TENSOR NOTATION

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If you are interested in using tensor notation:

- A variable without indexes is a scalar.
- An index that is repeated in the same term indicates summation over that index. For example

$$a_i b_i = \sum_{i=1}^{n} a_i b_i \tag{1}$$

  where n is the number dimensions you are working with. This shorthand is often referred to as Einstein's summation convention.

- Repeated indexes are called *dummy indexes*. As the name implies there is nothing special about what variable is used for the dummy index, and as long as it does not conflict with another index in the same term a dummy index can be renamed. For example the equation

$$s = x_i x_i + a_{ij} y_i z_j \tag{2}$$

  is fully equivalent to

$$s = x_j x_j + a_{ij} y_i z_j \tag{3}$$

  and to

$$s = x_j x_j + a_{kl} y_k z_l . \tag{4}$$

- An index that is not repeated means that you should imagine the equation written out n times with that index ranging over $1..n$. For example we can define the components of a vector by

$$v_i = a_{ij} x_j \tag{5}$$

  and of a matrix by

$$a_{ij} = a_i b_j \tag{6}$$

- Non-repeated indexes are called *free indexes*. The variable used for a free index can be renamed but the change must be across all terms in the equation. Also, every term should have the same free indexes. If you find yourself with an equation where different terms have different free indexes, something is probably wrong.
- The Kronecker delta, δ_{ij}, is basically the identity matrix. The value of δ_{ij} is 1 if $i = j$ and 0 if $i \neq j$.
- You can often simplify terms that contain deltas. For example

$$s\delta_{ii} = 3s \tag{7}$$

$$a_i b_j \delta ij = a_i a_i . \tag{8}$$

- The permutation symbol ϵ_{ijk} is 1 when i, j, and k take on values that are clockwise permutations of 123, is -1 when the permutation is counter clockwise, and 0 if any value is repeated. For example

$$\begin{aligned}\epsilon_{123} &= 1 &(9)\\ \epsilon_{231} &= 1 &(10)\\ \epsilon_{321} &= -1 &(11)\\ \epsilon_{213} &= -1 &(12)\\ \epsilon_{112} &= 0. &(13)\end{aligned}$$

- The identities

$$\begin{aligned}\epsilon_{ijk} = \epsilon_{jki} = \epsilon_{kij} &\quad (14)\\ \epsilon_{ijj} = \epsilon_{jij} = \epsilon_{jji} = 0 &\quad (15)\\ -\epsilon_{ijk} = \epsilon_{kji} = \epsilon_{jik} = \epsilon_{ikj} &\quad (16)\end{aligned}$$

  should all be obvious.

- The vector cross product can be expressed with the permutation symbol. The vector equation

$$\boldsymbol{a} = \boldsymbol{b} \times \boldsymbol{c} \quad (17)$$

  is equivalent to

$$a_i = \epsilon_{ijk} b_j c_k. \quad (18)$$

- The identity

$$\epsilon_{ijk}\epsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km} \quad (19)$$

  can be shown by expansion.

- The above just talks about the *notation* used for Cartesian tensors. The really neat thing about tensors is how they transform and how one can write equations that are invariant with respect to changes of coordinate systems. Just because you write an equation using this notation does not necessarily mean you are using tensors. There are also things called general tensors which let you do powerful things easily even when your coordinate system is not orthonormal. For more information see "Introduction to Vector Analysis" by Davis and Snider and "A Brief on Tensor Analysis" by Simmonds.

Show that the following are true:

1. $\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{c} \cdot (\boldsymbol{a} \times \boldsymbol{b}) = \boldsymbol{b} \cdot (\boldsymbol{c} \times \boldsymbol{a})$
2. $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{b}(\boldsymbol{a} \cdot \boldsymbol{c}) - \boldsymbol{c}(\boldsymbol{a} \cdot \boldsymbol{b})$
3. $(\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c} = \boldsymbol{b}(\boldsymbol{a} \cdot \boldsymbol{c}) - \boldsymbol{a}(\boldsymbol{c} \cdot \boldsymbol{b})$
4. $(\boldsymbol{a} \times \boldsymbol{b}) \cdot (\boldsymbol{c} \times \boldsymbol{d}) = (\boldsymbol{a} \cdot \boldsymbol{c})(\boldsymbol{b} \cdot \boldsymbol{d}) - (\boldsymbol{a} \cdot \boldsymbol{d})(\boldsymbol{b} \cdot \boldsymbol{c})$
5. $(\boldsymbol{a} \times \boldsymbol{b}) \times (\boldsymbol{c} \times \boldsymbol{d}) = \boldsymbol{b}(\boldsymbol{a} \cdot \boldsymbol{c} \times \boldsymbol{d}) - \boldsymbol{a}(\boldsymbol{b} \cdot \boldsymbol{c} \times \boldsymbol{d})$