \beta_B^T S_1^T \beta_B^{-T} = P_{11}$$

*** $\rightarrow$ $\boxed{H_1\ P\ H_1^T = P_{11}}$

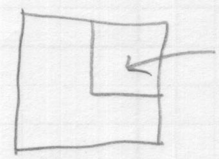
$$x'(u', v') = U' \beta_B P_{11} \beta_B^T V'^T = U' \beta_B H_1 P H_1^T \beta_B^T V'^T$$

$$H_1 = \frac{1}{8}\begin{bmatrix} 4 & 4 & 0 & 0 \\ 1 & 6 & 1 & 0 \\ 0 & 4 & 4 & 0 \\ 0 & 1 & 6 & 1 \end{bmatrix}$$

Can do same for other quadrants

<u>ie</u> $u = \frac{1}{2} + \frac{1}{2}u'$

$V = \frac{1}{2} + \frac{1}{2}V'$



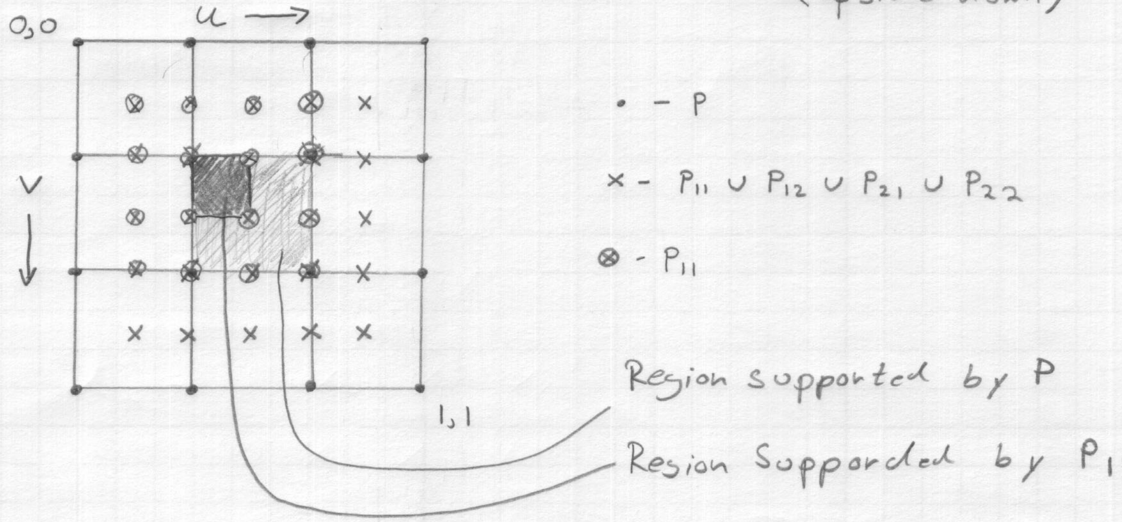
$$S_2 = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} \\ 0 & \frac{1}{2} & \frac{1}{2} & \frac{3}{8} \\ 0 & 0 & \frac{1}{4} & \frac{3}{8} \\ 0 & 0 & 0 & \frac{1}{8} \end{bmatrix}$$

$$H_2 = \frac{1}{8}\begin{bmatrix} 1 & 6 & 1 & 0 \\ 0 & 4 & 4 & 0 \\ 0 & 1 & 6 & 1 \\ 0 & 0 & 4 & 4 \end{bmatrix}$$

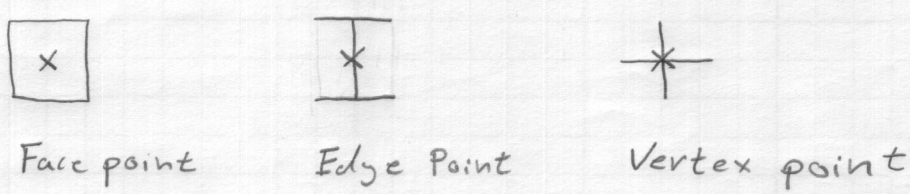
$P_{11} = H_1 P H_1^T$, $P_{12} = H_1 P H_2^T$, $P_{21} = H_2 P H_1^T$, $P_{22} = H_2 P H_2^T$

P_{12}	P_{22}
P_{11}	P_{21}

(upside down)

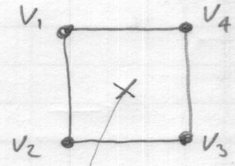


* If you expand $P_{ij} = H_i P H_j$, you will see a pattern:



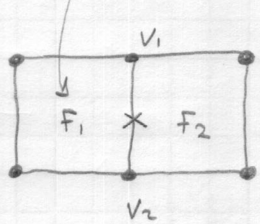
[Note that P_{11}, P_{12}, P_{21} & P_{22} share many common points...

Face Point



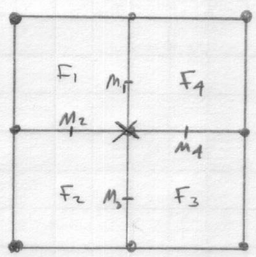
= Average of points defining original Face = $\frac{V_1+V_2+V_3+V_4}{4}$

Edge Point



= Average of adjacent Face & Edge points = $\frac{V_1+V_2+F_1+F_2}{4}$

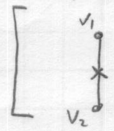
Vertex Point



$$= \frac{\frac{F_1+F_2+F_3+F_4}{4} + \frac{M_1+M_2+M_3+M_4}{4}\,2 + P}{4}$$

old vertex point

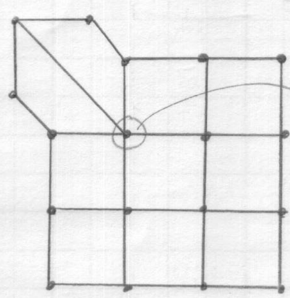
* M_1 is midpoint $\neq$ Edge point !



$M_1 = \frac{V_1+V_2}{2}$

Regular

Irregular



" Extraordinary Vertex "

* Generic rules must be consistant for ordinary vertex

F = Avg of verts defining Face

e = Avg of 2 verts & 2 Face points

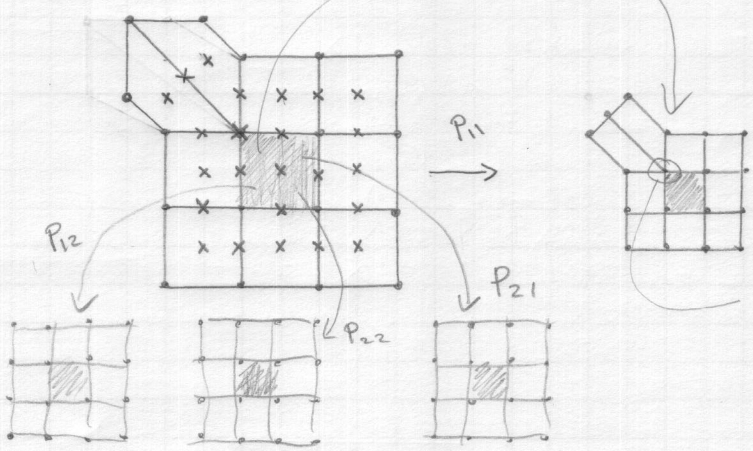
$$v = \frac{\bar{F}}{n} + \frac{2\,\bar{M}}{n} + \frac{P(n-3)}{n}$$

$\bar{F}$ = Avg of Face points

$\bar{M}$ = Avg of mid points

P = Orig Vert

n = Valence of P



P_{11} is still irregular

Still extraordinary vertex w/ same valance.

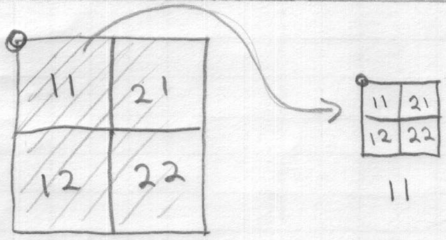
P_{12}, P_{22}, P_{21} - All regular

* New region is smaller version of the old.

Rearrange P & H matrices so you get:

$$\overline{P_{ij} = H_i P H_j^T}^{\text{old}} \iff P_{ij} = A_{ij} P$$

ie make the P's into a vector
and glom H_i & H_j into A_{ij}

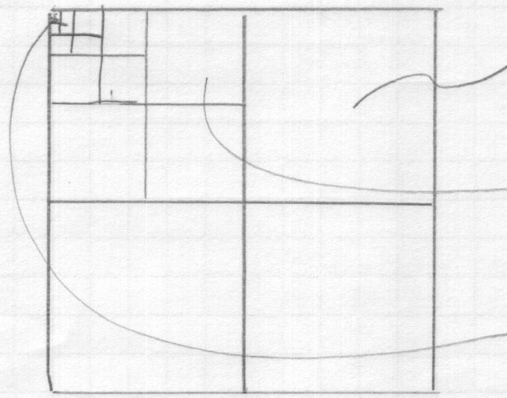


Since ext. vertex has same valance,

A_{11} is the same for the next round.

"This is a stationary scheme."

(2) Recall eigen systems? See Strang if you don't!



$P_{21} = A_{21} P$

$P_{21|1} = A_{21} A_{11} P$

$P_{ij(11)^n} = A_{ij} A_{11}^n P$

Let $A_{11} = V \Lambda V^{-1}$ ← eigen decomp, Λ is diag.

↳ $A_{11}^n = V \Lambda^n V^{-1}$

See paper by Jos Stam in Siggraph 98